MATH 3330-002: Matrices & Linear Algebra Hua Shan
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Final Review

Gauss-Jordan Elimination [1.2]
Reduced row-echelon form (rref)

Rank [1.3]
rank(A) = number of leading 1's in rref(A)
dim(im A) = rank(A)

Linear Transformation in Geometry [2.1, 2.2]
Linear transformation: $T:\mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$
Scaling/Orthogonal projection/Reflection/Rotation/Shears

Inverse of matrix [2.3]
Fine the reduced row-echelon form of an augmented matrix $[A \mid I_n]$.

Matrix multiplication [2.4]

Properties of matrix multiplication:

$AB \neq BA$.

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

If $BA = I_n$, then $B = A^{-1}$.

Consider $T:\mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$
Image: [3.1]
Def.: $\mathrm{im}(T) = \mathrm{im}(A) = \{\vec{y} : \vec{y} = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$
Properties:

- $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^n$ ($\mathrm{im}(A)$ contains zero and is closed under linear combination)

Kernel: [3.1]
Def.: $\ker(T) = \ker(A) = \{\vec{x} \in \mathbb{R}^m : A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^m$
Properties:

- $\ker(A)$ is a subspace of $\mathbb{R}^m$ ($\ker(A)$ contains zero and is closed under linear combination)

Applications:

- Find $\mathrm{im}(A)$ and $\ker(A)$.
- Find dimensions of $\mathrm{im}(A)$ and $\ker(A)$.
- Find bases of $\mathrm{im}(A)$ and $\ker(A)$.

Span: **[3.1]**
Consider $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. $\text{span}(\vec{v}_1, \cdots, \vec{v}_m) = \{c_1\vec{v}_1 + \cdots + c_m \vec{v}_m,$ for all $c_1, \cdots, c_m \in \mathbb{R}\}$

Subspace of $\mathbb{R}^n$ **[3.2]**
$W \subseteq \mathbb{R}^n$ and W contains the zero vector of $\mathbb{R}^m$ and W is closed under linear combination

Linear independence: **[3.2]**
Vectors $\vec{v}_1, \cdots, \vec{v}_m$ are linearly independent if none of them is redundant.

Basis of a subspace: **[3.2]**
Consider $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. Vectors $\vec{v}_1, \cdots, \vec{v}_m$ form a basis of a subspace V of $\mathbb{R}^n$ if V=span$(\vec{v}_1, \cdots, \vec{v}_m)$ and $\vec{v}_1, \cdots, \vec{v}_m$ are linearly independent.

Find a basis of $\mathbb{R}^n$ or a subspace of $\mathbb{R}^n$.
Vectors $\vec{v}_1, \cdots, \vec{v}_m$ form a basis of $\mathbb{R}^n$ iff matrix $[\vec{v}_1, \cdots, \vec{v}_n]$ is invertible.

Linear relation: **[3.2]** $c_1\vec{v}_1 + \cdots + c_m \vec{v}_m = \vec{0}$
Trivial linear relation: $c_1\vec{v}_1 + \cdots + c_m \vec{v}_m = \vec{0}$ iff $c_1 = \cdots = c_m = 0$

Dimension: **[3.3]**
Consider a subspace V of $\mathbb{R}^n$. The basis of V is formed by vectors $\vec{v}_1, \cdots, \vec{v}_m$.
$\dim(V) = m$ (number of vectors in basis of V)

Consider $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$
$\dim(\text{im } A) = \text{rank}(A)$
$\dim(\ker A) = m - \text{rank}(A)$
and $\dim(\text{im } A) + \dim(\ker A) = m$

$\dim(\mathbb{R}^n) = n$

Summary

Consider a linear transformation $T:\mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$. A is an $n \times m$ transformation matrix with $m \le n$. The following statements are equivalent:

