1. We want to use the $M_1$ spatial domain ring windows $h_{rk}(r,\theta)$ and the $M_2$ spatial domain wedge windows $h_{wk}(r,\theta)$ to calculate new features, where $f(r,\theta)$ in the inner product is replaced by $|F(R,\varphi)|^2$, where $F(R,\varphi)$ is the Fourier transform of $f(r,\theta)$ expressed in polar coordinates.

(a) If $M_2$ is even, how many unique frequency domain wedge features do we have?
(b) Give a post-processing method for introducing rotation invariance to these wedge features.
(c) What is the minimum required range of integration for the frequency domain ring features?

2. Let $FS\{f_d(x,y)\}$ denote a deformation-invariant feature set calculated from the image $f_d(x,y)$. Deformation types include scaling, rotation, shifting, oblique, and amplitude. Amplitude invariant features, for example, satisfy $FS\{a \cdot f(x,y)\} = FS\{f(x,y)\}$ for $a > 0$. We want to construct a proof of the conjecture: All deformation-invariant feature set operators, $FS$, are nonlinear. To prove that a given deformation-invariant operator is nonlinear, we must show that $LHS \neq RHS$ where

$$LHS \equiv FS\{ a \cdot f(x, y) + b \cdot f_d(x, y) \},$$

$$RHS \equiv a \cdot FS\{ f(x, y) \} + b \cdot FS\{ f_d(x, y) \}$$

(LHS means the right-hand-side and RHS denotes the right-hand-side).

(a) What values of $a$ and $b$ should be chosen for this proof?
(b) What happens to LHS and RHS for deformation-invariant feature sets?
(c) For which types of deformation (amplitude, scale, rotation, shift, oblique) are the corresponding deformation-invariant feature set operators nonlinear?

3. Our goal here is to evaluate the rotation, scale, and shift invariant moments, $\varphi_1$ and $\varphi_2$, where the input image $f_d(x,y)$ is $f(x,y)$ rotated by an angle $\theta_1$. Assume that there is no shift or scaling. If $c$ denotes $\cos(\theta_1)$ and $s$ denotes $\sin(\theta_1)$, we know that

$$n_{pq} = \iint (c \cdot u - s \cdot v)^p (s \cdot u + c \cdot v)^q f(u,v)du \cdot dv$$

Let $m'_{pq}$ denote moments found from the undeformed image $f(x,y)$.

(a) Find $n_{20}$, $n_{02}$, and $n_{11}$ in terms of the appropriate $m'_{pq}$.
(b) Find $\varphi_1$ ($n_{20} + n_{02}$) in terms of the appropriate $m'_{pq}$.
(c) Find $\varphi_2$ ($n_{20}^2 - n_{02}^2 + 4n_{21}^2$) in terms of the appropriate $m'_{pq}$. 
4. For an arbitrary deformation and feature set, assume that the kth feature is modelled as

\[ g_{di}(k) = g_i(k) \bullet d(k) + e(k) \]

where i is a class number between 1 and \( N_c \), d(k) is a function of the random deformation parameter s (s could be a random shift or random rotation angle for example), and where e(k) is zero-mean random noise which is independent of d(k). E[d(k)] = m(k), var(d(k)) = \( \sigma^2_d(k) \), and var(e(k)) = \( \sigma^2(k) \). These quantities are not functions of i. \( g_i(k) \) is deterministic and different for each class. \( P_i = 1/N_c \).

(a) Evaluate \( E[g_{di}(k)] \) and \( g_{di}(k) \). \( g_{di}(k) \) is the kth element of \( g_{do} \), which is the average deformed feature vector taken over all classes.

(b) Evaluate \( s_{dw}(k,k) \), the kth diagonal element of the deformed within-class scatter matrix \( S_{dw} \), in terms of \( g_i(k) \), m(k), \( \sigma^2_d(k) \), and \( \sigma^2(k) \).

(c) Evaluate \( s_{db}(k,k) \), the kth diagonal element of the deformed between-class scatter matrix \( S_{db} \), in terms of \( g_i(k) \) and m(k).

(d) Give \( J_4 \) for the deformed feature case.

5. Traditional Law’s features are defined as

\[ g_{ij}(m,n) = \frac{1}{(1+2L)^2} \sum_{k=-L}^{L} \sum_{p=-L}^{L} (Y_{ij}(k,p))^2 \]

where \( Y_{ij}(z_1,z_2) = H_{ij}(z_1,z_2)F(z_1,z_2) \) and \( H_{ij}(z_1,z_2) \) is one of twentyfive separable filters formed from the 1-D component filters:

- \( L(z) = z^2 + 4 \cdot z + 6 + 4 \cdot z^{-1} + z^{-2} \)
- \( E(z) = -z^2 + -2 \cdot z + 2 \cdot z^{-1} + z^{-2} \)
- \( S(z) = -z^2 + 2 - z^{-2} \)
- \( W(z) = -z^2 + 2 \cdot z + -2 \cdot z^{-1} + z^{-2} \)
- \( R(z) = z^2 - 4 \cdot z + 6 - 4 \cdot z^{-1} + z^{-2} \)

For example, \( H_{24}(z_1,z_2) = E(z_1)W(z_2) \)

(a) Are the Law’s features rotation, scale, or shift invariant? (yes or no)

(b) Are the Law’s features invariant to the signs of the individual filter coefficients in L(z), E(z), etc?

