IV Discrete –Time Fourier transform

DTFT.

A. Basic Definitions

The discrete-time Fourier transform (DTFT) of \( x(n) \) is

\[
X(e^{jw}) = \sum x(n) e^{-jwn}
\]

where \( w \) is in radians. \( X(e^{jw}) \) is periodic with period \( 2\pi \), since a function of a periodic function is periodic, and has the same period. Since the forward transform is a Fourier series, the inverse transform,

\[
x(n) = \frac{1}{2\pi} \int X(e^{jw}) e^{jwn} \, dw
\]

is the formula for the Fourier series coefficient. The frequency response \( H(e^{jw}) \) is the DTFT of the impulse response \( h(n) \). As with the continuous-time
Fourier transform, the DTFT is used because of the existence of a convolution theorem.

**Amplitude and Phase of DTFT.**

\[ X(e^{jw}) = \text{Re} \{ X(e^{jw}) \} + j \text{Im} \{ X(e^{jw}) \} \]

\[ = |X(e^{jw})| e^{j\phi(w)} \]

\[ |X(e^{jw})| = \sqrt{\text{Re}^2 \{X(e^{jw})\} + \text{Im}^2 \{X(e^{jw})\}} \]

\[ \phi(w) = \arg(X(e^{jw})) \]

\[ = \tan^{-1} \frac{\text{Im}(X(e^{jw}))}{\text{Re}(X(e^{jw}))} \]

\[ + \pi \ u (-\text{Re}(X(e^{jw}))) \]
\[ X(e^{jw}) = \begin{bmatrix} N(e^{jw}) \\ D(e^{jw}) \end{bmatrix} \]

then,

\[ |X(e^{jw})| = \begin{bmatrix} |N(e^{jw})| \\ |D(e^{jw})| \end{bmatrix} \]

If

\[ X(e^{jw}) = \begin{bmatrix} 1 \\ D(e^{jw}) \end{bmatrix} \]

Then,

\[ |X(e^{jw})| = \begin{bmatrix} 1 \\ |D(e^{jw})| \end{bmatrix} \]
\[
\phi(w) = \text{arg} \left( N(e^{jw}) \right) - \text{arg} \left( D(e^{jw}) \right)
\]

**B. Simple Examples of Forward and Inverse Transforms**

**Example** \( x(n) = \delta(n) \)

\[
X(e^{jw}) = \sum x(n) e^{-jwn} = \sum \delta(n) e^{-jwn}
\]

\[
= 1 = X(e^{jw})
\]

**Now Find** \( x(n) \)

\[
x(n)=1/2\pi \int_{-1}^{1} e^{jwn} \, dw
\]

\[
= e^{jwn}/2\pi jn \mid \text{ for } n \neq 0
\]

\[
= 1 \text{ for } n = 0
\]
\[(e^{j\pi n} - e^{-j\pi n})\]
\[= 2n\pi j\]
\[= \sin(\pi n)\text{ for } n \neq 0.\]
\[= n\text{ for } n = 0\]
\[= \delta(n)\]

**Example**  Find Transfer function or frequency response for a filter

\[h(n) = \begin{cases} 
1 & , \quad 0 \leq n \leq N-1 \\
0 & \text{ elsewhere.}
\end{cases}\]

\[H(e^{jw}) = \sum 1 \cdot (e^{-jw})^n = 1 - (e^{-jw})^N\]

\[1 - e^{-jw}\]
for \((e^{-jw}) \neq 1\) or \(w \neq 2\pi k\)

\[= N\] for \(w = 2\pi k\)

**Find Amplitude and Phase**

\[
H(e^{jw}) = \frac{1 - e^{-jwN}}{1 - e^{-jw}}
\]

\[
1 - \cos(wN) + j \sin(wN)
\]

\[
= 1 - \cos(w) + j \sin(w)
\]

\[
|H(e^{jw})| = [1 - \cos(wN)]^2 + \sin^2(wN)
\]

\[
[1 - \cos(w)]^2 + \sin^2(w)
\]
1 + \cos^2 + \sin^2 - 2\cos(wN)

= 1 + \cos^2 + \sin^2 - 2\cos(w)

2(1 - \cos(wN))

= 2(1 - \cos(w))

