A New Family of Recursive Power-Series Distributions

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Abstract

A new family of recursive power-series distributions involving the generalized hypergeometric function is proposed. The probability generating and moment generating functions are given; and extensions of the Poisson, geometric, negative binomial, log-series, and generalized Waring distributions are presented as examples.

Keywords: discrete distributions, power-series distributions, generalized hypergeometric function
1. Introduction

The hypergeometric function introduced by Gauss can be obtained from the differential equation

\[ \theta(1-\theta)\frac{d^2y}{d\theta^2} + [c - (a + b + 1)\theta] \frac{dy}{d\theta} - aby = 0, \]

where \(a, b, c,\) and \(\theta\) are the parameters. It is shown in Redheffer and Port (1992) that a solution of the form

\[ y(\theta) = \sum_{n=0}^{\infty} a_n \theta^n \]

reduces to

\[ y(\theta) = a_0 \left[ a + \frac{ab}{c} \theta + \frac{a(a+1)b(b+1)}{2!(c+1)} \theta^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!(c+1)(c+2)} \theta^3 + \ldots \right] . \] (1)

The bracketed expression in (1) is called the hypergeometric function and can be expressed as

\[ F(a, b, c; \theta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n \theta^n}{(c)_n n!} , \] (2)

where \((c)_n\) is the shifted factorial, or Pochhammer symbol (Seaborn, 1991), defined as \((c)_0 = 1,\) and \((c)_n = (c)(c+1)\cdots(c+n-1), n = 1, 2, 3, \ldots.\) Note that \(c\) does not have to be an integer.

In this paper we consider a family of discrete random variables \(\{X_k : k = 0, 1, 2, \ldots\}\), each with parameter \(\theta\) and values \(n = 0, 1, 2, \ldots.\) Denote the probability mass function (p.m.f.) of \(X_k\) by \(p_k(n) = P_r[X_k = n].\) We present here a family of power-series distributions \(p_k(n)\) involving the generalized hypergeometric function (Andrews, 1985) defined by

\[ _{s} F_{t}(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n \theta^n}{(b_1)_n \cdots (b_t)_n n!} , \] (3)

where \(s, t\) are non-negative integers. Obviously (2) is a special case. When \(s = t = 0,\) (3) remains valid if in the appropriate position no symbols appear. For example, \(0F_1(-; b_1; \theta) = \sum_{n=0}^{\infty} \frac{\theta^n}{(b_1)_n n!}.\) The series (3) converges for all \(\theta\) if \(s < t + 1.\) When \(s = t + 1,\) it converges for \(|\theta| < 1,\) diverges for \(|\theta| > 1,\) and may do either for \(\theta = 1.\) If \(s > t + 1,\) (3) diverges for all \(\theta\) except \(\theta = 0.\) We assume here that \(s \leq t + 1\) and \(a_1, \ldots, a_s, b_1, \ldots, b_t > 0\) are fixed, with the parameter \(\theta > 0\) such that (3) converges to a positive number. If \(s = t = 0,\) all results remain valid except that, respectively, no \(a_i\) or \(b_j\) appears.

In Section 2 we use (3) to develop a family of recursive power-series distributions and derive their probability and moment generating functions. Then in Section 3 we present five special cases: (i) a recursive Poisson distribution, (ii) a recursive geometric distribution, (iii) a recursive negative
binomial distribution, (iv) a recursive log-series distribution, and (v) a recursive generalized Waring distribution. Each of these special cases yields new discrete distributions. Section 4 contains some concluding remarks and suggestions for further research.

2. Recursive Power-Series Distributions

2.1 Development

We now present several lemmas and a theorem to develop a class of p.m.f involving the generalized hypergeometric function and the recursive operator \( T_k \) (Schwatt, 1962; Comtet, 1973). For a differentiable function \( G(\theta) \) of \( \theta \), define the operator \( T_k \) by

\[
T_k[G(\theta)] = \frac{d^k}{d\theta^k} \left\{ \frac{d}{d\theta} \left\{ \frac{d}{d\theta} \left\{ \cdots \frac{d}{d\theta} G(\theta) \right\} \right\} \right\}, \quad k = 1, 2, 3, \ldots ,
\]

where the operator \( \frac{d^k}{d\theta^k} \) involves the argument \( \theta \) of \( G \). \( T_1 \) is a special case of the Lie derivative, and \( T_k \) represents \( k \) successive applications. Lemma 1 is proved in Schwatt (1962, pp. 81–83).

