General procedure for Design of Frequency selective filters.

(1) Specify or determine the digital filter desired frequency response in radians.
   (a) This could be directly specified in radians, or
   (b) Given T and the desired analog frequency response, we can get the desired digital frequency response.

(2) Pick the filter type FIR or IIR.
   (a) FIR, if a specific phase response is needed, such is zero or linear phase, or matched filter.
   (b) IIR, if phase is not important or zero phase IIR acceptable (non-causal)
(3) Pick order and design method
   (a) IIR (Butterworth, Chebyshev, Bessel, Eliptic). Convert to H(z) using Impulse Invariance or Bilinear Transform
   (b) FIR
      (1) Pick window type or
      (2) Use alternate technique
   (4) Design Filter

A. Causal IIR Filters from Analog Filters

1. General requirements
   (a) A stable s-plane filter H(s) (poles in LHP) must be transformed into a stable z-plane filter (poles inside unit circle). Imaginary axis in s-plane map to unit circle in z-plane.
(b) The mapping of the analog frequency response to the digital frequency response must be well understood (LP to LP, BP to BP, HP to HP, BR to BR etc)

2. Impulse Invariance Design Method

(a) Given the analog filters impulse response, sample it to get the digital filter unit pulse response.

\[ h(n) = T \ h_a(nT) \]

(b) Frequency Response Relation is found as

\[ H_s(j\Omega) = T \ (1/T) \ \sum H_a(j \ (\Omega - (2\pi/T)k)) \]

Note that the T factor is used to cancel out the 1/T factor.
\[ H(e^{jw}) = \sum H_a( j(w/T) + j(2\pi/T) k ) \]

\[ H(e^{jw}) = H_a( j(w/T) ) \]

iff \( H_a( j(w/T) ) = 0 \) for \( |w| \geq \pi \)

HP Case:
Therefore used for lowpass and bandpass but not for HP or BR filters. Aliasing damages II performance, even for LP and BP cases.

(c) z-plane, s-plane stability relationship

Starting with

\[
H_s(j\Omega) = \sum H_a(j (\Omega - (2\pi/T)k))
\]

Substitute \(s\) for \(j\Omega\), which gives

\[
H(z) \mid_{z = e^{sT}} = \sum H_a(s + j(2\pi/T)k)
\]

Need \(|z| < 1\) if \(\text{Re}(S) < 0\) for stability

\[
s = \sigma + j\Omega
\]

\[
z = e^{sT} = e^{(\sigma+j\Omega)T} = e^{\sigma T} e^{j\Omega T}
\]
Let $s$ be a pole of $H_a(s)$ so $z$ is a pole of $H(z)$.

**Case 1:** $\sigma = 0$, marginal stability case for analog filter

$$|z| = |e^{\sigma T}| |e^{j\Omega T}| = 1$$

∴ Imaginary axis maps to unit circle, the marginal stability region for digital filters.

**Case 2:** $\sigma < 0$, stable region for analog transfer function poles

$$|z| = |e^{-|\sigma|T} e^{j\Omega T}| = |e^{-|\sigma|T}| |e^{j\Omega T}| < 1 \cdot 1$$

$$|z| < 1$$

Stable region for poles of $H(z)$

**Case 3:** $\sigma > 0$, unstable region for analog transfer function poles
\[ |z| = |e^{\sigma T} e^{i\Omega T}| = e^{\sigma T} |e^{i\Omega T}| > 1 \]

Unstable region for poles of \( H(z) \)

(d) **Examples**

**Example** \( h_a(t) \) is the analog impulse response of a filter.

\[ h_a(t) = e^{-0.9t} u(t) \]

Let \( h(n) \) and \( H(z) \) denote the impulse response and transfer function of a digital filter designed by impulse invariance.

\[ \therefore h(n) = T \cdot h_a(nT) \]

(a) Determine \( H(z) \), including \( T \) as a parameter, and show that for any positive value of \( T \), the digital filter is stable
(b) Indicate whether the digital filter is LP, HP, or BP.

Solution:

(a) $a = .9$

$h(n) = e^{-anT} u(n)T$

\[
H(z) = T \sum e^{-anT} z^{-n} = T \sum (e^{-aT} z^{-1})^n
\]

\[
= \frac{T}{(1 - e^{-aT} z^{-1})} = H(z)
\]

Stable if $|\text{pole}| < 1$ in magnitude

$H(z) = \infty$ if $z^{-1} = e^{aT}$ or $z = e^{-aT} = \text{pole}$
\[ |e^{-aT}| < 1 \text{ for } T > 0 \]

\[ \therefore \text{ Stable for all positive } T \]

(b) Type of frequency response?

\[ |H(e^{jw})| = \]

\[ T \]

\[ \sqrt{(1 - e^{-0.9T} e^{-jw})(1 - e^{-0.9T} e^{jw})} \]

\[ = \]

\[ T \]

\[ \sqrt{1 + e^{-1.8T} - 2e^{-0.9T} \cos(w)} \]
| $w$ | $\cos(w)$ | denominator | $|H(e^{jw})|$ |
|-----|-----------|-------------|----------------|
| 0   | 1         | Smallest    | Largest        |
| $\pi$ | -1       | Largest     | Smallest       |
| $\pi/2$ | 0        | In Between  | In between     |

Obviously Lowpass

**Example: The general case**

Suppose

$$H_a(s) = \sum A_k$$

$$s - s_k$$

after a partial fraction expansion, where $s_k$ are poles, some of which are complex.

Find $H(z)$
Solution:

\[ h_a(t) = \sum A_k \exp(s_k \cdot t) \ u(t) \]

\[ h(n) = T \ h_a(nT) = T \sum A_k \exp(s_k \cdot nT) \ u(n) \]

\[ H(z) = \sum h(n) \ z^{-n} \]

\[ = T \sum \sum A_k \exp(s_k \cdot nT) \ z^{-n} \]

\[ = T \sum A_k \sum \left( e^{s \cdot T} z^{-1} \right)^n \]

\[ = \sum \ T A_k \]

\[ (1 - \exp(s_k \cdot T) \ z^{-1}) \]
Example Given the analog filter

\[ H_a(s) = \frac{s+2}{(s+2)^2 + 9} \]

which has a center frequency = 3 rad/sec and a bandwidth of \( a/\pi \) in Hz

(a) Find the corresponding digital filter \( H(z) \) by impulse invariance, where
\( T = \) 0.1 sec.

(b) Develop an algorithm which calculates \( N_{\text{pts}} \) points of the filter's amplitude and phase responses.
(c) Develop PseudoCode for an algorithm which applies this filter to an input sequence \( x(n) \) to get an output sequence \( y(n) \).