\phi(w) = \tan^{-1}\left(\frac{\sin(wN)}{1 - (\cos(wN)) + \pi u (-(1 - (\cos(wN)))} \frac{\sin(w)}{\tan^{-1}(1 - (\cos(w))) + \pi u(-(1-(\cos(w))))}\right)
Find better amplitude and phase response expressions, starting from:

\[ H(e^{jw}) = \frac{1-e^{-jwN}}{1-e^{-jw}} \]
Example Find the DTFT of $x(n) = .5^n u(n)$

$$X(e^{jw}) = \sum .5^n (e^{-jw})^n = \sum (.5 e^{-jw})^n$$

$$= \frac{1}{1 - .5 e^{-jw}}$$

$$= \frac{1}{[1 - .5 \cos(w)]^2 + [ .5 \sin(w)]^2}$$

$$\exp -j \tan^{-1} \left( \frac{.5 \sin(w)}{1 - .5 \cos(w)} \right)$$
Example Find the forward and inverse transforms of $x(n) = \delta(n) + \delta(n-1)$

$$X(e^{j\omega}) = 1 + e^{-j\omega}$$

$$x(n) = \frac{1}{2\pi} \int (1+e^{-j\omega}) e^{jwn} \, dw$$

$$= \frac{1}{2\pi} \int e^{jwn} \, dw + \frac{1}{2\pi} \int e^{jw(n-1)} \, dw$$

$$= e^{jwn} + e^{jw(n-1)} |_{2\pi jn}^{2\pi j(n-1)}$$

$$= \sin(\pi n) + \sin(\pi(n-1)) |_{\pi n}^{\pi(n-1)}$$

$$= \text{sinc}(n) + \text{sinc}(n-1) = \delta(n) + \delta(n-1)$$
**Frequency Response From Difference Equation**

**Shift Theorem:** $F\{ x(n-n_o) \} = e^{-jw_n} \cdot X(e^{jw})$

Proof: $\sum x(n-n_o) \cdot e^{-jw_n} \mid n \leftarrow n + n_o = \uparrow$

Given the difference equation,

$$\sum a_k \cdot y(n-k) = \sum b_k \cdot x(n-k)$$

find the frequency response $H(e^{jw})$

Taking the DTFT of both sides,

$$F\{ \sum a_k \cdot y(n-k) \} = F\{ \sum b_k \cdot x(n-k) \},$$

$$\sum a_k \cdot F\{ y(n-k) \} = \sum b_k \cdot F\{ x(n-k) \},$$

Using the shift theorem,

$$Y(e^{jw})\sum a_k \cdot e^{-jwk} = X(e^{jw})\sum b_k \cdot e^{-jwk}$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$
\[
\sum b_k e^{-jwk} = \sum a_k e^{-jw}
\]

**Properties of the DTFT**

(1) \(X(e^{jw})\) is a periodic function of \(w\), with period \(2\pi\)

(2) If \(x(n)\) is a real sequence, then
   \[\text{Re } (X(e^{jw}))\] is an even function of \(w\)
   and \(\text{Im } (X(e^{jw}))\) is odd

**Proof:** \(\text{Re } \{\sum x(n) e^{-jwn}\} = \sum x(n) \text{Re } \{e^{-jwn}\}\)

\[
= \sum x(n) \cos(wn) = \sum x(n) \cos(-wn)
\]

\[
= \text{Re } (X(e^{j(-w)})) = \text{Re } (X(e^{jw}))
\]
\[ \text{Im} \{ X(e^{jw}) \} \]

\[ = \sum \text{Im} \{ x(n) [ \cos(wn) - j \sin(wn)] \} \]

\[ = - \sum \text{Im} \{ x(n) [ \cos(wn) + j \sin(wn)] \} \]

\[ = - \text{Im} \ X(e^{-jw}) \]

\( (3) \) if \( x(n) \) is a real sequence, then \( |X(e^{jw})| \)

is an even function of \( w \) and \( \arg \{ X(e^{jw}) \} \)

is an odd function of \( w \).