**Lemma 1** For \( k=1,2,3, \ldots , \)

\[
T_k[G(\theta)] = \sum_{n=0}^{\infty} a_{n,k} \theta^n \cdot \frac{d^n G(\theta)}{d\theta^n},
\]

where \( a_{n,k} = \frac{(-1)^k}{k!} \sum_{i=1}^{\infty} (-1)^i \binom{k}{i} i^n \).

Next, in Lemma 2, we find \( \frac{d^k G(\theta)}{d\theta^k} \) in Lemma 1 for \( G(\theta) = {}_s F_t(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta) \).

**Lemma 2** For \( k=1,2,3, \ldots , \)

\[
\frac{d^k}{d\theta^k} {}_s F_t(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta) = \frac{(a_1)_k \cdots (a_s)_k}{(b_1)_k \cdots (b_t)_k} {}_s F_t(a_1+k, \ldots, a_s+k; b_1+k, \ldots, b_t+k; \theta).
\]

**Proof.** For \( k = 1, 2, 3, \ldots , \) and \( n = 0, 1, 2, \ldots , \) the definition of \( (c)_n \) immediately gives \( (a_i+k)_n = \frac{(a_i)_n (a_i+1) \cdots (a_i+n)}{(a_i)_n} \), \( i = 1, \ldots, s \), and \( (b_j+k)_n = \frac{(b_j)_n (b_j+1) \cdots (b_j+n)}{(b_j)_n} \), \( j = 1, \ldots, t \). It follows from (3) that

\[
{}_s F_t(a_1+k, \ldots, a_s+k; b_1+k, \ldots, b_t+k; \theta) = \frac{(b_1)_k \cdots (b_t)_k}{(a_1)_k \cdots (a_s)_k} \sum_{n=0}^{\infty} \frac{(a_1)_n+k \cdots (a_s)_n+k \theta^n}{(b_1)_n+k \cdots (b_t)_n+k \cdot n!}.
\]
Or,

\[
\frac{(a_1)k \cdots (a_s)k}{(b_1)k \cdots (b_t)k} \cdot F_i(a_1 + k, \ldots, a_s + k; b_1 + k, \ldots, b_t + k; \theta)
\]

\[
= \sum_{n=0}^{\infty} \frac{(a_1)_{n+k} \cdots (a_s)_{n+k} \theta^n}{(b_1)_{n+k} \cdots (b_t)_{n+k} n!}.
\]

But directly from (3),

\[
\frac{d^k}{d\theta^k} F_i(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta)
\]

\[
= \sum_{n=k}^{\infty} \frac{(a_1)_n \cdots (a_s)_n \theta^{n-k}}{(b_1)_n \cdots (b_t)_n (n-k)!}.
\]

\[
= \sum_{n=0}^{\infty} \frac{(a_1)_{n+k} \cdots (a_s)_{n+k} \theta^n}{(b_1)_{n+k} \cdots (b_t)_{n+k} n!}.
\]

The lemma now follows.

Lemma 1 and 2 yield the closed form of \( T_k \left[ F_i(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta) \right] \) by direct substitution.

**Lemma 3** For \( k=1,2,3, \ldots, \)

\[
T_k \left[ F_i(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta) \right] = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{(a_1)_{i} \cdots (a_s)_i (-1)^{i+j}}{(b_1)_i \cdots (b_t)_i i!} \frac{1}{j!} j^k \theta^j F_i(a_1 + i, \ldots, a_s + i; b_1 + i, \ldots, b_t + i; \theta).
\]

We now state our main result as Theorem 1, which establishes that normalizing the result of Lemma 3 yields a new family of discrete recursive power series distributions.