- The linear system $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$.
- $\text{rank}(A) = m$ (number of free variables $= m - \text{rank}(A) = 0$).
- $\ker(A) = \{\vec{0}\}$
- There is only the trivial relation among the column vectors $\vec{v}_1, \cdots, \vec{v}_m$ of matrix A (The column vectors $\vec{v}_1, \cdots, \vec{v}_m$ of matrix A are linearly independent.).
- $\dim(\ker A) = m - \text{rank}(A) = 0$ (zero vector is redundant)
- $\dim(\text{im } A) = \text{rank}(A) = m$

If $m = n$, then we can add a few more to the list

- A is invertible
- $\text{rref}(A) = I_n$
- $\text{im}(A) = \mathbb{R}^n$

Coordinates: [3.4]

Consider vectors $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ that form a basis of a subspace V of $\mathbb{R}^n$, i.e. $\mathcal{B}=(\vec{v}_1, \cdots, \vec{v}_m)$. The $\mathcal{B}$–coordinate of any vector $\vec{x} \in V$ is $[\vec{x}]_{\mathbf{B}} = [c_1, \cdots, c_m]^T$ if

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m \vec{v}_m$$

or in matrix form

$$\vec{x} = [\vec{v}_1 \cdots \vec{v}_m]\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = S[\vec{x}]_{\mathbf{B}}$$

Two types of questions: $S[\vec{x}]_{\mathbf{B}} = \vec{x}$ or $\vec{x} = S[\vec{x}]_{\mathbf{B}}$

$\mathcal{B}$ –matrix B and standard matrix A: [3.4]

$\mathcal{B}=(\vec{v}_1, \cdots, \vec{v}_n)$ of $\mathbb{R}^n$. Consider a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$,.

A is the standard matrix of the transformation. ($T(\vec{x}) = A\vec{x}$)

B is the $\mathcal{B}$ –matrix of the transformation. ($\left[T(\vec{x})\right]_{\mathbf{B}} = B\left[\vec{x}\right]_{\mathbf{B}}$)

$$B = \left[\left[T(\vec{v}_1)\right]_{\mathbf{B}} \cdots \left[T(\vec{v}_n)\right]_{\mathbf{B}}\right]$$

Let $S = [\vec{v}_1 \cdots \vec{v}_n]$, we have

$$AS = SB, \quad A = SBS^{-1}, \quad B = S^{-1}AS \quad (A \text{ and } B \text{ are similar matrices})$$

Subspace of a linear space [4.1]
A subset W of linear space V is a subspace of V if 1) W contains the neutral element of V and 2) W is closed under linear combination.

Dimension of a subspace: [4.1]
Consider a subspace W of a linear space V. The basis of W is formed by elements $f_1, \cdots, f_m$, then

$$\dim(W) = m \text{ (number of elements in basis of } W\text{)}$$

Orthonormal vectors and orthonormal basis: [5.1]
Vectors $\vec{u}_1, \cdots, \vec{u}_n \in \mathbb{R}^n$ are orthonormal if they are all unit vectors and orthogonal to each other. $\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$

Vectors $\vec{u}_1, \cdots, \vec{u}_n \in \mathbb{R}^n$ form an orthonormal basis of a $\mathbb{R}^n$, i.e. $\mathcal{U} = (\vec{u}_1, \cdots, \vec{u}_n)$.

Orthogonal projection on a subspace V: [5.1]
The orthogonal projection of $\vec{x}$ onto a subspace V spanned by $\mathcal{U} = (\vec{u}_1, \cdots, \vec{u}_m)$

$$\text{proj}_V(\vec{x}) = \vec{x}^{\parallel} = \vec{x} - \vec{x}^{\perp}$$
$$= \vec{x} - \left[(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n)\right]$$

Orthonormal Complement: [5.1]
Let V be a subspace of $\mathbb{R}^n$, the orthogonal complement $V^{\perp}$ is defined as

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0 \text{ for all } \vec{v} \in V\}$$

Angle between two vectors: [5.1]

$$\theta = \cos^{-1} \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|}$$

Gram-Schmidt Process: [5.2]