Solution:

(a) \[ h_a(t) = e^{-2t} \cos(3t) \ u(t) \]

\[ h(n) = T \cdot e^{-2nT} \cos(3nT) \ u(nT) \]

\[
\begin{align*}
T &= e^{-2nT} \left( e^{j3nT} + e^{-j3nT} \right) \ u(n) \\
&= \frac{1}{2} \left( e^{-2T+j3T} \right)^n + \left( e^{-2T-j3T} \right)^n \\
&= \frac{1}{2} \\
\end{align*}
\]
\[ H(z) = \frac{1}{2} \left( (e^{-2T+j3T} z^{-1})^n + (e^{-2T-j3T} z^{-1})^n \right) \]
\[ + \left(1-e^{-2T+j3T} z^{-1}\right) \left(1-e^{-2T-j3T} z^{-1}\right) \]

\[ = T \left( 1 \right. \left. - e^{-2T} \left( \frac{e^{j3T} + e^{-j3T}}{2} \right) z^{-1} \right) \]
\[ = \left( 1 \right. \left. - e^{-2T} \left( e^{j3T} + e^{-j3T} \right) z^{-1} + e^{-4T} z^{-2} \right) \]

\[ = T \left( 1 \right. \left. - e^{-2T} \cos(3T) z^{-1} \right) \]
\[ = \left( 1 \right. \left. - 2e^{-2T} \cos(3T) z^{-1} + e^{-4T} z^{-2} \right) \]

\[ T = 0.1 \]
\[
0.1 \left( 1 - e^{-0.2} \cos(0.3) \, z^{-1} \right)
\]

\[
\therefore H(z) =
\left( 1 - 2e^{-0.2} \cos(0.3) \, z^{-1} + e^{-0.4} \, z^{-2} \right)
\]

(b) Algorithm for amplitude and phase response
(c) PseudoCode for applying the filter
3. Bilinear Transformation

Impulse invariance aliasing problems are solved by using the bilinear transform,

\[ s = \frac{2 \left(1 - z^{-1}\right)}{T \left(1 + z^{-1}\right)} \]

\[ T s (1 + z^{-1}) = 2 \left(1 - z^{-1}\right) \]

\[ z^{-1} = \frac{2 - T \cdot s}{2 + T \cdot s} \]

\[ z = \frac{2 + T \cdot s}{2 - T \cdot s} \]
\[ H(z) = H_a(s) \]

Question, Does a stable \( H_a(s) \) become a stable \( H(z) \)? Yes. Let the pole of \( H_a(s) \) be

\[ s = a + jb \]

The corresponding pole of \( H(z) \) is

\[ z = \frac{2 + T(a+jb)}{2 - T(a+jb)} \]

\[ = \frac{(2 + aT) + jbT}{(2 - aT) - jbT} \]

\[ |z|^2 = \frac{(2 + aT)^2 + b^2T^2}{(2 - aT)^2 + b^2T^2} \]
Case 1: $H_a(s)$ is marginally stable and $a=0$

Then $|z|^2$ and $|z|$ are 1 and the $j\ \Omega$ axis in the $s$ plane maps to the unit circle in the $z$ plane. $z$ is a marginally stable pole

Case 2: $H_a(s)$ is stable and $a < 0$

Then

$$|z|^2 = \frac{(2 + aT)^2 + b^2T^2}{(2 - aT)^2 + b^2T^2} < 1$$

and the left-half $s$ plane maps to the inside of the unit circle in the $z$ plane. $z$ is a stable pole and $H(z)$ is stable.

Case 3: $H_a(s)$ is unstable and $a > 0$

$$|z|^2 = \frac{(2 + aT)^2 + b^2T^2}{(2 - aT)^2 + b^2T^2} > 1$$
Then \( |z|^2 \) and \(|z| > 1\) and the right-half s plane maps to the outside of the unit circle in the z plane. \( z \) is an unstable pole and \( H(z) \) is unstable.

\[ \therefore \text{Bilinear transform maps stable S domain filters to stable Z domain.} \]

(a) Question 2, How does Bilinear transform map \( H_a(j\Omega) \) into \( H(e^{jw}) \)?

Let \( z = e^{jw} \),

\[
s = \frac{2(1-e^{-jw})}{T(1+e^{-jw})} \left( e^{jw/2} - e^{-jw/2} \right) /2j = \left( e^{jw/2} + e^{-jw/2} \right) /2
\]
\[ 2j \sin(w/2) = T \cos(w/2) \]

So,
\[ 2 \Omega = \tan(w/2) \]
\[ T \]

Now let \( \Omega_c \) be a continuous system's frequency variable in rad/sec and let \( \Omega_d \) be a digital system's frequency variable in rad/sec

Then \( \Omega_d T = w \) and

\[ 2 \]
\[ \Omega_c = \tan \left( \frac{\Omega_d T}{2} \right) \]
\[ T \]

and
\[ w = \Omega_d T = 2 \tan^{-1} (\Omega_c T / 2) \]

Analog to Digital Frequency Mappings

**Impulse Invariance**

**Bilinear Transform**
Preliminary Example to illustrate the mechanics of the transformation

\[ H_a(s) = \frac{1}{s+2}, \]

find \( H(z) \) as a function of the parameter \( T \).

Answer acceptable but before being used, we must normalize \( H(z) \) so is 1.
(c) Butterworth Polynomials and filters

First let's specify a "prototype" lowpass filter which has a cut-off frequency of 1 rad./sec. This filter $H_1(s)$ can then be transformed into any frequency selective filter of our choice.

$$H_1(s) = \frac{1}{a_ns^n + a_{n-1}s^{n-1} + \ldots + 1}$$

$n$ is filter order \hspace{1cm} $\Omega_c = 1$

<table>
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<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
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<td>4</td>
<td>2.613</td>
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Magnitude squared characteristic of a Butterworth filter with cut-off frequency $\Omega_c$ is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2n}}$$

For the prototype filter with cut-off frequency of 1 rad./sec., we have

$$|H_1(j\Omega)|^2 = \frac{1}{1 + \Omega^{2n}}$$

Sketch
Derivation of Butterworth $H_1(s)$ Poles

$s = j\Omega$, \quad \Omega = s/j$ so

$$|H_1(s)|^2 = \frac{1}{1 + (s/j)^{2n}}$$

$$= \frac{1}{1 + (-1)^n s^{2n}}$$

We need to factor $|H_1(s)|^2$ into $H_1(s)H_1^*(s)$, where $H_1(s)$ has the stable poles

**Case 1:** $n$ is even

$$|H_1(s)|^2 = \frac{1}{1 + s^{2n}}$$
Poles: \( s^{2n} = -1 = e^{j(\pi + k \cdot 2\pi)} \)

and

\[ s_k = e^{j(\pi/2n + k \pi/n)} \quad , \quad k = 0, 1, 2….2n-1 \]

**Case 2:** \( n \) is odd

\[ |H_1(s)|^2 = \frac{1}{1 - s^{2n}} \]

Poles: \( s^{2n} = 1 = e^{j \cdot k \cdot 2\pi} \), and

\[ s_k = e^{jk \pi/n} \quad , \quad k = 0, 1, 2….2n-1 \]

**Properties of \( H_1(s) \) Poles**

1. They are on the unit circle
2. Adjacent poles are separated by an angle of \( \pi/n \)
3. For any filter with real coefficients, the poles come in complex conjugate pairs
4. No poles on \( j\Omega \) axis.
We need expressions for $s_k$ satisfying:
1. $k = 1, 2, 3, \ldots$ gives the stable poles
2. The expressions are easy to remember
3. The expressions are the same for $n$ odd and $n$ even

Sketch for $n=4$

For $n$ even, the stable poles are

$$s_k = e^{j\theta(k)}$$

$$\theta(k) = \frac{\pi}{2} + \frac{\pi}{2n} + (k-1) \frac{\pi}{n}$$

and $k = 1, 2, \ldots, n$
Sketch for n=3

For n odd, the stable poles are again

\[ s_k = e^{j\theta(k)} \]

\[ \theta(k) = \pi/2 + \pi/2n + (k-1) \pi/n \]

and \( k = 1,2,\ldots,n \)
Note that there is one pole at -1 on the real axis.