**Theorem 1** Let \( \{X_k : k = 0, 1, 2, \ldots\} \) be a family of discrete random variables, each with parameter \( \theta \) and values \( n = 0, 1, 2, \ldots \). Then

\[
p_0(n) = \frac{\theta^n}{\sum_{n=0}^{\infty} \frac{(a_1)_{n} \cdots (a_s)_n \theta^n}{(b_1)_n \cdots (b_t)_n n!}},
\]

\[
p_k(n) = \frac{\sum_{i=1}^{k} \sum_{j=1}^{i} \frac{(a_1)_{i} \cdots (a_s)_i (-1)^{i+j}}{(b_1)_i \cdots (b_t)_i i!} \frac{1}{j!} j^k \theta^j F_i(a_1 + i, \ldots, a_s + i; b_1 + i, \ldots, b_t + i; \theta)}{\sum_{n=0}^{\infty} \frac{(a_1)_{n} \cdots (a_s)_n \theta^n}{(b_1)_n \cdots (b_t)_n n!}}.
\]

\( k = 1, 2, 3, \ldots, \) is a p.m.f. for \( X_k, k = 0, 1, 2, \ldots \).
Proof. Apply $T_k$ directly to both sides of (3), divide by the resulting left side, and use Lemma 3. For $k = 0$, these steps yield

$$
\frac{T_0\left[\sum_{n=0}^{\infty} \frac{(a_1n\cdots a_s)n^n}{(b_1n\cdots b_t)n^n} \theta^n\right]}{T_0\left[sF_t(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta)\right]} = \frac{\sum_{n=0}^{\infty} \frac{(a_1n\cdots a_s)n^n}{(b_1n\cdots b_t)n^n} \theta^n}{sF_t(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta)}
$$

$$
= \sum_{n=0}^{\infty} p_0(n)
= 1.
$$

Similarly, for $k = 1, 2, 3, \ldots$,

$$
\frac{T_k\left[\sum_{n=0}^{\infty} \frac{(a_1n\cdots a_s)n^n}{(b_1n\cdots b_t)n^n} \theta^n\right]}{T_k\left[sF_t(a_1, \ldots, a_s; b_1, \ldots, b_t; \theta)\right]} = \frac{\sum_{n=0}^{\infty} \frac{(a_1n\cdots a_s)n^n}{(b_1n\cdots b_t)n^n} n^k \theta^n}{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(a_1j\cdots a_sj)(-1)^{j-1}}{n!} (\theta)^i j^k \theta^i F_t(a_1 + i, \ldots, a_s + i; b_1 + i, \ldots, b_t + i; \theta)}
$$

$$
= \sum_{n=0}^{\infty} p_k(n)
= 1.
$$

Since $p_k(n) \geq 0$ for $k = 0, 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$, the proof is complete. 

We call (6) and (7) a family of recursive power-series distributions since the application of (5) in Lemma 3 yields (6) and (7). For $k = 0$ and particular values of $s, t, a_1, \ldots, a_s, b_1, \ldots, b_t, \theta$, some common discrete distributions cases will be shown as special cases of $p_k(n)$ in Section 3. Moreover, we extend these common distributions when $k = 1, 2, 3, \ldots$. An intuitive interpretation is that increasing $k$ represents an evolution of the phenomenon modeled by the stochastic process $\{X_k : k = 0, 1, 2, \ldots\}$ with its fixed parameters $s, t, a_1, \ldots, a_s, b_1, \ldots, b_t, \theta$. For example, increasing $k$ could be viewed as an index of time, dispersion, or transmission. Alternately, each distribution $p_k(n)$ with fixed $s, t, a_1, \ldots, a_s, b_1, \ldots, b_t, \theta$ can be used individually to model phenomena, where $k$ then becomes simply another parameter to be chosen.

### 2.2 The Probability and Moment Generating Functions

The probability generating function (p.g.f.) and moment generating function (m.g.f.) of the general recursive power-series distributions are now presented, from which various moments of the random
variables can be obtained. For fixed \(s, t, a_1, \ldots, a_s, b_1, \ldots, b_t\), we note that the denominator for \(p_k(n)\) in (6) and (7) can also be expressed as

\[
D_k(\theta) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_t)_n} \frac{n^k \theta^n}{n!}, \quad k = 0, 1, 2, \ldots
\]

Now let \(P_k(z)\) be the p.g.f. \(E[Z^{N_k}]\) of the random variable \(N_k, k = 0, 1, 2, \ldots\). Then by definition,

\[
P_k(z) = \frac{D_k(\theta z)}{D_k(\theta)}, \quad k = 0, 1, 2, \ldots \tag{8}
\]

