Optimality Conditions for Maximizations of Set-Valued Functions

H. W. Corley

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Abstract. The maximization with respect to a cone of a set-valued function into possibly infinite dimensions is defined, and necessary and sufficient optimality conditions are established. In particular, an analogue of the Fritz John necessary optimality conditions is proved using a notion of derivative defined in terms of tangent cones.

Key Words. Optimality conditions, set-valued functions, cones, tangent cones.

1. Introduction

In this paper, we define the maximization of a set-valued function with respect to a cone into possibly infinite dimensions and establish optimality conditions. Such problems have apparently not been explicitly studied heretofore, although in the duality theories of Tanino and Sawaragi (Ref. 1) and Corley (Ref. 2) for multiobjective programming the dual problems took this form. The main result here is an analogue of the Fritz John necessary optimality conditions of mathematical programming, which is stated in terms of derivatives for set-valued functions defined via tangent cones. Sufficient optimality conditions requiring a type of concavity are also given. Duality and existence are considered in Ref. 3.

It is anticipated that an optimization theory for set-valued functions will provide a useful analytical tool because of the range of application of such functions. For example, Klein and Thompson (Ref. 4) survey their use in economics, in addition to presenting their theory. Zangwill (Ref. 5) uses them to present a unified treatment of convergence of nonlinear programming algorithms, while Hogan (Ref. 6) studies their properties from

1 Professor, Department of Industrial Engineering, University of Texas at Arlington, Arlington, Texas.
this viewpoint. Generalized equations (Ref. 7) and differential inclusions (Ref. 8) are other applications. Frequently occurring examples of set-valued functions include inverses of functions, cones of tangents, and subgradients.

2. Maximization of Set-Valued Functions

Let \( X, Y, Z \) be real normed linear spaces, and let \( F: X \rightarrow 2^Y \), \( G: X \rightarrow 2^Z \) be relations. In optimization literature, relations are often called set-valued functions, multifunctions, or point-to-set maps. We refer to them as set-valued functions. The domain of \( F: X \rightarrow 2^Y \) is given by

\[
\text{Dom}(F) = \{ x \in X : F(x) \neq \emptyset \}.
\]

A set \( K \) in \( Y \) is a cone if

\[
\lambda y \in K, \quad \text{for all } y \in K \text{ and } \lambda \geq 0.
\]

A pointed cone \( K \) is one for which \( K \cap -K = \{0\} \). A convex cone \( K \) is one for which

\[
\lambda_1 y_1 + \lambda_2 y_2 \in K, \quad \text{for all } y_1, y_2 \in K \text{ and } \lambda_1, \lambda_2 \geq 0.
\]

The following notions of optimality are used here. Let \( K \) be a pointed cone in \( Y \) and \( B \subseteq Y \). For \( y_1, y_2 \in Y \), write \( y_1 \leq_s y_2 \), if \( y_2 - y_1 \in K \). If \( y_2 - y_1 \in K \setminus \{\theta\} \), write \( y_1 <_s y_2 \); if \( y_2 - y_1 \in K^0 \) (the interior of \( K \)), write \( y_1 <_w y_2 \). The point \( y_o \in B \) is a strong maximal element of \( B \) with respect to \( K \), denoted \( y_o \in \text{max} B \) [or \( y \in \text{max}(B; K) \) when more specificity is required] if there exists no \( y \in B \) for which \( y_o <_s y \). Similarly, \( y_o \in B \) is a weak maximal element of \( B \) with respect to \( K \), denoted \( y_o \in \text{wmax} B \) [or \( y \in \text{wmax}(B; K) \)], if there exists no \( y \in B \) for which \( y_o <_w y \). Maximal and weak maximal elements are related and characterized in the next result given in Ref. 9.

Result 2.1. Let \( K \) be a pointed convex cone in \( Y \), \( B \subseteq Y \), and \( y_o \in B \). Then:

(a) \( y_o \in \text{max} B \) if and only if \( B \cap [K + y_o] = \{y_o\} \), and \( y_o \in \text{wmax} B \) if and only if \( B \cap [K^0 + y_o] = \emptyset \).

(b) \( \text{wmax}(B; K) = \text{max}(B; K^0 \cup \{\theta\}) \) for the pointed cone \( K^0 \cup \{\theta\} \).