For n even, a complex conjugate pair of pole factors is
\[(s - e^{j\theta(k)})(s - e^{-j\theta(k)})\]

\[= (s^2 - 2 \cos(\theta(k)) s + 1 )\]

so

\[H_1(s) = \frac{1}{\Pi (s^2 - 2 \cos(\theta(k)) s + 1 )}\]

For \(n\) odd we have

\[H_1(s) = \frac{1}{(s+1) \Pi (s^2 - 2 \cos(\theta(k)) s + 1 )}\]

\[(d) \text{ Chebyshev Filter and Pole locations}\]

First calculate to \(\beta = \tanh \left[ \frac{1}{n} \sinh^{-1}(1/\varepsilon) \right] \)

\(s_k = s_k\) but with real part multiplied by \(\beta\)
\[ s_k = \beta \cos \left( \frac{\pi}{2} + \frac{\pi}{2n} + (k-1) \frac{\pi}{n} \right) + \text{j} \cdot \sin \left( \frac{\pi}{2} + \frac{\pi}{2n} + (k-1) \frac{\pi}{n} \right) \]

\[ H_1(s) = K \prod \left( s - s_k \right) \]

Find K such that \( H_1(0) = 1 \).

\[ \therefore K = \prod ( -s_k ) \]

Typical values for \( \varepsilon \) are 0.3, 0.2, 0.1, 0.5 etc.

(e) Given \( H_1(s) \), next step is to apply the proper transformation and get \( H_a(s) \).

Case 1 : Lowpass from \( H_1(s) \) and \( \Omega_{cc} \)

\[ H_a(s) = H_1(s) \]
Case 2: Highpass from $H_1(s)$ and $\Omega_{cc}$

$$H_a(s) = H_1(s)$$

Case 3: Bandpass from $\Omega_{c1}$, $\Omega_{c2}$, $H_1(s)$

$$\Omega_o = \sqrt{\Omega_{c2} \cdot \Omega_{c1}}$$
$$BW = \Omega_{c2} - \Omega_{c1}$$

$$H_a(s) = H_1(s)$$

Case 4: Bandreject

$$H_a(s) = H_1(s)$$

Given $H_a(s)$ design the filter $H(z)$ using Impulse invariance or Bilinear Transform
(f) Examples

Example Design a Butterworth lowpass filter $H_a(s)$ with 2$^{nd}$ order and cutoff $\Omega_{cc} = 5 \text{ rad/sec}$

$$H_a(s) = \frac{1}{s^2 + \sqrt{2} s + 1}$$

$$= \frac{1}{(s/5)^2 + (\sqrt{2} s / 5) + 1}$$

$$= \frac{25}{(s)^2 + \sqrt{2} \cdot s \cdot 5 + 25}$$
**Example**  Design a 2\textsuperscript{nd} order Butterworth band pass filter $H_a(s)$ with cut-offs $\Omega_{c1} = 2$ and $\Omega_{c2} = 5$,  $\text{BW} = 5-2 = 3$

$\Omega_o = \sqrt{10} = 3.162$

$$H_a(s) = \frac{1}{s+1}$$

$$= \frac{(3s/3s)}{((s^2+10)/3s) + (3s/3s)}$$

$$= \frac{3s}{(s^2+3s+10)}$$
(4) **Deciding Filter order n for Butterworth Lowpass Case**

Assume $H(z)$ specifications in db

(a) Given digital cut off $\Omega_{dc}$, calculate $\Omega_{cc}$ for Bilinear case, or use $\Omega_{cc} = \Omega_{dc}$ for impulse invariant case.

Magnitude is

$$|H_a(j\Omega)| = \frac{1}{\left[1 + (\Omega/\Omega_{cc})^{2n}\right]^{1/2}}$$

(b) Express specifications using (a)
(c) Solve for $n$, if $n$ is a fraction, use the next largest value for $n$.

**Example** Find the order $n$ of a Butterworth lowpass filter $H_a(s)$ such that
(a) Response must be down by 3 dB at most at 1000 Hz (cut off frequency)

(b) Down by at least 10 dB at 3000 Hz,

\[
1 \leq 10 \log_{10} \left( \frac{1}{\left[ 1 + \left( \frac{\Omega}{2\pi 1000} \right)^{2n} \right]^{1/2}} \right)
\]

\[
0 - 20 \log_{10} \left( \left[ 1 + \left( \frac{\Omega}{2\pi 1000} \right)^{2n} \right]^{1/2} \right) \leq -10
\]

\[-10 \log \left( 1 + \left( \frac{\Omega}{2\pi 1000} \right)^{2n} \right) \leq -10
\]

\[\log \left[ 1 + \left( \frac{\Omega}{2\pi 1000} \right)^{2n} \right] \geq 1\]

Solve inequality:

\[
\left[ 1 + \left( \frac{\Omega}{2\pi 1000} \right)^{2n} \right] \geq 10
\]

\[
\left( \Omega / \pi 2000 \right)^{2n} \geq 9 \quad \text{for} \quad \Omega = 3000 \cdot 2\pi
\]

\[3^{2n} \geq 9\]
2n \ln 3 \geq \ln 9

n \geq \frac{1}{2} \left( \frac{\ln 9}{\ln 3} \right) = 1

\therefore \text{ First order Butterworth is O.K.}

\begin{align*}
    H_a(s) &= \frac{1}{s+1} \\
    |H_a(s)| &= \frac{1}{s+1}
\end{align*}

\textbf{Example}  \ What order must a Bilinear transformed lowpass Butterworth filter H(z) have to satisfy these conditions?

(a) \( \Omega_{dc} = 5 \text{ rad/sec} \)
(b) \( T = 0.1 \text{ sec} \)
(c) Response is down 20 db for \( \Omega_d = 10 \text{ rad/sec} \)
Design the filter

Find

\[ \Omega_{cc} = \frac{2}{T} \tan \left( \Omega_{dc} \cdot \frac{T}{2} \right) \]

\[ = 20 \tan (0.25) \]

\[ = 5.11 \]

Find

\[ \Omega_{c2} = 20 \tan \left( 10 \cdot \frac{T}{2} \right) \]

\[ = 20 \tan (0.5) \]

\[ = 10.93 \]

\[
\frac{1}{20 \log_{10} \left( \frac{1}{\left[ 1 + \left( \frac{\Omega_{c2}}{\Omega_{cc}} \right)^{2n} \right]^{1/2}} \right)} \leq -20
\]
\[ 20 \log_{10} 1 - 20 \log_{10} [1 + 2.139^{2n}]^{1/2} \leq -20 \]

\[-10 \log_{10} [1 + 2.139^{2n}] \leq -20 \]

\[ \log_{10} [1 + 2.139^{2n}] \geq 2 \]

\[ [1 + 2.139^{2n}] \geq 100 \]

\[ 2.139^{2n} \geq 99 \]

\[ 2n \ln 2.139 \geq \ln 99 \]

\[ n \geq \frac{1}{2} \ln 99 / \ln 2.139 = 3.02, \text{ pick } n = 4 \]
\[ H_a(s) = \frac{1}{s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1} \]