\(P_k(z)\) can also be expressed recursively. With \(u = \theta z\), for \(k = 1, 2, 3, \ldots\), write

\[
P_k(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_t)_n} \frac{n^k u^n}{n!} D_k(\theta) = \frac{u \frac{d}{du} \left(\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_t)_n} \frac{n^k u^n}{n!}\right)}{D_k(\theta)}
\]

From (5) and (9) it therefore follows that

\[
P_k(z) = \frac{u \frac{d}{du} D_{k-1}(u)}{\theta \frac{d}{d\theta} D_{k-1}(\theta)} = \frac{T_k[D_0(u)]}{T_k[D_0(\theta)]} \bigg|_{u=\theta z}, \quad k = 1, 2, 3, \ldots \tag{10}
\]

Equations (8) and (10) also immediately gives the m.g.f. \(M_k(t) = P_k(e^t), k = 0, 1, 2, \ldots\).

### 3. A Family of Recursive Power-Series Distributions

We now present five families of recursive power-series random variables characterized by (6) and (7). These extensions of some standard distributions have apparently not been studied and do not appear, for example, in Patil (1985).

**Case 1: Recursive Poisson family.** Let \(s = t = 1 \) and \(a_1 = b_1\) in Theorem 1 to give \(\text{e}_1 F_1(a_1; a_1; \theta) = e^\theta\). Then (6) and (7) become for \(\theta > 0\)

\[
p_0(n) = \frac{e^{-\theta} \theta^n}{n!}, \quad n = 0, 1, 2, \ldots ,
\]

\[
p_k(n) = \frac{n^k e^{-\theta} \theta^n}{\sum_{i=1}^{k} \sum_{j=1}^{i} \frac{(-1)^{i+j}}{i^j} (j) j^k \theta^i}, \quad n, k = 1, 2, 3, \ldots
\]
Each $X_k$, $k = 0, 1, 2, \cdots$, is said to have a recursive Poisson distribution with exponential parameter $k$ (referring to its being an exponent of $n$). As special cases, $X_0$ and $X_1$ have standard Poisson distributions.

**Case 2: Recursive geometric family.** Next let $s = 2, t = 1$ and $a_1 = 1, a_2 = 1$, and $b_1 = 1$. Immediately, $\sum_F (1, 1; 1; \theta) = (1 - \theta)^{-1}$, $0 < \theta < 1$. Hence for $0 < \theta < 1$

$$p_0(n) = \theta^n(1 - \theta), \quad n = 0, 1, 2, \ldots,$$

$$p_k(n) = \frac{n^k\theta^n(1 - \theta)}{\sum_{i=1}^{k} \sum_{j=1}^{i} (-1)^{i+j} \frac{(i)}{j} j^k \left( \frac{\theta}{1 - \theta} \right)^i}, \quad n, k = 1, 2, 3, \ldots.$$

Then $X_k$, $k = 0, 1, 2, \cdots$, is said to have a recursive geometric distribution with exponential parameter $k$. Obviously $X_0$ has a standard geometric distribution.

**Case 3: Recursive negative binomial family.** Now let $s = 1, t = 0$ and $a_1 = r$ for a positive integer $r$. It readily follows that $\sum_F (0, r; -; \theta) = (1 - \theta)^{-r}$, $0 < \theta < 1$. As before, for $0 < \theta < 1$

$$p_0(n) = \binom{n + r - 1}{n} \theta^n(1 - \theta)^r, \quad n = 0, 1, 2, \ldots,$$

$$p_k(n) = \frac{(n + r - 1)^k \theta^n(1 - \theta)^r}{\sum_{i=1}^{k} \sum_{j=1}^{i} (-1)^{i+j} \frac{(i+j-1)}{i} \frac{(i)}{j} j^k \left( \frac{\theta}{1 - \theta} \right)^i}, \quad n, k = 1, 2, 3, \ldots.$$

Here $X_k$, $k = 0, 1, 2, \cdots$, is said to have a recursive negative binomial distribution with exponential parameter $k$. $X_0$ and $X_1$ have standard negative binomial distributions with parameters $r$ and $r + 1$, respectively.