We are concerned with the following problems. Throughout the paper, let \( Y \) be ordered by the pointed convex cone \( K \subseteq Y \), let \( Z \) be ordered by the pointed convex cone \( D \subseteq Z \), and let \( A \subseteq X, F: X \rightarrow 2^Y, G: X \rightarrow 2^Z \). Also, denote

\[
F(A) = \bigcup_{x \in A} F(x) \quad \text{and} \quad G^-(U) = \{x : G(x) \cap U \neq \emptyset\}.
\]
The basic problem is to

\[ \text{maximize } F(x), \quad \text{subject to } x \in A \]  

i.e., to find all \( x_0 \in A \) for which there exists a \( y_0 \in F(x_0) \) such that \( y_0 \in \max F(A) \) (or \( y_0 \in \text{wmax } F(A) \) if weak maximal elements are desired). A special case of (1) is to

\[ \text{maximize } F(x), \quad \text{subject to } x \in E \]  

(2a)

\[ \text{s.t. } G(x) \cap D \neq \emptyset, \]  

(2b)

i.e., to find all \( x_0 \in E \cap G^{-1}(D) \) for which there exists a \( y_0 \in F(x_0) \) such that \( y_0 \in \max F[E \cap G^{-1}(D)] \) (or \( y_0 \in \text{wmax } F[E \cap G^{-1}(D)] \) if weak maximal elements are desired).

Since the constraint in (2) reduces to \( G(x) \geq 0 \) when \( G \) is a real-valued singleton function and \( D \) is the cone of nonnegative reals, problem (2) is a generalization of a standard nonlinear programming problem. Any \( x_0 \) solving (1) or (2) is called a (weak) maximal point at \( y_0 \) for the problem.

3. Tangent Cones, Derivatives, Concavity

Some preliminary definitions are needed. For \( B \subset Y \), define the sequential tangent cone or contingent cone \( S(B, y_0) \) to \( B \) at \( y_0 \in B \) (only \( y_0 \in \bar{B} \), the closure of \( B \), need actually be required) as the set of limits of the form

\[ y = \lim \lambda_n (y_n - y_0), \]

where \( \langle \lambda_n \rangle \) is a sequence of nonnegative real numbers and \( \langle y_n \rangle \) is a sequence in \( B \) with limit \( y_0 \). This definition of a sequential tangent cone is used in Refs. 10–12, Ref. 13, and Ref. 14 among other places. Equivalent definitions appear, for example, in Refs. 15–17 and are called contingent cones or sets of adherent displacements.

A more restrictive concept, and a particularly useful one since it is convex, is the tangent cone \( T(B, y_0) \) to \( B \) at \( y_0 \in B \) defined in Refs. 15 and 8 (where equivalent definitions are also given). A point \( y \in T(B, y_0) \) if and only if, for every sequence \( y_n \in B \) converging to \( y_0 \) and for every sequence \( h_n > 0 \) converging to 0, there is a sequence of elements \( v_n \in Y \) converging to \( y \) such that \( y_n + h_n v_n \in B \) for all \( n \).

A result proved in Ref. 15 summarizing some properties of \( S(B, y_0) \) and \( T(B, y_0) \) to be used later is stated below.

**Result 3.1.** (a) \( S(B, y_0) \) is a closed cone.

(b) \( T(B, y_0) \) is a closed convex cone.
(c) \( T(B, y_0) \subset S(B, y_0) \subset \bigcup_{h>0} (1/h)(B - y_0) \).

(d) If \( B \) is convex, the three sets in (c) coincide and \( B - X_0 \subset T(B, Y_0) \).

Derivatives for the set-valued function \( F \) are defined in Ref. 15, as in calculus, from tangents to its graph. The contingent derivative \( CF(x_0, y_0) \) of \( F \) at \( (x_0, y_0) \in Gr(F) = \{(x, y): y \in F(x), x \in \text{Dom}(F)\} \) is the set-valued function from \( X \) to \( 2^Y \) whose graph is the sequential tangent (contingent) cone to the graph of \( F \) at \( (x_0, y_0) \). Similarly, the derivative \( DF(x_0, y_0) \) of \( F \) at \( (x_0, y_0) \in Gr(F) \) is the set-valued function whose graph is the tangent cone to the graph of \( F \) at \( (x_0, y_0) \). In other words, \( y \in CF(x_0, y_0)(x) \) if and only if \( (x, y) \in S[Gr(F), (x_0, y_0)] \), and \( y \in DF(x_0, y_0)(x) \) if and only if \( (x, y) \in T[Gr(F), (x_0, y_0)] \). We also define the \( K \)-directed contingent derivative \( CKF(x_0, y_0) \) of \( F \) at \( (x_0, y_0) \) to be the contingent derivative of the set-valued function

\[
F(x) - K = \{y - k: y \in F(x), k \in K\}.
\]