Vectors $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ form a basis of a subspace V of $\mathbb{R}^n$, i.e. $\mathcal{B} = (\vec{v}_1, \cdots, \vec{v}_m)$. The Gram-Schmidt process converts $\mathcal{B} = (\vec{v}_1, \cdots, \vec{v}_m)$ to $\mathcal{U} = (\vec{u}_1, \cdots, \vec{u}_m)$

$$\text{proj}_V(\vec{x}) = \vec{x}^{\parallel} = \vec{x} - \vec{x}^{\perp}$$
$$= \vec{x} - \left[(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n)\right]$$

$\vec{v}_1$		$\vec{u}_1 = \dfrac{\vec{v}_1}{\lVert\vec{v}_1\rVert}$
$\vec{v}_2$	$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$	$\vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\lVert\vec{v}_2^{\perp}\rVert}$
…	…	…
$\vec{v}_j$	$\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{\parallel} = \vec{v}_j - \left[(\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 + \cdots + (\vec{u}_{j-1} \cdot \vec{v}_j)\vec{u}_{j-1}\right]$	$\vec{u}_j = \dfrac{\vec{v}_j^{\perp}}{\lVert\vec{v}_j^{\perp}\rVert}$

Determinant: [6.1, 6.3]

2×2 matrices

3×3 matrices: Sarrus's Rule

$n \times n$ matrices:

Laplace expansion down the i-th row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \text{ for fixed } i$$

Laplace expansion down the j-th column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \text{ for fixed } j$$

Geometrical meaning of determinant [6.3]

In 2D,

Area of parallelogram formed by vectors $\vec{v}_1$ and $\vec{v}_2$

$$= \left|\det(A)\right| = \left|\det[\vec{v}_1\ \vec{v}_2]\right| = \lVert\vec{v}_1\rVert \lVert\vec{v}_2^{\perp}\rVert$$

In 3D,

Volume of parallelepiped formed by vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$

$$= \left|\det(A)\right| = \left|\det[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\right| = \lVert\vec{v}_1\rVert \lVert\vec{v}_2^{\perp}\rVert \lVert\vec{v}_3^{\perp}\rVert$$

Properties of Determinants: [6.2]

$$\det(AB) = \det(A)\det(B)$$
$$\det(A^{-1}) = 1/\det(A)$$
$$\det(A^T) = \det(A)$$

$$\det(diag(a_i)) = \prod_{i=1}^{n} a_i$$

Summary

Consider a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$. A is an $n \times n$ transformation matrix. The following statements are equivalent:

- A is invertible
- $\text{rref}(A) = I_n$
- $\det(A) \neq 0$
- $\text{rank}(A) = n$ (number of free variables $= n - \text{rank}(A) = 0$).
- The linear system $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$.
- $\ker(A) = \{\vec{0}\}$
- $\text{im}(A) = \mathbb{R}^n$
- There is only the trivial relation among the column vectors $\vec{v}_1, \cdots, \vec{v}_n$ of matrix A (The column vectors $\vec{v}_1, \cdots, \vec{v}_n$ of matrix A are linearly independent.). $\vec{v}_1, \cdots, \vec{v}_n$ form a basis of $\mathbb{R}^n$
- $\dim(\ker A) = n - \text{rank}(A) = 0$ (zero vector is redundant)
- $\dim(\text{im } A) = \text{rank}(A) = n$

Orthogonal Matrix: [6.3]

An $n \times n$ matrix Q is an orthogonal matrix if $Q^T Q = I_n$.

$\det(Q) = \pm 1$

Cramer's Rule: [6.3]

$A\,\vec{x} = \vec{b}$. A is an <u>invertible</u> $n \times n$ matrix ($\det(A) \neq 0$)

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)} \quad i = 1, \ldots, n$$

Eigen Problem: [7.1, 7.2, 7.3]

Consider an $n \times n$ matrix A. $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ that solves $A\,\vec{x} = \lambda\,\vec{x}$ for some scalar λ is called the eigenvector of A. λ is called the eigenvalue associated with the eigenvector.