Proof: Prove it for \( |X(e^{jw})|^2 \)

\[ |X(e^{jw})|^2 = X(e^{jw}) \cdot X(e^{jw})^* \]

but \( X(e^{jw})^* = X(e^{-jw}) \)

so
\[ |X(e^{jw})|^2 = X(e^{jw}) \cdot X(e^{-jw}) \]
\[ |X(e^{-jw})|^2 = X(e^{-jw}) \cdot X(e^{jw}) \]

- \( \arg(X(e^{-jw})) \)

\[ -\sum x(n)\sin(-wn) \]

\[ = -\tan^{-1} \sum x(n)\cos(wn) \]

\[ + \pi u(-\sum x(n)\cos(wn)) \]

\[ -\sum x(n)\sin(wn) \]

\[ = \tan^{-1} \sum x(n)\cos(wn) \]

\[ -\pi u(-\sum x(n)\cos(wn)) \]

\[ = \arg(X(e^{jw})) \]

\[ \therefore \arg(X(e^{jw})) \text{ is an odd function.} \]
(4) let \( x(n) \) be a real, even sequence, 
\[ x(n) = x(-n) \] . Then \( X(e^{jw}) \) is real and
\[ \text{Im} \{ X(e^{jw}) \} = 0. \]

Proof: 
\[ X(e^{jw}) = \sum x(n) e^{-jwn} \]
\[ = x(0) + \sum x(n) e^{-jwn} + \sum x(n) e^{-jwn} \]
\[ = x(0) + \sum x(-n) e^{jwn} \]
\[ = x(0) + 2 \sum x(n) \cos(wn) \]
which is real and even.
(5) Let $x(n)$ be odd and real, \[ x(n) = -x(-n) \]
\[ x(0) = 0. \]
Then $X(e^{jw})$ is odd and imaginary, so
\[ \text{Re}\{X(e^{jw})\} = 0. \]
Proof: $X(e^{jw}) = \sum x(n) e^{-jwn}$
\[ = \sum x(n) e^{-jwn} + \sum x(n) e^{jwn} \]
\[ = \sum x(n) e^{-jwn} - \sum x(n) e^{jwn} \]
\[ = 2j \sum x(n) (e^{-jwn} - e^{jwn}) / 2j \]
\[ = -2j \sum x(n) \sin(wn) \]
which is odd and imaginary

(6) $F\{ x(n-m) \} = e^{-jwm} X(e^{jw})$
(7) $x(n) = e^{jwn}$ is an eigenfunction of the system, $y(n) = h(n) \cdot x(n)$, the corresponding eigenvalue is $H(e^{jw})$

$$y(n) = \sum h(k) e^{jw(n-k)}$$

$$= e^{jwn} \sum h(k)e^{-jwk}$$

$$= e^{jwn} H(e^{jw})$$

Note: $\sum x(g(n)) e^{-jf(w)g(n)} = X(e^{jf(w)})$

**Convolution Theorems for the DTFT**

(8) If $x(n)$, $h(n)$ and $y(n)$ have DTFT’s $X(e^{jw})$, $H(e^{jw})$ and $Y(e^{jw})$, and

If $y(n) = \sum h(k) x(n-k)$,

then $Y(e^{jw}) = H(e^{jw}) X(e^{jw})$
Proof: Take the Fourier Transforms of both sides as

\[ Y(e^{jw}) = \sum \sum h(k) x(n-k) e^{-jwn} \]

\[ = \sum \sum h(k) \ e^{-jwk} \ x(n-k) \ e^{-jw(n-k)} \]

\[ = \sum \ h(k) \ e^{-jwk} \ \sum x(m) \ e^{-jwm} \]

\[ = H(e^{jw}) \cdot X(e^{jw}) \]

(9) \[ F\{x(n)\cdot h(n)\} = 1/2\pi \int X(e^{j(w-u)}) \ H(e^{iu})du \]

First Proof:

Let \[ X(e^{j(w-u)}) = \sum x(n) \ e^{-jn(w-u)} \text{ and} \]

\[ H(e^{iu}) = \sum h(m) \ e^{-jum} \text{ on the right} \]
hand side above. This gives

\[ 1/2\pi \int \sum \sum x(n) h(m) e^{-jnw} e^{ju(n-m)} \, du \]

\[ = 1/2\pi \sum \sum x(n) h(m) e^{-jnw} \int e^{ju(n-m)} \, du \]

\[ = \sum x(n) h(n) e^{-jnw} \]

\[ = F \{ x(n) \cdot h(n) \} \]