(c) Assume that we want to construct new Law’s features and filters. Are the filters \( -H_{ij}(z_1,z_2) \) useful?

(d) Replacing z by \( z_1 \) or \( z_2 \) in the component filters, we get ten new, useful Law’s filters \( H_{ij}(z_1,z_2) \) as \( E(z_1), E(z_2), L(z_1), L(z_2) \) etc. Replacing z in the component filters by functions of both \( z_1 \) and \( z_2 \), give ten additional new Law’s filters.

(Hint: do not raise \( z_1 \) or \( z_2 \) to a power greater than 1 or less than -1.)
6. The first equation in problem 5 can be approximated as
\[ g_{ij}(m,n) = E[(y_{ij}(m,n))^2] = r_{y_{ij}y_{ij}}(0,0) \]
where \( r_{y_{ij}y_{ij}}(0,0) \) is the autocorrelation of \( y_{ij}(m,n) \) at shift 0 in both the \( m \) and \( n \) directions. In other words, a Laws feature is the local autocorrelation of a filtered image, at shift 0. In general the autocorrelation of the filtered output \( y_{ij}(m,n) \) is
\[ r_{y_{ij}y_{ij}}(m,n) = v_{ij}(m,n) * r_{ff}(m,n) \]
where \( r_{ff}(m,n) \) is the local autocorrelation of \( f(m,n) \) and \( v_{ij}(m,n) \) is the finite energy autocorrelation (FEA) of \( h_{ij}(m,n) \), defined as
\[ v_{ij}(m,n) = h_{ij}(m,n) * h_{ij}(-m,-n) \].

(a) Give an exact equation for \( v_{ij}(m,n) \) in terms of \( h_{ij}(m,n) \), keeping in mind that the component filters have only 5 coefficients.

(b) How many nonzero coefficients can \( v_{ij}(m,n) \) have?

(c) If the FEA \( v_{ij}(m,n) \) satisfies \( v_{ij}(m,n) = v_{ij}(-m,-n) \), how many unique samples \( M \) does it have?

(d) Express the feature \( r_{y_{ij}y_{ij}}(0,0) \) as a double summation in terms of \( v_{ij}(m,n) \) and \( r_{ff}(m,n) \). Note that this is one equation in \( M \) unknowns (the uniques samples of \( r_{ff}(m,n) \)) (Hint: First express \( r_{y_{ij}y_{ij}}(m,n) = v_{ij}(m,n) * r_{ff}(m,n) \) using a double summation)

(e) Keeping in mind your results from parts (c) and (d), the Laws feature vector \( g \) can be expressed as
\[ g = V \cdot r \]

where the Laws feature vector \( g \) is \( N_F \) by 1, \( V \) is \( N_F \) by \( M \), and \( r \) is \( M \) by 1. Here, \( N_F \) is the number of Laws features, which could be 25 or more. How many Laws features would be needed to reconstruct the local input autocorrelation \( r_{ff}(m,n) \)? For partial credit, give a symbol for the required value of \( N_F \). For full credit, give a numerical value for \( N_F \).
Fourier descriptor features calculated from contour points \((x(v), y(v))\) are

\[
a(n) = \frac{i}{L} \int_0^L u(v) \cdot e^{i \frac{2\pi}{L} n v} \, dv
\]

\[
u(v) = x(v) + j \cdot y(v)
\]

The complex Zernike features are calculated as

\[
v_{nm} = \int_0^{2\pi} \int_0^r f(r, \theta) \tilde{h}_{nm}(r, \theta) \cdot r \cdot d\theta dr, \quad |m| \leq n,
\]

\[
h_{nm}(r, \theta) = R_{nm}(r) \cdot e^{i \theta \cdot n \cdot \theta}
\]

The 2-D DFT of \(F_d(i, j)\) is

\[
F_d(i, j) = \sum_{m=-N}^{N-1} \sum_{n=-N}^{N-1} f_d(m, n) W_N^{im} W_N^{jn}
\]

\[
W_N = \exp(-j2\pi / N)
\]

If \(C_{di}\) denotes the covariance matrix for deformed features from the ith class, then the within-class scatter matrix is

\[
S_w = \frac{1}{N_c} \sum_{i=1}^{N_c} C_{di}
\]

If \(g_o\) denotes the mean feature vector for all classes, the between-class scatter matrix is

\[
S_b = \frac{1}{N_c} \sum_{i=1}^{N_c} (E[g_{di}] - g_o)(E[g_{di}] - g_o)^T
\]

Now,

\[
J_4 = \frac{\text{tr}(S_b)}{\text{tr}(S_w)}
\]