Digital filter

\[ s \leftarrow \frac{2}{T} \frac{(1-z^{-1})}{(1+z^{-1})} \]

\[ = 20 \frac{1-z^{-1}}{1+z^{-1}} \]
Finding Necessary Order of Butterworth Bandpass filters

(1) Convert digital frequencies in radians to analog frequencies. The analog cut-off frequencies are \( \Omega_{c1} \) and \( \Omega_{c2} \)

(2) Map bandpass characteristics back to low pass characteristics
\[
BW = \Omega_{c2} - \Omega_{c1},
\]
\[
\Omega_0 = \sqrt{\Omega_{c1} \Omega_{c2}}
\]
\[
\Omega_{BP}^2 - \Omega_o^2
\]
\[
\Omega_{LP} = \Omega_{BP} BW
\]

(3) Find order \( n \) of lowpass Butterworth filter \( H_1(s) \) satisfying conditions.
(4) Round the order up to an integer value, and then double it for the order of the bandpass filters \( H_a(s) \) and \( H(z) \).

**Example**  What order must a bilinearly transformed bandpass Butterworth filter \( H(z) \) have to satisfy these conditions;
(a) \( \Omega_{d1} = 0.8 \) rad/sec
(b) \( \Omega_{d2} = 1.5 \) rad/sec
(c) \( T = 1 \) sec
(d) Response is down 30 db or more for \( \Omega_d = 0.4 \) rad/sec and for \( \Omega_d = 2.5 \) rad/sec

(1) Find \( H_a(s) \) cut-off frequencies
\[
\Omega_{c1} = \frac{2}{T} \tan \left( \frac{\Omega_{d1}}{2} \frac{T}{2} \right) = 0.8456
\]
\[ \Omega_{c2} = \frac{2}{T} \tan (\Omega_{d2} T/2) = 1.8632 \]

\[ \Omega_{c3} = 2 \tan (0.4/2) = 0.405 \]

\[ \Omega_{c4} = 2 \tan (2.5/2) = 6.019 \]

\[ \Omega_{BP}^2 - \Omega_o^2 = \]

\[ \Omega_{LP} = \Omega_{BP} BW \]

Test try \( \Omega_{BP} = 0.846 \quad \Omega_{LP} = -1 \)

Try \( \Omega_{BP} = 1.8632 \quad \Omega_{LP} = 1 \)

Try \( \Omega_{c3} = \Omega_{BP} \quad \Omega_{LP3} = -3.425 \)

Try \( \Omega_{c4} = \Omega_{BP} \quad \Omega_{LP4} = 5.657 \)
(3) 
\[ 20 \log \left( \frac{1}{\sqrt{1 + 3.425^{2n}}} \right) \leq -30 \]
\[ = -10 \log (1+3.425^{2n}) \leq -30 \]
\[ 1+3.425^{2n} \geq 1000 \]
\[ 3.425^{2n} \geq 999 \]
\[ 2n \ln(3.425) \geq \ln 999 \]
\[ \ln 999 \]
\[ n \geq \frac{\ln 999}{2 \ln(3.425)} = 2.805 \]

(4) Round up to 3 for H₁(s) and double to get n = 6 for Hₐ(s) and H(z)
Example

An analog lowpass filter has the following characteristics:

\[
| |H_c(j\Omega)| - 1| \leq \delta_1 \quad \text{for} \quad |\Omega| \leq \Omega_p
\]
\[
|H_c(j\Omega)| \leq \delta_2 \quad \text{for} \quad |\Omega| \geq \Omega_s
\]

\[H(z) = H_c(s)\]

(a) For constant \(\Omega_p\) find \(T_d\) such that

\[w_p = \pi/2\]

\[
\Omega_p = \underbrace{\frac{2}{T_d}}_{T_d} \tan(w_p/2) = \frac{2}{T_d} \tan(\pi/4)
\]

\[T_d = \frac{2}{\Omega_p}\]
(b) With $\Omega_p$ fixed, sketch $w_p$ as a function of $T_d$

$$w_p = 2 \tan^{-1}(\Omega_p \ T_d / 2)$$

(c) With $\Omega_p$ and $\Omega_s$ fixed, sketch $\Delta w = (w_s - w_p)$ as a function of $T_d$

$$w_s - w_p = 2\tan^{-1}(\Omega_s T_d / 2) - 2\tan^{-1}(\Omega_p T_d / 2)$$

The slope at $T_d = 0$ is $(\Omega_s - \Omega_p)$
Example  The analog filter \( H_c(s) = \frac{1}{s} \) corresponds to an ideal integrator

\[
y_c(t) = \int x_c(T) \, dT
\]

(a) Find \( h(n) \) if

\[
H(z) = \begin{cases} \frac{1}{s} & \text{if } s \\ (1+z^{-1}) &= (T/2) \\ (1- z^{-1}) \end{cases}
\]

\[
h(n) = \frac{T}{2} \left( \delta(n) + \delta(n-1) \right) * u(n)
\]
\[ = T/2 \left[ u(n) + u(n-1) \right] \]

(b) Find the difference equation implementing \( H(z) \), and describe the system's stability.

\[ y(n) = T/2 \left[ x(n) + x(n-1) \right] + y(n-1) \]
Marginally stable. Pole at \( z=1 \) on unit circle

(c) Find \( H(e^{jw}) \) and relate it to \( H_c(j\Omega) \)

\[
H(e^{jw}) = \frac{(1+e^{-jw}) e^{jw/2}}{(1- e^{-jw}) e^{jw/2}}
\]
\[
\begin{align*}
\cos(w/2) &= (T/2j) \\
\sin(w/2) &= (T/2j) \left[ \cot(w/2) \right] \\
 &= (T/2j)(2/w - 2w/12 \ldots)
\end{align*}
\]

Using \(\Omega T = w\), the first term in \(H(e^{j\Omega T})\) is \(1/(j\Omega)\) which equals \(H_c(j\Omega)\)

**Example** A lowpass filter \(H_c(s)\) satisfies

\[
1 - \delta_1 \leq |H_c(j\Omega)| \leq 1 + \delta_1 \quad \text{for} \quad |\Omega| \leq \Omega_p
\]
\[ |H_c(j\Omega)| \leq \delta_2 \quad \text{for} \quad \Omega_s \leq |\Omega| \]

Digital filters \(H_1(z)\) and \(H_2(z)\), with cut-offs \(w_{p1}\) and \(w_{p2}\), are designed using

\[
H_1(z) = H_c(s) \quad \text{is a LPF}
\]

\[
H_2(z) = H_c(s) \quad \text{is a HPF}
\]

(a) Find the \(w_{p1}\) and \(\Omega_p\) relationships

\[
-j\Omega = \frac{(1-e^{-jw}) e^{jw/2}}{(1+ e^{-jw}) e^{jw/2}}
\]

\[
\Omega_p = \tan \left(\frac{w_{p1}}{2}\right)
\]
\[ w_{p1} = 2 \tan^{-1}(\Omega_p) \]