**Case 4: Recursive log-series family.** For $s = 2, t = 1$, $a_1 = a_2 = 1$, and $b_1 = 2$, it can be shown that $\sum_F (1, 1; 2; \theta) = \theta^{-1}\{\log(1 - \theta)\}, |\theta| < 1$. Hence for $|\theta| < 1$

$$p_0(n) = \frac{-\theta^{n+1}}{(n+1)\log(1 - \theta)}, \quad n = 0, 1, 2, \ldots,$$

$$p_k(n) = \frac{n^k\theta^{n+1}}{\sum_{i=1}^{k} \sum_{j=1}^{i} (-1)^{i+j} \frac{(i)}{j} j^k \left( \log(1 - \theta)^{-1} + \sum_{p=1}^{i} \frac{(-1)^p}{p} \left( \frac{1}{(1-\theta)^p} - 1 \right) \right)}, \quad n, k = 1, 2, 3, \ldots.$$

In this case, $X_k$, $k = 0, 1, 2, \cdots$, is said to have a recursive log-series distribution with exponential parameter $k$, where $X_0$ has a standard log-series distribution.
Case 5: Recursive generalized Waring family. When $s = 2$, $t = 1$, $a_1, a_2$, $b_1$, $\theta = 1$, and $b_1 = a_1 + a_2 + \rho$, $2F_1(a_1, a_2; b_1; 1) = \frac{\Gamma(a_1 + a_2 + \rho)\Gamma(\rho)}{\Gamma(a_2 + \rho)\Gamma(a_1 + \rho)}$. Thus,

\[
p_0(n) = \frac{(a_1)_n(a_2)_n}{(a_1 + a_2 + \rho)_n} \frac{(\rho)_{a_2}}{(a_1 + \rho)_{a_2}} \frac{1}{n!}, \quad n = 0, 1, 2, \ldots ,
\]

\[
p_k(n) = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{(a_1)_i(a_2)_j}{(a_1+a_2+\rho)_n} \frac{(-1)^{i-j}}{i!} \left(\frac{j}{\rho+1}\right)^k \frac{(a_2+\rho)(a_1+i)}{(\rho+2)(a_1-i)}, \quad n, k = 1, 2, 3, \ldots .
\]

Now $X_k$, $k = 0, 1, 2, \ldots$, is said to have a recursive generalized Waring distribution with exponential parameter $k$. In particular, $X_0$ has a standard generalized Waring distribution, with details found in Irwin (1963, 1975).

Figures 1-5 illustrate the p.m.f.’s of Cases 1-5 for $k = 1, 4, 7, 10$ and the given parameters. Note that each p.m.f tends toward a bell shape reminiscent of the binomial distribution’s relation to the Bernoulli for fixed $p$, a consequence of the central limit theorem. Hence, it is possible that $N_k$ for the general recursive power-series distribution is a sum of $k$ independent, identically distributed random variables. However, this conjecture remains unproved.

[Figures 1-5 about here]

4. Conclusions

We have presented here new discrete distributions derived from the generalized hypergeometric function. For fixed parameters, these distributions can be viewed as a family of recursive power-series distributions over $k = 0, 1, 2, \ldots$, with the p.g.f. and m.g.f. defined recursively on $k$. We gave examples where such families extend some standard distributions. Future work should include the estimation of $\theta$, which is difficult in general, and establishing whether $N_k$ for each family is a sum of $k$ independent, identically distributed random variables. In addition, values of the parameters in (6) and (7) should be determined that give further families – in particular, new distributions that are not extensions of standard ones. Finally, applications of specific cases of our recursive power-series distributions should be identified.
References


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Figure 1: Probability mass function of the recursive Poisson distributions with $\theta = 1$ for $k = 1, 4, 7, 10$. 
Figure 2: Probability mass function of the recursive geometric distributions with $\theta = 0.3$ for $k = 1, 4, 7, 10$. 
Figure 3: Probability mass function of the recursive negative binomial distributions with $r = 5$ and $\theta = 0.3$ for $k = 1, 4, 7, 10$. 
Figure 4: Probability mass function of the recursive log-series distributions with $\theta = 0.3$ for $k = 1, 4, 7, 10$. 