The \( K \)-directed derivative at \( (x_0, y_0) \) is analogously defined to be \( DKF(x_0, y_0) \).

The function \( F(x) - K \) also appears in the following definition of concavity for set-valued functions and thereby in subsequent sufficient optimality conditions. If \( A \subset X \) is a convex subset of \( \text{Dom}(F) \), then \( F \) is \( K \)-concave on \( A \) if, for any \( x_1, x_2 \in A \) and \( \lambda \in [0, 1] \),

\[
\lambda F(x_1) + (1 - \lambda) F(x_2) \subset F[\lambda x_1 + (1 - \lambda) x_2] - K. \tag{3}
\]

This definition is the obvious counterpart of convexity used, for example, in Ref. 12. Concavity and differentiation are related in the next theorem.

**Theorem 3.1.** Let \( F \) be \( K \)-concave on the convex set \( A \subset \text{Dom}(F) \). Then, for all \( x', x'' \in A \) and any \( y' \in F(x'') \),

\[
F(x'') - y' \subset CKF(x', y')(x'' - x'). \tag{4}
\]

**Proof.** Let \( x', x'' \in A \) and \( y' \in F(x') \). For \( \lambda_n \in (0, 1) \) and \( \lambda_n \to 0 \), define

\[
x_n = x' + \lambda_n (x'' - x') = (1 - \lambda_n) x' + \lambda_n x'' \in A.
\]

Then,

\[
y_n = y' + \lambda_n (y'' - y') = (1 - \lambda_n) y' + \lambda_n y'' \in F_K(x_n),
\]

by (3). But

\[
x_n \to x', \quad y_n \to y', \quad [(x_n, y_n) - (x', y')]/\lambda_n \to (x'' - x', y'' - y'),
\]

so it follows that \((x'' - x', y'' - y') \) is in \( S[Gr(F - K), (x', y')] \). Thus, \( y'' - y' \in CKF(x', y')(x'' - x') \) to establish (4). \( \square \)
4. Optimality Conditions for Problem (1)

Differential optimality conditions for problem (1) are now developed. The necessary conditions in this and the next section are actually derived for weak maximal points, but they hold for maximal points since a maximal point is also a weak maximal point. Tangent cones apparently cannot distinguish between the two concepts, so it seems unlikely that stronger differential necessary can be developed for maximal points. The notation $F_A$ is used to denote the restriction of $F$ to $A$.

**Theorem 4.1.** If $x_0$ is a (weak) maximal point at $y_0$ for (1), then

$$C_F(x_0, y_0)(x) \cap K^0 = \emptyset, \quad \text{for all } x \in A,$$

and hence

$$D_F(x_0, y_0)(x) \cap K^0 = \emptyset, \quad \text{for all } x \in A.$$

**Proof.** To establish the first conclusion, suppose to the contrary that, for some $\hat{x} \in A$, there exists $\hat{y} \in C_F(x_0, y_0)(\hat{x}) \cap K^0$. Thus, $\hat{y} \neq \emptyset$. By definition, $(\hat{x}, \hat{y}) \in S[Gr(F_A), (x_0, y_0)]$, and hence there are sequences $(x_n) \subset A$, $(y_n) \subset Y$, $(\alpha_n) \subset R^1$, such that

$$x_n \to x_0, \quad y_n \to y_0, \quad y_n \in F(x_n), \quad \alpha_n \geq 0, \quad \alpha_n(x_n - x_0, y_n - y_0) \to (\hat{x}, \hat{y}).$$

It follows that there exists $N$ for which $\alpha_n > 0$ and $\alpha_n(y_n - y_0) \in K^0 \subset K\{\theta\}$, for $n \geq N$. Since $K$ is a cone, then $y_n - y_0 \in K^0 \subset K\{\theta\}$, for $n \geq N$. Thus, $y_N \in F(x_N)$ and $y_N - y_0 \in K^0 \subset K\{\theta\}$, in contradiction to $x_0$ being a (weak) maximal point, and the first conclusion is established. The second follows immediately from the fact that $D_F(x_0, y_0)(\hat{x}) \subset C_F(x_0, y_0)(\hat{x})$ as a consequence of Result 3.1(c) and the definitions of the two derivatives. \( \square \)

Sufficient conditions based on concavity are now stated for (1).