(b) Find the \( w_{p2} \) and \( \Omega_p \) relationships

\[ j\Omega = \frac{1}{j \tan\left(\frac{w}{2}\right)} \]

\[ j\Omega_p = -j \cot\left(\frac{w_{p2}}{2}\right) \]

\[ \Omega_p = -\cot\left(\frac{w_{p2}}{2}\right) \]

\[ \cot^{-1}(-\Omega_p) = \frac{w_{p2}}{2} \]

\[ w_{p2} = 2 \cot^{-1}(-\Omega_p) \]

(c) Find the relationship between \( w_{p1} \) and \( w_{p2} \)
\[
\sin(\frac{w_{p1}}{2})
\]
\[
\Omega_p = \tan(\frac{w_{p1}}{2}) = \cos (\frac{w_{p1}}{2})
\]

- \cos(\frac{w_{p2}}{2})
\[
\Omega_p = - \cot(\frac{w_{p2}}{2}) = \sin (\frac{w_{p2}}{2})
\]

\[
\sin(\frac{w_{p2}}{2} - \frac{\pi}{2}) = \cos (\frac{w_{p2}}{2} - \frac{\pi}{2})
\]

\[
(w_{p2}/2) - (\pi/2) = w_{p1}/2
\]

\[
w_{p1} = w_{p2} - \pi
\]
B. Allpass Filters

**Goal:** Given a filter or system $H(z)$ with an undesirable phase $\phi(w)$, we want to design a filter $H_{ap}(z)$ that corrects $H(z)$ so that the overall system has the desired phase $\phi_d(w)$ and $|H_{ap}(e^{jw})| = 1$.

1. Allpass Filter’s Phase

Find $\phi_{ap}(w)$ for $H_{ap}(z)$ as follows:

$\phi(w) + \phi_{ap}(w) = \phi_d(w)$, so

$\phi_{ap}(w) = \phi_d(w) - \phi(w)$

2. IIR Implementation

If the kth pole out of N poles is $z_0(k)$,

$$H_{ap}(z) = \prod_{k=1}^{N/2} \frac{c_k + b_k z^{-1} + z^{-2}}{1 + b_k z^{-1} + c_k z^{-2}},$$

$$b_k = -2 \text{Re}\{z_0(k)\}, \quad c_k = |z_0(k)|^2$$
C. FIR Digital Filters

1. Properties of FIR Digital filters
   (a) FIR filters can be any reasonable amplitude or phase whereas IIR filters can have band pass, high pass or low pass amplitude phase is non-linear (other responses very hard to get)
   (b) FIR filters have far more coefficients than IIR filters. \( N >> (N_b + N_a) \) maybe ten times greater.
       \(.\) FIR difference equation takes more time to apply in time domain
   (c) FIR filters may be applied in the frequency domain using DFT (FFT i.e name of a fast DFT algorithm). This sometimes makes them as fast to apply as IIR filters.
   (d) Always stable.
2. Desired Phase Responses

(a) Zero Phase

H(e^{jw}) is real, \text{Im} (H(e^{jw})) = 0

h(n) = \frac{1}{2\pi} \int H(e^{jw}) e^{jwn} \, dw

= \frac{1}{2\pi} \int H(e^{jw}) \cos(wn) \, dw

h(-n) = \frac{1}{2\pi} \int H(e^{jw}) \cos(-wn) \, dw

= h(n)

\therefore \text{Filter with } h(n) = h(-n) \text{ has zero phase.}
(b) Linear Phase

Let $H(e^{jw}) = H_z(e^{jw}) e^{-jwK}$

$h(n) = \frac{1}{2\pi} \int H(e^{jw}) e^{jwn} \, dw$

$$= \frac{1}{2\pi} \int H_z(e^{jw}) (e^{jw(n-K)}) \, dw$$

$$= \frac{1}{2\pi} \int H_z(e^{jw}) \cos(w(n-K)) \, dw$$

Let $n \leftrightarrow -n + 2K$ : $h(2K - n) =$

$$1/2\pi \int H_z(e^{jw}) \cos(-w(n-K)) \, dw$$

$$= h(n)$$

K is usually $\frac{1}{2}$ the order of the filter.
\[ H(z) = \sum_{K=0}^{N-1} h(n) z^{-n} \]

\[ \therefore h(n) = h(N-1-n) \]

**Effect in time domain**
(1) Output is symmetric about $n_0$ if input is symmetric about $n_0$ centered at the same time.

(2) The zero-phase filter is non-causal, since $h(n) \neq 0$ for some negative $n$. Not applied in real time. Store input and filter and then use output.

**Example** From another viewpoint

$$H(e^{jw}) = \sum h(n) e^{-jwn}$$

$$= [ (h(0) + h(N-1) e^{-jw(N-1)} )$$
\[ + (h(1) e^{-j\omega} + h(N-2)e^{-j\omega(N-2)}) \ldots \]

\[ + h((N-1)/2) e^{-j\omega((N-1)/2)} \cdot e^{j\omega((N-1)/2)} e^{-j\omega((N-1)/2)} = e^{-j\omega((N-1)/2)} \cdot h((N-1)/2) + 2 \sum h(n) \cos(w(n-(N-1)/2))] \]

Most common case is \( N \) odd, time delay is position of middle coefficient.

IIR filters can have 0 phase if

\[ H(z) = H(z) \cdot H(z^{-1}) \]

Apply filter \( H(z) \) in forward time.
\[ H(e^{jw}) = H(e^{jw}) H(e^{-jw}) = |H(e^{jw})|^2 \]

Apply filter \( H(z^{-1}) \) backward in time.

**Example**

\[ H(z) = \frac{1}{1 + 0.5z^{-1}} \]

\[ H(z^{-1}) = \frac{1}{1 + 0.5z} \]

\[ y_1(n) = x(n) - 0.5y_1(n-1) \]

\[ n = 0, 1, 2, \ldots, N \]

Causal stable filter

\[ y_2(n) = y_2(n) - 0.5y_2(n + 1) \]
for \( n = N, N-1, N-2, \ldots, 0 \)
Anti-Causal stable filter.

(3) Design of FIR filters using Windows

\[
h_d(n) = \frac{1}{2\pi} \int H_d(e^{jw}) e^{jwn} \, dw
\]
for all values on \( n \)

If \( h_d(n) \) is truncated to \( N \) coefficients as
\[h(n) = h_d(n) \, w(n), \text{ where } w(n) \text{ is the rectangular window function.}\]

\[
w(n) = \begin{cases} 
1 & \text{for } 0 \leq n \leq N-1 \\
0, & \text{else}
\end{cases}
\]
Then

$$H(e^{jw}) = \frac{1}{2\pi} \int H_d(e^{j\theta}) W(e^{jw-\theta}) \, d\theta$$

where

$$W(e^{jw}) = \frac{\sin(wN/2)}{\sin(w/2)} e^{-jw(N-1)/2}$$

This is the circular convolution of $W(e^{jw})$ with $H_d(e^{jw})$
Truncation of $h_d(n)$ to $N$ coefficients, 
Gibbs phenomenon goes away as $N \rightarrow \infty$

Non-Rectangular Windows

Standard way of fixing this is

$$h(n) = h_d(n) \cdot w(n), \quad \text{where } w(n) \text{ is not the rectangular truncation window function.}$$

Want side lobes small and positive if possible

Several windows specified in the book.