Figure 5: Probability mass function of the recursive Waring distributions with $a_1 = 2, a_2 = 3$, and $b_1 = 40$ for $k = 1, 4, 7, 10$. 
DERIVATIONS IN SECTION 3 FOR REFEREES

Recursive Poisson family.

\[ 1F_1(a_1 + i; a_1 + i; \theta) = \sum_{n=0}^{\infty} \frac{(a_1 + i)_n \theta^n}{(a_1 + i)_n n} \]
\[ = \sum_{n=0}^{\infty} \frac{\theta^n}{n} \]
\[ = e^\theta. \]

Recursive geometric family.

\[ 2F_1(1 + i, 1 + i; 1 + i; \theta) = \sum_{n=0}^{\infty} \frac{(1 + i)_n(1 + i)_n \theta^n}{(1 + i)_n n} \]
\[ = \sum_{n=0}^{\infty} \frac{(i+n)!}{n!} \frac{\theta^n}{n!} \]
\[ = \frac{1}{i!} \sum_{n=0}^{\infty} \frac{(i+n)!}{n!} \theta^n \]
\[ = \frac{1}{i!} \frac{i!}{(1 - \theta)^{i+1}} \]
\[ = (1 - \theta)^{-(i+1)}. \]

Recursive negative binomial family.

\[ 1F_0(r + i; -; \theta) = \sum_{n=0}^{\infty} \frac{(r + i)_n \theta^n}{n!} \]
\[ = (1 - \theta)^{-(r+i)}. \]
Recursive log-series family.

$$2F_1(1+i, 1+i; 2+i; \theta) = \sum_{n=0}^{\infty} \frac{(1+i)_n(1+i)_n \theta^n}{(2+i)_n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(i+1)_{i+n}! \theta^n}{i+n+1 n!}$$

$$= \frac{i+1}{i!} \sum_{n=0}^{\infty} \frac{(i+n)! \theta^n}{n! i + n + 1}$$

$$= \frac{i+1}{i!} \frac{1}{\theta^{i+1}} \sum_{n=0}^{\infty} \frac{(i+n)! \theta^{n+i+1}}{n! n + i + 1}$$

$$= \frac{i+1}{i!} \frac{1}{\theta^{i+1}} \int_0^\theta t^{n+i} dt$$

$$= \frac{i+1}{i!} \frac{1}{\theta^{i+1}} \int_0^\theta t^i \frac{t^n}{(1-t)^{i+1}} dt$$

Let $w = 1-t, \ dw = -dt$

$$= -\left(\frac{i+1}{\theta^{i+1}}\right) \int_1^{1-\theta} \frac{(1-w)^i}{w^{i+1}} dw$$

$$= -\left(\frac{i+1}{\theta^{i+1}}\right) \int_1^{1-\theta} \sum_{p=0}^{i} \frac{(i)}{p} \frac{(-1)^{i-p}}{w^{i+1}} dw$$

$$= -\left(\frac{i+1}{\theta^{i+1}}\right) \int_1^{1-\theta} \sum_{p=0}^{i} \frac{(i)}{p} \frac{(-1)^{i-p}}{w^{i+1}} dw$$

$$= \left(\frac{i+1}{\theta^{i+1}}\right) (-1)^i \int_1^{1-\theta} \sum_{p=0}^{i} \frac{(i)}{p} \frac{1}{(-w)^{i+1}} dw$$

By integrating a finite sum, we obtained as follows:

$$2F_1(1+i, 1+i; 2+i; \theta) = \left(\frac{i+1}{\theta^{i+1}}\right) (-1)^i \left[ \log(1-\theta)^{-1} + \sum_{p=1}^{i} \frac{(-1)^p}{p} \left( \frac{1}{(1-\theta)^p} - 1 \right) \right]$$

□
Recursive generalized Waring family.

\[ _2F_1(a_1 + i, a_2 + i; b_1 + i; 1) = \frac{\Gamma(b_1 + i)\Gamma(b_1 - a_1 - a_2 - i)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)} \]
\[ = \frac{\Gamma(a_1 + a_2 + \rho + i)\Gamma(\rho + i)}{\Gamma(a_2 + \rho)\Gamma(a_1 + \rho)} \]
\[ = \frac{(a_1 + a_2 + \rho + i - 1)!(\rho + i - 1)!}{(a_2 + \rho - 1)!(a_1 + \rho - 1)!} \]
\[ = \frac{(a_2 + \rho)^{(a_1+i)}}{(\rho+i)(a_1-i)}. \]

□