**Theorem 4.2.** Let $F$ be $K$-concave on the convex set $A \subset \text{Dom}(F)$. If

$$K \cap D_K F(x_0, y_0)(x - x_0) = \{\theta\}, \quad \text{for all } x \in A,$$

then $x_0$ is a maximal point at $y_0$ for (1). If

$$K^0 \cap D_K F(x_0, y_0)(x - x_0) = \emptyset, \quad \text{for all } x \in A,$$

then $x_0$ is a weak maximal point at $y_0$ for (1).
Proof. Let \( x \in A \). It is easily established that \( Gr(F-K) \) is convex, since \( F \) is \( K \)-concave, and hence
\[
D_K F(x_0, y_0)(x) = C_K F(x_0, y_0)(x),
\]
from Result 3.1(b) and the definition of the two derivatives. It now follows by hypothesis and Theorem 3.1 that, for all \( x \in A \),
\[
K \cap [F(x) - y_0] \subset K \cap D_K F(x_0, y_0)(x - x_0) = \{ \theta \}. \tag{5}
\]
Thus, \( x_0 \) is a maximal point at \( y_0 \) to (1); otherwise, for some \( x_1 \in A \), there exists \( y_1 \in F(x_1) \) for which \( y_1 - y_0 \in K \setminus \{ \theta \} \), in contradiction to (5). Using the cone \( K^0 \cup \{ \theta \} \) instead of \( K \) in (5) similarly establishes the other conclusion.

Maximum-principle type necessary optimality criteria involving multiplier functionals can be readily obtained for (1) by applying to Theorem 4.1 a standard separation result; Theorem 4.2 also has an analogue. These results are stated below without proof. The following terminology is used. Let \( Y^* \) denote the dual space of \( Y \), and let
\[
K^+ = \{ l \in Y^* : l(y) > 0, \text{ for all } y \in K \}
\]
denote the nonnegative dual cone of \( K \). Then, \( l \in K^+ \) is definitely positive if \( l(y) > 0 \), for all \( y \in K^0 \), and strictly positive if \( l(y) > 0 \), for all \( y \in K \setminus \{ \theta \} \). We write \( lF(x) \leq 0 \) to mean \( l(y) \leq 0 \), for all \( y \in F(x) \).

**Theorem 4.3.** (a) If \( x_0 \) is a (weak) maximal point at \( y_0 \) for (1), then there exists a definitely positive \( l \in K^+ \) such that \( lDF(x_0, y_0)(x_0) \leq 0 \).

(b) Let \( F \) be \( K \)-concave on the convex set \( A \subset \text{Dom}(F) \). If there exists a strictly (definitely) positive \( l \in K^+ \) such that \( lD_K F(x_0, y_0)(x - x_0) \leq 0 \), for all \( x \in A \), then \( x_0 \) is a (weak) maximal point at \( y_0 \) for (1).

5. Optimality Conditions for Problem (2)

A generalization of the Fritz John necessary optimality conditions is established for (2) in this section. The notation \( (F, G)(x) \) is used for \( F(x) \times G(x) \).

**Theorem 5.1.** Let \( K^0 \neq \emptyset \) and \( D^0 \neq \emptyset \), and suppose that \( x_0 \) is a (weak) maximal point at \( y_0 \) for (2). Then, for any \( z_0 \in G(x_0) \cap D \), there exist \( l \in C^+ \) and \( u \in D^+ \), both dependent on \( z_0 \), but not both being zero functionals, such that
\[
u(z_0) = \theta, \tag{6}
\]
\[
l(y) + u(z) \leq 0, \tag{7}
\]
for all 

\[(y, z) \in D(F_E, G_E)(x_0, y_0, z_0)(x)\]

and 

\[x \in \text{Dom}[D(F_E, G_E)(x_0, y_0, z_0)].\]