Hamming is good one, commonly used.

The causal form is
\[ w(n) = 0.54 - 0.46 \cos(2\pi n/M) \]

\[ 0 \leq n \leq M \]

The zero phase form for \(-(M/2) \leq n \leq M/2\) is

\[ w(n) = 0.54 - 0.46 \cos(2\pi n/M - \pi) \]

\[ = 0.54 + 0.46 \cos(2\pi n/M) \]
Types of Linear Phase Filters

\[ H(z) = \sum h(n) z^{-n} \]

Type I: M Even and \( h(n) = h(M-n) \)

\( N \) is odd.

\[ H(e^{j\omega}) = \left[ h(0) \left( 1 + e^{-j\omega M} \right) \right. \]
\[ + \left( h(1) \left( e^{-j\omega} + e^{-j\omega(M-1)} \right) \right) \ldots \]
\[ + h(M/2) \] \[ \left. e^{j\omega M/2} e^{-j\omega M/2} \right] \]

\[ \left[ h(M/2)+2\sum h(n) \cos(\omega((M/2) - n)) \right] e^{-j\omega M/2} \]

This is a delayed zero-phase filter, which is good for LP, HP, BP, and BR cases.
Type II: M Odd and \( h(n) = h(M-n) \)

\( N \) is even.

\[
H(e^{jw}) = [ h(0) \left( 1 + e^{-jwM} \right) \\
+ (h(1) \left( e^{-jw} + e^{-jw(M-1)} \right) \ldots ] e^{jwM/2} e^{-jwM/2} \\
[ 2 \sum h(n) \cos(w((M/2) - n)) ] e^{-jwM/2}
\]

\( w=0 : H(e^{jw}) = [ 2 \sum h(n) ] e^{-jwM/2} \)

\( w=\pi : H(e^{jw}) = 0 \), so \( H(z) \) has zero at \( z=-1 \)

Bad for HP and BR cases
Type III: M Even and $h(n) = -h(M-n)$

$N$ is odd. $h(M/2) = 0$

$$H(e^{jw}) = [ h(0) \left( 1 - e^{-jwM} \right)$$

$$+ (h(1) \left( e^{-jw} - e^{-jw(M-1)} \right) \ldots] e^{jwM/2} e^{-jwM/2}$$

$$[ 2j \sum h(n) \sin(w((M/2)-n)) ] e^{-jwM/2}$$

$w=0 : H(e^{jw}) = 0$

$w=\pi : H(e^{jw}) = 0$

$H(z)$ has zeroes at $z=1$ and $z=-1$

**Bad for LP, HP and BR cases**
Type IV : M Odd and \( h(n) = -h(M-n) \)

\( N \) is even.

\[
H(e^{jw}) = [ 2j \sum h(n) \sin(w((M/2) - n)) ] e^{-jwM/2}
\]

\( w=0 : H(e^{jw}) = 0 \)

\( w=\pi : H(e^{jw}) \) is not 0

\( H(z) \) has a zero at \( z=1 \)

**Bad for LP and BR cases**
Symmetry Determination in FIR filters

Given input sequences \( h(n) \) and \( x(n) \), which can be any of the four filter types, how do we determine the symmetry and type for the output \( y(n) \) when \( y(n) = h(n) \cdot x(n) \) or \( y(n) = h(n) \ast x(n) \)?

**Product Case**

For this case, assume that \( M \) is the same for \( h(n) \) and \( x(n) \). If \( h(n) \) and \( x(n) \) are both type I or both type III, then \( y(n) \) is respectively type I or III, etc.

If \( h(n) \) is type I and \( x(n) \) is type III, then \( y(n) \) is type III. The basic idea is:

1. Even times Even yields Even
2. Even times Odd yields Odd
3. Odd times Odd yields Even
4. \( M \) for \( y(n) \) equals \( M_1 = M_2 \).
Convolution Case

Because any two sequences can be convolved, \( h(n) \) and \( x(n) \) don’t need to have the same value of \( M \).

The basic idea is:
(1) Even convolved with Even yields Even
(2) Even convolved with Odd yields Odd
(3) Odd convolved with Odd yields Even
(4) \( M \) for \( y(n) \) equals \( M_1 + M_2 \).

Ex:
D. Advanced Topics

1. Gradient Approaches for Error Minimization

Uses
(a) Filter design in DSP
(b) Filter design in statistical signal processing
(c) Adaptive filtering
(d) Training of neural networks
(e) Solve ill-conditioned sets of linear equations

Basic Idea
Let $E(w)$ be an error function of the coefficient $w$. See sketch.
Given a value for \( w \) (not the correct value), the gradient of \( E(w) \) with respect to \( w \) is defined as

\[
g = \frac{\partial E(w)}{\partial w}
\]

In steepest descent, \( w \) is updated as

\[
w = w - B_2 \cdot g
\]

where \( B_2 \) is a small positive number. Given the current values of \( w \) and \( g \), the optimal \( B_2 \) is found by solving one equation in one unknown,

\[
\frac{\partial E(w - B_2 \cdot g)}{\partial B_2} = 0
\]

Then \( w \) is updated as before.

**Gradient Example, One Unknown**

\( x_p \) : pth example input vector of dimension \( N \) (\( N=1 \) for this example)
M : number of elements in an output vector (M=1 for this example)
p : row or example number from the training data
\( t_p \) : desired output vector for the pth example
\( y_p \) : pth actual output vector
\( N_v \) : Number of example patterns in the training data file

**Training Data**

Training data \( \{x_p, t_p\} \) consists of a file or matrix of dimensions \( N_v \) by \( (N+M) \), which contains example input and desired output vectors.
Training Data for $N_v = 7$

<table>
<thead>
<tr>
<th>p</th>
<th>$x_p$</th>
<th>$t_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.1</td>
<td>4.2</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>-2.1</td>
</tr>
<tr>
<td>3</td>
<td>4.3</td>
<td>8.4</td>
</tr>
<tr>
<td>4</td>
<td>3.5</td>
<td>7.2</td>
</tr>
<tr>
<td>5</td>
<td>-3.7</td>
<td>-7</td>
</tr>
<tr>
<td>6</td>
<td>8.2</td>
<td>16.1</td>
</tr>
<tr>
<td>7</td>
<td>-5.4</td>
<td>-11.1</td>
</tr>
</tbody>
</table>

Linear Model for the Output

$$y_p = w \cdot x_p$$

Error Function to be Minimized

$$E = \frac{I}{N_v} \sum_{p=1}^{N_v} [t_p - w \cdot x_p]^2$$
Gradient of E With Respect to w

\[ \frac{\partial E}{\partial w} = g = -2 \sum_{p=1}^{N_v} \left[ t_p - w \cdot x_p \right] \cdot x_p \]

\[ g = -2 \cdot \left[ \frac{1}{N_v} \sum_{p=1}^{N_v} t_p \cdot x_p - w \cdot \frac{1}{N_v} \sum_{p=1}^{N_v} x_p \cdot x_p \right] \]

\[ = -2 \cdot [c - w \cdot r] \]