**Second Proof for Property (9)** Let

\[ x(n) = 1/2\pi \int X(e^{jv}) e^{jnv} \, dv \] and

\[ h(n) = 1/2\pi \int H(e^{ju}) e^{jnu} \, du \] to get
\[ F \{ x(n) \cdot h(n) \} = \sum x(n) h(n) e^{-jnw} \]

\[ = \frac{1}{4\pi^2} \sum \int \int X(e^{jv}) H(e^{ju}) e^{j(n+u-v-w)} \, du \, dv \]

\[ = \frac{1}{4\pi^2} \int \int X(e^{jv}) H(e^{ju}) \left[ \sum e^{j(n+u-v-w)} \right] \, du \, dv \]

\[ 2\pi \sum \delta(u+v-w-2\pi n) \text{ since} \]

\[ \sum e^{-jnTw} = 2\pi/ T \sum \delta(w-2\pi n / T) \]

\[ = \frac{1}{2\pi} \int \int X(e^{jv}) H(e^{ju}) \left[ \delta(u + v - w) + \delta(u+v-w-2\pi) \right] \, du \, dv \]

\[ u = w-v, \, u = w + 2\pi - v \]
\[ = \frac{1}{2\pi} \int X(e^{jv}) \ H(e^{j(w-v)}) \ dv \]

**Third Proof for property (9)**

Let \( y(n) = h(n) \ast x(n) \).

Find \( Y(e^{jw}) \) as a function of \( H(e^{jw}) \) and \( X(e^{jw}) \)

\[
h(n) = \frac{1}{2\pi} \int H(e^{j\theta}) \ e^{jn\theta} \ d\theta
\]

\[
Y(e^{jw}) = \sum x(n) \ h(n) \ e^{-jn\omega}
\]

\[
= \sum 1/2\pi \int H(e^{j\theta}) \ e^{jn\theta} \ d\theta \ x(n) \ e^{-jn\omega}
\]

\[
= 1/2\pi \int H(e^{j\theta}) \left[ \sum x(n) \ e^{-jn(\omega-\theta)} \right] \ d\theta
\]

\[
= 1/2\pi \int X(e^{j(\omega-\theta)}) \ H(e^{j\theta}) \ d\theta
\]
(10) Parseval’s Equation

\[
\sum_{n=-\infty}^{\infty} h(n)x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw})X(e^{jw})dw
\]

\[x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{-jwn}dw \quad \text{so LHS =} \]

\[
\sum_{n=-\infty}^{\infty} h(n)\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{-jwn}dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) \sum_{n=-\infty}^{\infty} h(n)e^{-jwn}dw
\]

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})H(e^{jw})dw = \text{RHS}
\]

**Ex.** Let \(H(e^{jw})\) be a causal, stable allpass filter, i.e. \(|H(e^{jw})| = 1\) for all \(w\). Prove that \(h(n)\) is shift-orthogonal, i.e.

\[
\sum_{n=-\infty}^{\infty} h(n)h(n+m) = \delta(m)
\]

From (10),

\[
\sum_{n=-\infty}^{\infty} h(n)h^*(n+m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{jw})|^2 e^{-jwm}dw
\]

\[= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jwm}dw = \delta(m)\]
Example IIR and FIR filters

Ex. Zero Phase averaging filter, FIR, non-recursive

\[ y(n) = \frac{1}{1+2M} \sum x(n-k) \]

\[ h(k) = \frac{1}{1+2M} \text{ if } |k| \leq M \]

\[ H(e^{jw}) = \frac{1}{1+2M} \sum e^{-jnw} \]
\[
\begin{align*}
(e^{jMw} - e^{-jw(M+1)}) &= 1 \\
&= (1 - e^{-jw}) \quad (1+2M) \\
(e^{j(M+1/2)w} - e^{-jw(M+1/2)}) &= 1 \\
&= (e^{jw/2} - e^{-jw/2}) \quad (1+2M) \\
1 &= \sin((M+1/2)w) \\
&= (1+2M) \quad \sin(w/2)
\end{align*}
\]

**Example**  Find \( G(e^{jw}) \) if \( g(n) = x(2n) \)

\[
g(n) = 1/ 2\pi \int X(e^{j\theta}) e^{j(2n)\theta} \, d\theta
\]
\[ G(e^{jw}) = \sum g(n) \ e^{-jnw} \]

\[ = \sum 1/ 2\pi \int X(e^{j\theta}) \ e^{j2n\theta} \ d\theta \ e^{-jnw} \]

\[ = 1/ 2\pi \int X(e^{j\theta}) \left[ \sum e^{-jn(w-2\theta)} \right] \ d\theta \]