**Proof.** Let \(z_0 \in G(x_0) \cap D\), and define 

\[B = \left[ \bigcup_{x \in \Omega} D(F_E, G_E)(x_0, y_0, z_0)(x) \right] + (0, z_0),\]

where 

\[\Omega = \text{Dom}[D(F_E, G_E)(x_0, y_0, z_0)].\]

We first show that \(B\) is convex by showing that \(B_1 = B - (\theta, z_0)\) is convex. Let \((y_1, z_1), (y_2, z_2) \in B_1\). Then, there exist \(x_1, x_2 \in \Omega\) such that 

\[(y_i, z_i) \in D(F_E, G_E)(x_0, y_0, z_0)(x_i), \quad i = 1, 2,\]

and thus 

\[(x_i, y_i, z_i) \in T[Gr(F_E, G_E)(x_0, y_0, z_0)], \quad i = 1, 2.\]

But \(T[Gr(F_E, G_E)(x_0, y_0, z_0)]\) is a convex cone. Therefore, 

\[\lambda (x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) \in T[Gr(F_E, G_E)(x_0, y_0, z_0)], \quad \text{for all } \lambda \in [0, 1],\]

so 

\[(\lambda y_1 + (1 - \lambda)y_2, \lambda z_1 + (1 - \lambda)z_2) \in D(F_E, G_E)(x_0, y_0, z_0)(\lambda x_1 + (1 - \lambda)x_2), \quad \text{for all } \lambda \in [0, 1].\]

It follows that \(B_1\) and its translate \(B\) are convex.

We next show that \(B \cap [K^0 \times D^0] = \emptyset\). To arrive at a contradiction, assume the contrary. Then, there exists \((\hat{x}, \hat{y}, \hat{z})\) such that 

\[(\hat{y}, \hat{z} + z_0) \in [D(F_E, G_E)(x_0, y_0, z_0)(\hat{x}) + (\theta, z_0)] \cap [K^0 \times D^0]. \quad (8)\]

Thus, 

\[(\hat{x}, \hat{y}, \hat{z}) \in T[Gr(F_E, G_E)(x_0, y_0, z_0)];\]

and, as a consequence of Result 3.1(c), there exists a sequence \(\langle (x_n, y_n, z_n) \rangle\), with 

\[x_n \in E, \quad y_n \in F(x_n), \quad z_n \in G(x_n), \quad x_n \to x_0,\]

\[y_n \to y_0, \quad z_n \to z_0,\]
and a sequence \( \langle \alpha_n \rangle \), with all \( \alpha_n > 0 \), such that
\[
\alpha_n (x_n - x_0, y_n - y_0, z_n - z_0) \to (\hat{x}, \hat{y}, \hat{z}).
\]
But \( \hat{y} \in K^0 \), from (8). Thus, there exists \( N_1 \) such that
\[
\alpha_n (y_n - y_0) \in K^0, \quad \text{for } n \geq N_1,
\]
\[
y_n - y_0 \in K^0, \quad \text{for } n \geq N_1,
\]
since \( \alpha_n > 0 \) and \( K \) is a cone. Similarly, since \( D \) is a cone, \( z + z_0 \in D^0 \) and
\[
\alpha_n (z_n - z_0) \to \hat{z}
\]
implies that there exists \( N_2 \) for which
\[
\alpha (z_n - z_0) + z_0 \in D^0, \quad \text{for } n \geq N_2.
\]
Moreover, there exists \( N \geq \max(N_1, N_2) \), such that \( \alpha N > 1 \). Otherwise,
\[
\alpha_n (y_n - y_0) \leq 0,
\]
in contradiction to \( \hat{y} \in K^0 \). It now follows from (9) that
\[
\alpha_N (z_N - z_0) + z_0 = \alpha_N [z_N - (1 - 1/\alpha_N)z_0] \in D^0,
\]
and hence
\[
z_N - (1 - 1/\alpha_N)z_0 \in D^0.
\]
But \( z_0 \in D \) and \( 1 - 1/\alpha_N > 0 \), since \( \alpha_N > 1 \), so \( (1 - 1/\alpha_N)z_0 \in D \). Upon adding
\[
z_n (1 - 1/\alpha_N)z_0 \in D^0 \text{ to } (1 - 1/\alpha_N)z_0 \in D,
\]
an elementary property of convex cones yields that \( z_N \in D^0 \). We have thus established that \( x_N \in E, z_N \in G(x_N) \cap D, y_N \in F(x_N), \) and \( y_N - y_0 \in K^0 \), in contradiction to Result 2.1
because \( x_0 \) is a (weak) maximal point at \( y_0 \) for (1). Therefore,
\[
B \cap [K^0 \times D^0] = \emptyset.
\]