Linear Equation Solution for w

Equating g to 0, we get

\[ c = w \cdot r \]

and

\[ w = c/r. \]
Gradient or Steepest Descent Solution, for $N=M=1$

Idea: Pick an initial value for $w$, a number of iterations $N_{it}$, and a small positive number $B_2$. Then repeatedly change it using the gradient.

$$w = 0$$
Calculate $c$ and $r$
Pick values for $N_{it}$, and $B_2$

For $1 \leq i_t \leq N_{it}$

Calculate $g$
Change $w$ as: $w = w - B_2 \cdot g$
Calculate $E$
Print $i_t$, $E$, $w$, $g$

End
Optimal Learning Factor for Steepest Descent, for N=M=1

Problem: The previous algorithm can take a long time to converge. Therefore we develop an optimal learning factor as follows.

\[ E(w - B_2 g) = \frac{I}{N_v} \sum_{p=1}^{N_v} [t_p - (w - B_2 g) \cdot x_p]^2 \]

\[ \frac{\partial E}{\partial B_2} = \frac{2}{N_v} \sum_{p=1}^{N_v} [t_p - (w - B_2 g) \cdot x_p] \cdot g \cdot x_p \]

\[ g[c - w \cdot r] = -B_2 g^2 r \]

\[ B_2 = \frac{-g \cdot [c - w \cdot r]}{g^2 r} = \frac{-[c - w \cdot r]}{g \cdot r} \]

We can test this learning factor as follows.

\[ w \leftarrow w - B_2 g = w + \frac{[c - w \cdot r]}{r} = \frac{c}{r} \]
Note that this yields the previous solution in one iteration.

**Gradient or Steepest Descent Solution**

**With Optimal Learning Factor, for N=M=1**

\[ w = 0 \]
Calculate c, r, and \( E_t \)

Pick a value for \( N_{it} \)

For \( 1 \leq i_t \leq N_{it} \)

Calculate g
Calculate \( B_2 \) from c, w, g, and r
Change w as: \( w = w - B_2 \cdot g \)
Calculate E
Print \( i_t, E, w, g \)

End
2. Gradient Algorithms for Many Unknowns

If $w$ is a vector of unknown coefficients, the $n$th element of the gradient vector $g$ is

$$g(n) = \frac{\partial E(w)}{\partial w(n)}$$

Now $w$ is updated as

$$w = w - B_2 \cdot g$$

Final Steepest Descent algorithm is then:

Initialize $w$ and pick $N_{it}$
For $i_t = 1$ to $N_{it}$
Calculate $g$
Calculate $B_2$ by solving

$$\frac{\partial E(w - B_2 \cdot g)}{\partial B_2} = 0$$
Update $w$ as $w = w - B_2 \cdot g$
End
Conjugate Gradient Algorithm

**Problem:** Steepest descent can require a very large number of iterations to converge to a good solution.

**Solution:** Use Conjugate gradient which requires $N_{it}$ on the order of the number of unknowns.

Initialize $w$ and pick $N_{it}$

Initialize the direction vector as $d = 0$

$X_d = 1$

For $i_t = 1$ to $N_{it}$

Calculate $g$ and its energy $X_n$

Calculate $B_1$ as $X_n/X_d$

Calculate $d$ as $-g + B_1 \cdot d$

Calculate $B_2$ by solving

$$\frac{\partial E(w + B_2 \cdot d)}{\partial B_2} = 0$$

Update $w$ as $w = w + B_2 \cdot d$

$X_d = X_n$

End
3. Application: Design of IIR Filters with Arbitrary Amplitude Response

Consider the error function,

$$E = \sum_{i=1}^{M} \left[ |H(e^{jw_i})| - |H_d(e^{jw_i})| \right]^2$$

where

$$H(z) = A \cdot G(z),$$

$$G(z) = \prod_{k=1}^{K} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}}$$

The frequencies $w_i$ do not have to be evenly spaced. As an example of a gradient element,
\[
\frac{\partial E}{\partial a_m} = 2 \sum_{i=1}^{M} \left[ | H(e^{jw_i}) | - | H_d(e^{jw_i}) | \right] \frac{\partial | H(e^{jw_i}) |}{\partial a_m},
\]

\[
\frac{\partial | H(e^{jw_i}) |}{\partial a_m} = \frac{\partial [H(e^{jw_i})H^*(e^{jw_i})]^{1/2}}{\partial a_m}
\]

\[
= | H(e^{jw_i}) | \text{Re}\left\{ \frac{e^{jw_i}}{1 + a_m e^{jw_i} + b_m e^{j2w_i}} \right\}
\]
4. Discrete Time Random Signals (1/23/2012)

Power versus Energy:

\[ E_x = x^2(n) \]

\[ P_x = \lim_{N \to \infty} \frac{1}{1+2N} x^2(n) \]

Many signals \( x(n) \) are non-periodic, unpredictable, infinite energy signals with finite power.

Transform techniques are convenient.
However special averages of $x(n)$ may have finite energy, just as $P_x$ can be finite even if $E_x$ is $\infty$.

a. Probability Densities and Distributions [1]

(1) Continuous Random Variables

$f_x(x)$ is a non-negative function such that

then $f_x(x)$ is a probability density function (pdf) of the random variable $X$.

$F_x(x)$, the probability distribution of $X$ is defined as

$F_x(x) = \int f_x(u) \, du$,

also monotonically non-decreasing
\[ f_x(x) = \frac{F_x(x)}{x}, \]

Note: \( X \) is a random variable and \( x \) denotes a value of \( X \).

(2) Discrete Random Variables

Probability Mass Function:

Here \( f_x(x) = P(X = x) \) where \( X \) has a countably infinite set of possible values, which may be integer, rational, or real numbers. Denote possible values of \( X \) as \( x(k) \).

\[ f_x(x(k)) = 1 \]

\[ F_x(x) = f_x(x(n)) \]
(3) Relationships Between Two Random Variables:

Joint Probability Distribution:

\[ F_{XY}(x, y) = P(X \leq x, Y \leq y) \]

\[ f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \]

For discrete random variable case,

\[ f_{XY}(x, y) = P(X = x, Y = y) \]

\( X \) and \( Y \) are **statistically independent** if

\[ F_{XY}(x, y) = F_X(x) F_Y(y), \]

or distribution of \( X \) and \( Y \) is distribution of \( X \bullet \) distribution of \( Y \). Alternately,

\[ f_{XY}(x, y) = f_X(x) f_Y(y) \]
(4) Stationarity

Replace X and Y by $X_n$ and $X_m$, where $n$ and $m$ denote times $n$ and $m$. Then the distribution is $F_{X_nX_m}(x_n, x_m)$. The stationarity of the random process $\{X_n\}$ implies:

$$F_{X_nX_m}(x, y) = F_{X_{n+k}X_{m+k}}(x, y)$$

for all $n, m, k$. In other words, joint distribution is shift invariant. Generally $X_n$ is stationary iff:

$$F_{X_{n1}X_{n2} \ldots X_{nM}}(x_1, x_2, \ldots, x_M) = F_{X_{n1+k}X_{n2+k} \ldots X_{nM+k}}(x_1, x_2, \ldots, x_M)$$