If $\vec{v}$ is an eigenvector of matrix A, then $\vec{v}$ is an eigenvector of matrices $A^2, A^3, \ldots$ with $A^2\vec{v} = \lambda^2\vec{v},\ A^3\vec{v} = \lambda^3\vec{v}, \ldots$

Characteristic Polynomial $f_A(\lambda) = \det(A - \lambda I_n)$

Find Eigenvalues and Eigenvectors: [7.2, 7.3]

Eigenvalues:

Solve $f_A(\lambda) = \det(A - \lambda I_n) = 0$

Algebraic multiplicity of eigenvalue

Eigenvectors:

Solve $A\,\vec{x} = \lambda\,\vec{x}$ or $(A - \lambda I_n)\,\vec{x} = \vec{0}$

Eigenspace and Eigenbasis: [7.3]

Eigenspace:

$E_\lambda = \ker(A - \lambda I_n)$

Geometric multiplicity of eigenvalue λ =

$$\dim(E_\lambda) = \dim(\ker(A - \lambda I_n)) = n - \operatorname{rank}(A - \lambda I_n)$$

Eigenbasis:

A basis of $\mathbb{R}^n$ that consists of eigenvectors $(\vec{v}_1, \cdots, \vec{v}_n)$ of A.

To find eigenspace and eigenbasis

1) Solve $f_A(\lambda) = \det(A - \lambda I_n) = 0$.
2) Find $E_\lambda = \ker(A - \lambda I_n)$ spanned by eigenvectors and $\dim(E_\lambda)$.
3) Compute $s = \sum_j \dim(E_{\lambda_j})$ and compare with n:

   If $s = n$, then the eigenbasis exists and it consists of all the eigenvectors found in step 2).
   If $s < n$, then the eigenbasis does not exist. But still, the eigenvectors found in step 2) are linearly independent.

Diagonalization: [7.4]

Consider a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$. A is an $n \times n$ transformation matrix. Suppose the eigenbasis for A exists: $\mathcal{D} = (\vec{v}_1, \cdots, \vec{v}_n)$. The $\mathcal{D}$–coordinate of any vector $\vec{x} \in V$ is $[\vec{x}]_{\mathbf{D}} = [c_1, \cdots, c_n]^T$ if

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\ \vec{v}_n$$

or in matrix form

$$\vec{x} = [\vec{v}_1 \cdots \vec{v}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S[\vec{x}]_{\mathbf{D}}$$

where $S = [\vec{v}_1 \cdots \vec{v}_n]$

Let D be the $\mathcal{D}$–matrix of the transformation. ($\left[T(\vec{x})\right]_{\mathbf{D}} = D\left[\vec{x}\right]_{\mathbf{D}}$),
As we learned from section 3.4, that

$$D = \left[\left[T(\vec{v}_1)\right]_{\mathbf{D}} \cdots \left[T(\vec{v}_n)\right]_{\mathbf{D}}\right]$$

and matrices A and D are similar, i.e.

$$AS = SD, \quad D = S^{-1}AS$$

Because $\vec{v}_1, \cdots, \vec{v}_n$ are eigenvectors of matrix A, we can show that matrix D is diagonal.

- An $n \times n$ matrix A is diagonalizable <u>if and only if</u> there exists an eigenbasis for A.
- If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Algorithm - Diagonalization: [7.4]

1) Solve $f_A(\lambda) = \det(A - \lambda I_n) = 0$.
2) Find $E_\lambda = \ker(A - \lambda I_n)$ spanned by eigenvectors and $\dim(E_\lambda)$.
3) Compute $s = \sum_j \dim(E_{\lambda_j})$ and compare with n:

   If $s = n$, then the eigenbasis exists and it consists of all the eigenvectors found in step 2). Let $S = [\vec{v}_1 \cdots \vec{v}_n]$ and compute $D = S^{-1}AS$.

   If $s < n$, then the eigenbasis does not exist thus matrix A is not diagonalizable.