We next separate \( B \) and \( K^0 \times D^0 \). By a standard separation theorem
(Ref. 18, p. 119), there exist \( l \in Y^*, u \in Z^*, \) not both zero functionals, and a real number \( \xi \) such that
\[
l(y) + u(z) \leq \xi, \quad (y, z) \in B, \quad \text{(10)}
\]
\[
l(y) + u(z) > \xi, \quad (y, z) \in K^0 \times D^0. \quad \text{(11)}
\]
But since \( (y, z) \in K^0 \times D^0 \), can be made arbitrarily close to \( (\theta, \theta) \), the continuity of \( l, u \) gives from (11) that \( \xi < 0 \). However, suppose that
\[
l(y) + u(z) < 0, \quad \text{for some } (y, z) \in K^0 \times D^0.
\]
Then,
\[
l(\beta y) + u(\beta z) < \xi, \quad \text{for sufficiently large } \beta > 0,
\]
while \( (\beta y, \beta z) \in K^0 \times D^0 \), in contradiction to (11). It follows now from (10) and (11) that \( \xi \) may be taken as 0. Hence from (11), upon letting \( y \in K^0 \)
get arbitrarily close to \( \theta \), the continuity of \( l, u \) gives that \( u \in D^+; \) similarly,
\( l \in K^+ \).
To establish (6), we first get \( u(z_0) \leq 0 \) from (10) and the fact that \((\theta, z_0) \in B\). But \( z_0 \in D \) and \( u \in D^* \), so \( u(z_0) \geq 0 \). Thus, \( u(z_0) = 0 \). Finally, to prove (7), let

\[ x \in \text{Dom}[D(F_E, G_E), (x_0, y_0, z_0)]. \]

Since

\[ D(F_E, G_E)(x_0, y_0, z_0)(x) + (\theta, z_0) \subset B, \]

from (10)

\[ l(y) + u(z + z_0) \leq 0. \quad (12) \]

But \( u(z_0) = 0 \), so (7) follows to complete the proof. \( \square \)

There are two possible modifications of Theorem 5.1 which will be mentioned, but not pursued. First, it seems likely that a constraint qualification could be placed on \( G \), perhaps as in the single-valued case in Ref. 11, to obtain as an analogue of the Kuhn–Tucker conditions that \( \lambda \) is definitely positive in (6). Second, by requiring \( E \) to be convex and \( F, G \) to be \( K \)-concave, one could deduce that a single pair \( \lambda, u \) suffices for all \( z_0 \in G(x_0) \) in (6) and (7). Such restrictions are rather severe for necessary conditions. However, weak sufficient conditions using such assumptions are next stated.

Theorem 5.2. Let \( F, G \) be \( K \)-concave on the convex set \( E \subset \text{Dom}(F) \cap \text{Dom}(G) \), and let

\[ A = E \cap G^{-1}(D). \]

Suppose that there exist \( x_0 \in A, y_0 \in F(x_0), z_0 \in G(x_0) \cap D \), strictly (definitely) positive \( \lambda \in K^+ \), and \( u \in T^+(D, z_0) \) such that

\[ l(y) + u(z) \leq 0, \quad (13) \]

for all \((y, z) \in D_K(F_A, G_A)(x_0, y_0, z_0)(x) \) and \( x \in T(A, x_0) \). Then, \( x_0 \) is a (weak) maximal point at \( y_0 \) for (2).

Proof. It is easily established that \( A \) is convex, so that

\[ x - x_0 \in T(A, x_0), \quad \text{for all } x \in A. \]

An argument as in the proof of Theorem 5.1 gives that

\[ DG_A(x_0, z_0)(x - x_0) \subset T(D, z_0), \]

from which

\[ uDG_A(x_0, z_0)(x - x_0) \geq 0, \quad \text{for all } x \in A. \]

The conclusion then follows from (13) and Theorem 4.3(b). \( \square \)
References