For all values of $M>0$ and $k$.
(5) Additional Properties [2]

(a) If RVs U, V, and X... are statistically independent, and \( Z = U + V + X... \), then
\[
f_Z(z) = f_U(z) \ast f_V(z) \ast f_X(z)... (* = convolution)
\]

(b) The Law of Large Numbers:
\[
\text{As } n \rightarrow \infty, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k = E[X]
\]

where the expected value is defined as
\[
E[X] \equiv \int f_X(x)dx
\]

(c) Central Limit Theorem (CLT): Let \( X_1, X_2, X_3, \ldots, X_n \) be a sequence of \( n \) (iid) random variables each having finite mean \( m_x \) and variance \( \sigma^2 \). If
\[
S_n = \sum_{k=1}^{n} X_k, \quad Z_n = \frac{S_n - n \cdot m_x}{\sigma \sqrt{n}}
\]
then \( f_{Z_n}(z_n) \) will converge in distribution to the standard normal \( N(0,1) \) as \( n \) approaches infinity.
b. Averages:

Let \{X_n\} be a random process, \(X_n\) is a random variable for each \(n\), \{X_n\} is a random process.

The **ensemble of** \(X_n\) would be a 2-D array as:
Vertical scale denotes all possible sequences for $x_n$.

Horizontal scale denotes all possible time values for $n$.

We can average in the ensemble (vertical)

$$m_{X_n} = E[x_n] = \int x \, f_{X_n}(x) \, dx$$

where $E[]$ denotes expected value. Let $g(x_n)$ be a function of the random variable $x_n$.

$$E[g(x_n)] = \int g(x) \, f_{X_n}(x) \, dx$$

**Discrete Case:**

$$m_{X_n} = E[x_n] = \sum x f_{X_n}(x)$$

$$E[g(x_n)] = \sum g(x) \, f_{X_n}(x)$$
For stationary process $x_n$, $f_{Xn}(x) = f_x(x)$, i.e., not a function of $n$.

Example Calculations of Averages:

**Ex:** Nonstationary process example.

1. $f_{Xn}(x)$ (a) Find $E[x_1] = \quad$
2. (b) Find $E[x_5] = \quad$
3. (c) Find $E[x_n] = \quad$
Ex: $f_{X_n}(x) = 1 - |x|, \, |x| \leq 1.$
Find $E[x_n]$ and $E[x_n^2]$

$E[x_n] = (1 - |x|)x \, dx$

$= (1-x)x \, dx + (1+x)x \, dx$

$E[x_n^2] = (1 - |x|)x^2 \, dx$
**Statistical Independence:**

Two random variables $X_n$ and $Y_m$ are statistically independent iff

$$f_{X_nY_m}(x, y) = f_{X_n}(x) f_{Y_m}(y)$$

for continuous and discrete random variables.

**Linear Independence:**

Two random variables are linearly independent or uncorrelated if

$$E[x_n y_m] = E[x_n] E[y_m]$$

**Ex:** Let $x_n$ and $y_m$ be statistically independent. Show that they are also linearly independent.
\[ E[x_n y_m] = \int xy f_{XnYm}(x, y) \, dx \, dy \]

\[ f_{Xn}(x) \, f_{Ym}(y) \]

**Ex: Prove**

(a) \( E[x_n + y_m] = E[x_n] + E[y_m] \)

(b) \( E[a x_n] = a E[x_n] \)
In general, let \( f(n) \) be a function of \( n \), but not a random function, i.e.,
\[ f(n) = 1 + n + 3n^2, \]

\[ E[f(n) \ g(x_n)] = f(n) \ E[g(x_n)] \]

\textbf{Ex:} \quad E[n \ x_n] = n \ E[x_n] = n \ m_{X_n}
\[ E[n^3 x_n^2] = n^3 E[x_n^2] \]

\( f(n) \) is deterministic, not random.

\[ E[f(n)] = f(n), \text{ i.e., each member of } f(n) \text{ ensemble is identical.} \]

\textbf{Properties of Expected Value}

(1) \[ E[x_n + y_m] = E[x_n] + E[y_m] \]
(2) \[ E[ax_n] = aE[x_n] \]
(3) \[ E[f(n) \ g(x_n)] = f(n) \ E[g(x_n)] \]
(4) \[ E[f(n)] = f(n) \]
(5) \[ E[E[g(x_n)]] = E[g(x_n)] \]
Variance Mean-Square and Autocorrelation of $x_n$:

Mean Square of R.V. $x$ is

$$E[x^2] = \int x^2 f_x(x) dx$$

For $x_n$, $E[x_n^2] = \int x^2 f_{X_n}(x) dx$

Variance = $E[(x_n - m_{X_n})^2]$ 

In general case, they are functions of parameter $n$, but for stationary process, they are numbers.
**Autocorrelation:**

\[ r_{xx}(n, m) = E[x_n x_m^*] \]

\[ = \int xy f_{x_n x_m}(x, y) \, dx \, dy \]

**Autocovariance:**

\[ c_{xx}(n, m) = E[(x_n - m_{Xn}) (x_m - m_{Xm})^*] \]

\[ = r_{xx}(n, m) - m_{Xn} m_{Xm} \]

**Crosscorrelation and Crosscovariance:**

\[ r_{XY}(n, m) = E[x_n y_m^*] \]

\[ c_{XY}(n, m) = E[(x_n - m_{Xn}) (y_m - m_{Ym})^*] \]

(1) Note that means are subtracted out for the covariances, forcing the quantities to be zero mean: \((x_n - m_{Xn}) = z_n\) is zero mean.
(2) Also time domain quantities will usually be real, so conjugate operation \((\cdot)^*\) will have no effect.

(3) Random variables we look at will usually be zero mean, so that correlation and covariance will be equal.

(4) Random processes we look at will often be stationary, so:

\[
\begin{align*}
  f_{X_n}(x, n) &= f_X(x) \\
  f_{X_nY_m}(x, y) &= f_{X_{n+k}Y_{m+k}}(x, y) \\
  r_{XX}(n, m) &= r_{XX}(m-n) \\
  r_{XX}(n, n+m) &= r_{XX}(m) = E[x_nx_{n+m}^*]
\end{align*}
\]
Wide Sense Stationary

A random process $x_n$ is wide sense stationary (WSS) if $\mathbb{E}[x_n]$ is not a function of $n$ and $r_{xx}(n,m) = r_{xx}(m-n)$, i.e., $m_{X_n}$ is constant and $r_{xx}$ is only a function of $n-m$.

Ex. $x_n = X \cos(w \cdot n + \phi)$, where $X$ and $\phi$ are statistically independent. $X$ is zero-mean and $\phi$ is uniformly distributed (has a uniform pdf) between $-\pi$ and $\pi$. Show that $x_n$ is WSS.
White Noise

\[ r_{xx}(m) = \delta(m)\sigma^2 \quad \text{(or)} \]
\[ r_{xx}(n-m) = \delta(n-m)\sigma^2 \]

1, if \( m = 0 \)

\[ \delta(m) = \]

0, otherwise

(Kronecker delta function)