**Change of variable to simplify the exponent.**

\[ u= w- 2\theta, \ du = -2d\theta \quad d\theta = -1/2 \ du. \]

\[ u_1 = w +2\pi = \text{lower limit} \]

\[ u_2 = w - 2\pi = \text{upper limit} \]

**Switch limits and change signs.**

\[ G(e^{jw}) = 1/ 4\pi \int X(e^{j(w-u)/2}) \left[ \sum e^{-jnu} \right] \ du \]

But
\[ \sum e^{-jnTw} = \frac{2\pi}{T} \sum \delta(w-2\pi n / T) \]

so

\[ \sum e^{-jnu} = 2\pi \sum \delta(u - 2\pi n) \]

\[ G(e^{jw}) = \frac{1}{2} \sum \int X(e^{j(w-u)/2}) \delta(u-2\pi n) \, du \]

\[ = \frac{1}{2} \left[ X(e^{jw/2}) + X(e^{i(w/2-\pi)}) \right] \]

\[ = \frac{1}{2} \left[ X(e^{jw/2}) + X(-e^{j(w/2)}) \right] \]

\[ = G(e^{jw}) \]
E. More Examples

Example \(y(n) - a \ y(n-1) = x(n) - b \ x(n-1)\)

\[ y(n) = x(n) - b \ x(n-1) + a \ y(n-1) \]

Given \(a\), find \(b\) such that the system is all-pass. Frequency response is

\[
H(e^{jw}) = \frac{(1-be^{-jw})}{(1-ae^{-jw})}
\]

\[
|H(e^{jw})|^2 = \frac{|(1-be^{-jw})|^2}{|(1-ae^{-jw})|^2}
\]

\[
= \frac{(1+b^2 - 2b \cos(w))}{(1+a^2 - 2a \cos(w))}
\]
\[
(1 + \frac{1}{b^2}) - 2 \left( \frac{1}{b} \right) \cos(w)
\]
\[
= b^2 (1 + a^2 - 2a \cos(w))
\]

**try b = 1/a**

\[
1 (1 + a^2 - 2a \cos(w))
\]
\[
= a^2 (1 + a^2 - 2a \cos(w))
\]

**Example**

Let \(X(e^{jw}) = F\{ x(n) \}\) Find the sequence \(y(n)\) in terms of \(x(n)\) if

\(Y(e^{jw}) = X^2(e^{jw})\)
\[ X^2(e^{j\omega}) = (\sum x(n) \ e^{-jnw}) \ (\sum x(m) \ e^{-jmw}) \]

\[ = \sum y(k) \ e^{-jkw} \text{ note difference indexes} \]

\[ \therefore \ \sum \sum x(n) \ x(m) \ e^{-jw(n+m)} \]

\[ = \sum y(k) \ e^{-jkw} \]

\[ e^{-jw(n+m)} = e^{-jkw} \] solve for \( n \) as

\[ n+m = k \]

\[ m = k-n \quad m \text{ is fixed now, and sum over } m \text{ disappears.} \]

\[ \therefore y(k) = \sum x(n) \ x(k-n) \]
\[ y(n) = \sum x(k) \, x(n-k) \]

Alternately; use convolution theorem.

**Example**

\[ g(n) = \begin{cases} x(n/2), & n \text{ even} \\ 0, & n \text{ odd.} \end{cases} \]

Find \[ G(e^{jw}) = \sum g(n) \, e^{-jnw} = \sum x(n/2) \, e^{-jnw} \]

\[ = \sum x(n) \, e^{-j(n/2)w} \text{ so } X(e^{j2w}) = G(e^{jw}) \]

**Hard Method**

\[ g(n) = \text{Same Definition.} \]

\[ g(n) = x(n/2) = \frac{1}{2\pi} \int X(e^{j\theta}) \, e^{jn/2}\theta \, d\theta \]
\[ G(e^{iw}) = \sum \frac{1}{2\pi} \int X(e^{j\theta}) e^{j(n/2)\theta} d\theta e^{-jnw} \]

\[ = \frac{1}{2\pi} \int X(e^{j\theta}) \left[ \sum e^{-j(n/2)(\theta - 2w)} \right] d\theta \]

\[ \sum e^{-jn(\theta - 2w)} = 2\pi \sum \delta(\theta - 2w - 2\pi n) \]

use
\[ \sum e^{-jnTw} = \frac{2\pi}{T} \sum \delta(w - 2\pi n / T) \]

\[ = \sum \int X(e^{j\theta}) \delta(\theta - 2w - 2\pi n) d\theta \]

