(Closed books and notes)

1. The NTSC composite video signal with 2:1 line interlace can be represented as

\[ u(x, y, t) = Y(x, y, t) + I(x, y, t) \cos(2\pi f_{sc} x + \phi) \cos[\pi(f_r t - f_i)] + Q(x, y, t) \sin(2\pi f_{sc} x + \phi) \cos[\pi(f_r t - f_i)] \]

This signal is sampled such that \( \Delta x = \frac{1}{(2\Delta x_0)} \), \( \Delta y = \frac{1}{f_1} \). At any given frame, say at \( t=0 \), what would be the spectrum of the sampled frame for the composite color signal spectrum shown in figure 1? Sketch the spectrum and identify clearly the important frequency locations.

![Figure 1 NTSC composite color signal spectrum.](image)

2. (a) For zero mean Gaussian random variables,

\[ P_u(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{u^2}{2\sigma^2}\right) \]

show that the compandor transformations are given by \( f(x) = \text{erf}\left(\frac{x}{\sigma\sqrt{6}}\right) \), \( x \geq 0 \), and \( g(y) = \sigma\sqrt{6}\text{erf}^{-1}\left(\frac{y}{2}\right) \), \( y \geq 0 \), where \( \text{erf}(x) = \left(\frac{2}{\sqrt{\pi}}\right) \int_0^x \exp(-t^2)dt \).
(b) For the Rayleigh density

\[ p_u(u) = \begin{cases} 
\frac{u}{\sigma^2} \exp\left(- \frac{u^2}{2\sigma^2}\right), & u > 0 \\
0, & u < 0
\end{cases} \]

show that the compandor transformation is

\[ f(x) = c \int_0^x \alpha^{\frac{1}{2}} \exp\left(- \frac{\alpha^2}{6\sigma^2}\right) d\alpha - 1 \]

where \( c \) is a normalization constant such that \( f(\infty) = 1 \).

3. The differential entropy (in bits) of a continuous random variable \( u \) is defined as

\[ H(u) = -\int_{-\infty}^{\infty} p_u(u) \log_2 p_u(u) du \]

show that for a Gaussian random variable \( u \) (described in problem 2a) whose variance is \( \sigma^2 \), \( H(u) = \frac{1}{2} \log_2(2\pi e \sigma^2) \).

given that

\[ \int_{-\infty}^{\infty} \frac{x^2}{\sigma \sqrt{2\pi}} dx = 1 \]

\[ \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma \sqrt{2\pi}} dx = \sigma^2 \]

and

\[ \log_2 x = \frac{\log_e x}{\log_e 2} \approx (1.4427) \ln x \]
A band limited image acquired by a practical sensor is observed as \( g(x, y) = f(x, y) + n(x, y) \), where \( \xi_{x0} = \xi_{y0} = \Delta \xi_f \) and \( n(x, y) \) is the wideband noise whose spectral density function, \( S_n(\xi_1, \xi_2) = \eta/4 \) is band limited to \( -\xi_n \leq (\xi_1, \xi_2) \leq \xi_n \), \( \xi_n = 2\xi_f \). The random field \( g(x, y) \) is sampled without prefiltering and with prefiltering at the Nyquist rate of the noiseless image and reconstructed by a low pass filter whose bandwidths are also \( \xi_{x0} \) and \( \xi_{y0} \) as shown in the figures 2 & 3 below. Show that:

a. SNR of the sensor output \( g \) over its bandwidth is \( \sigma_f^2 / 4(\pi \xi_f^2) \), where \( \sigma_f^2 \) is the image power.

b. the SNRs of the reconstructed image with and without prefiltering are \( \sigma_f^2 / (\pi \xi_f^2) \) and \( \sigma_f^2 / (4\pi \xi_f^2) \) respectively. What would be the SNR of the reconstructed image if the sensor output were sampled at the Nyquist rate of the noise without any prefiltering? Compare the preceding sampling schemes and recommend the best way for sampling noisy images.

**FIGURE 2:**

- Its region of support.

**FIGURE 3:**

- Fourier transform of a bandlimited function.

*Sampling of noisy images*
Note: For a compandor design, the functions $f$ and $g$ are determined so that the overall system approximates the Lloyd-Max quantizer. The result is given by

$$g(x) = f^{-1}(x)$$
$$f(x) = 2a\{\frac{\int_{t_1}^{x} [p_u(u)]^{1/3} \, du}{\int_{t_1}^{t_{L+1}} [p_u(u)]^{1/3} \, du}\} - a$$

Additionally, if $p_u(u)$ is an even function, that is, $p_u(u) = p_u(-u)$, we get

$$f(x) = a\{\frac{\int_{0}^{x} [p_u(u)]^{1/3} \, du}{\int_{0}^{t_{L+1}} [p_u(u)]^{1/3} \, du}\}, \quad x \geq 0$$

$$f(x) = -f(-x), \quad x < 0$$

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**Figure 2.** A compandor.

Input range is from $t_1$ to $t_{L+1}$. 