Question: how many values of \( n \) will generate a non-zero \( \delta(\theta - 2w - 2\pi n) \), given constant \( w \).
Answer; only one, use \( n = 0 \).

\[ \therefore \text{ use } \delta(\theta - 2w) \text{ and } \theta = 2w \]

so = \[ \int X(e^{j\theta}) \delta(\theta - 2w) \, d\theta = X(e^{j2w}) \]

**Ex. Ideal LP Filter**

\[
h(n) = \frac{1}{2\pi} \int e^{jwn} \, dw = \quad | e^{jwn} \\
2\pi jn \]

\[ e^{j(wc)n} - e^{-j(wc)n} = (2j) \pi n \]
\[
\sin(w_c n)
= 
\pi n
\]

Ex. Ideal BP Filter

\[
\sin(w_{c2} n) - \sin(w_{c1} n)
\]

\[
h(n) = 
\pi n
\]
How do we implement $y(n) = h(n) \ast x(n)$ in pseudocode if

$$h(n) = \pi n$$

$$\sin(w_{c2}n) - \sin(w_{c1}n)$$
F. Advanced Topic Number 1

Problems:

Applications, such as **speech recognition** and communications have a continuous stream of samples coming in, and spectral information is needed. Using past samples up to time $n$, we can calculate a **Sliding DTFT** of the data in several ways.

**Solution 1**

With samples starting at time 0, and continuing up to time $n$, we get

$$X_n(e^{j\omega}) = \sum_{m=0}^{n} x(m)e^{-j\omega m}$$

If $x(n)$ is real, the number of real multiplies is $N_M = 2(n+1)$. The problems here are that:
(1) $N_M$ quickly becomes too large to update in real time,
(2) The time variable $n$ causes overflow.

Solution 2

We can solve the first problem by defining a spectrum over a fixed window of $N$ samples, starting at time $n-(N-1)$, as

$$X_n(e^{jw}) = \sum_{m=n-(N-1)}^{n} x(m)e^{-jwm}$$

$N_M$ is $2N$ with $N$ fixed, and $n$ increases as new data comes in. Although the limits on the sum increase, this could be fixed by using a shift register that keeps only the most recent $N$ samples. However, the exponent of $e$ still grows without bound, leading to overflow.
Solution 3

We can solve the exponent problem by re-defining the spectrum as

\[ X_n(e^{jw}) = \sum_{m=n-(N-1)}^{n} x(m)e^{-jw(m-n)} \]

which can be re-written as

\[ X_n(e^{jw}) = e^{jw n} \sum_{m=n-(N-1)}^{n} x(m)e^{-jwm} \]

Now, since

\[ X_{n-1}(e^{jw}) = e^{jw(n-1)} \sum_{m=n-(N-1)-1}^{n-1} x(m)e^{-jwm} \]

we can write

\[ X_n(e^{jw}) = e^{jw} X_{n-1}(e^{jw}) + x(n) - x(n-N)e^{jwN} \]

Now the exponents are well-behaved, and we have \( N_M = 6 \) real multiplies.
**Ex:** Suppose that a signal $x(n)$ is being monitored, where

$$x(n) = \cos(w_o(n) \cdot n + \phi(n)) + n(n)$$

and where $n(n)$ represents noise. Here $w_o(n)$ denotes a frequency that is slowly changing with time.

(a) Indicate a method for calculating and updating a relevant feature vector, over a moving window of $N$ time samples.

(b) Give a method for estimating $w_o(n)$.

**Solution:** Given the number of features $N_F$, define evenly spaced frequencies as $w(k) = (\pi/(N_F-1))(k-1)$.

(a) For $k$ between 1 and $N_F$, the $k$th complex feature $X(k)$, in the feature vector $X$, is calculated and updated as

$$X(k) = X_n(e^{jw(k)})$$ on the previous page.
(b) At each time $n$, estimate $w_o$ as:

$$X_{\text{max}} = |X(1)|, \quad w_o = w(1)$$

For $2 \leq k \leq N_F$

If ($|X(k)| > X_{\text{max}}$) Then

$$X_{\text{max}} = |X(k)|$$

$$w_o = w(k)$$

Endif

End