

MATH 2425
Calculus II
Solutions to Exam 1

Problem 1. Evaluate the integral $\int_0^{\pi/2} 5\sqrt{\sin \nu} \cos \nu \, d\nu$.

Solution: Let $u = \sin \nu$. Then $du = \cos \nu \, d\nu$, and

$$\int_0^{\pi/2} 5\sqrt{\sin \nu} \cos \nu \, d\nu = \int_0^1 5\sqrt{u} \, du = 5 \cdot \frac{1}{\frac{1}{2} + 1} u^{\frac{1}{2} + 1} \Big|_0^1 = \frac{10}{3} u^{3/2} \Big|_0^1 = \frac{10}{3}.$$

Problem 2. Evaluate the integral $\int_2^4 \frac{2 \, dx}{x^2 - 6x + 10}$ by completing the square.

Solution: First notice $x^2 - 6x + 10 = x^2 - 6x + 3^2 - 3^2 + 10 = (x - 3)^2 + 1$ and then perform substitution $u = x - 3$ to get

$$\int_2^4 \frac{2 \, dx}{x^2 - 6x + 10} = \int_2^4 \frac{2 \, dx}{(x - 3)^2 + 1} = \int_{-1}^1 \frac{2 \, du}{u^2 + 1} = 2 \arctan u \Big|_{-1}^1 = \pi.$$

Problem 3. Find the area between the x -axis and the curve $y = \frac{1}{\sqrt{1 - x^2}}$ for $\frac{1}{2} \leq x \leq \frac{\sqrt{3}}{2}$.

Solution: The area is

$$\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1 - x^2}} = \arcsin x \Big|_{1/2}^{\sqrt{3}/2} = \frac{\pi}{6}.$$

Problem 4. Evaluate the integral $\int_0^1 x e^{-x} \, dx$.

Solution: Use integration by part: $u = x$ and $dv = e^{-x} \, dx$ (and thus $v = -e^{-x}$) to get

$$\int_0^1 x e^{-x} \, dx = x[-e^{-x}] \Big|_0^1 - \int_0^1 [-e^{-x}] \, dx = -e^{-1} - e^{-x} \Big|_0^1 = 1 - 2e^{-1}.$$

Problem 5. The partial-fraction expansion of $\frac{3x^2 - x + 7}{(x^2 - 2x + 1)(x^2 - 1)}$ is (where $A, B, C,$ and D are constants):

Solution:

$$\frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3},$$

since

- $(x^2 - 2x + 1)(x^2 - 1) = (x - 1)^2(x + 1)(x - 1) = (x - 1)^3(x + 1)$;
- there is no common factors in the numerator and the denominator;

- the degree of the numerator is less than the degree of the denominator.

Problem 6. Evaluate the integral $\int_0^{\pi/6} \sin^2 3\theta \cos^3 3\theta d\theta$.

Solution: We have

$$\begin{aligned} \int_0^{\pi/6} \sin^2 3\theta \cos^3 3\theta d\theta &= \int_0^{\pi/2} \sin^2 u \cos^3 u \frac{1}{3} du \quad (\text{substitution: } u = 3\theta, du = 3d\theta) \\ &= \frac{1}{3} \int_0^{\pi/2} \sin^2 u \cos^2 u \cos u du \\ &= \frac{1}{3} \int_0^1 v^2(1-v^2)dv \quad (\text{substitution: } v = \sin u, dv = \cos u du) \\ &= \frac{1}{3} \left(\frac{1}{3}v^3 - \frac{1}{5}v^5 \right) \Big|_0^1 \\ &= \frac{2}{45}. \end{aligned}$$

Problem 7. Evaluate the integral $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 t} dt$.

Solution: For $\pi/2 \leq t \leq \pi$, $\sqrt{1 - \sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = -\cos t$. So

$$\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 t} dt = \int_{\pi/2}^{\pi} (-\cos t) dt = (-\sin t) \Big|_{\pi/2}^{\pi} = 1.$$

Problem 8. Which is the most appropriate substitution to use in evaluating the integral $\int \frac{dx}{\sqrt{x^2 - 1}}$.

Solution: $x = \sec \theta$.

Problem 9. Use Simpson's Rule with $n = 4$ to estimate $\int_1^2 \frac{1}{x} dx$.

Solution: $a = 1$, $b = 2$, $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1}{4}$, $x_i = a + i(\Delta x)$ for $i = 0, 1, 2, 3, 4$, and $y_i = f(x_i) = \frac{1}{x_i}$:

i	x_i	y_i
0	1	1
1	5/4	4/5
2	3/2	2/3
3	7/4	4/7
4	2	1/2

The estimate by Simpson's Rule is

$$\frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{1}{12} \left(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2} \right).$$

Problem 10. The improper integral $\int_0^{\infty} \frac{1}{(x+1)^p} dx$

Solution: converges if $p > 1$ and diverges if $p \leq 1$.

Problem 11. Evaluate the integral $\int e^x \sin x dx$.

Solution: See Example 5 on Page 548 of the textbook.

Problem 12. Evaluate the integral $\int_{-1}^1 \frac{1}{(x+1)^{3/4}} dx$.

Solution: This is a type II improper integral. We have

$$\begin{aligned} \int_{-1}^1 \frac{1}{(x+1)^{3/4}} dx &= \lim_{A \rightarrow -1^+} \int_A^1 \frac{1}{(x+1)^{3/4}} dx \\ &= \lim_{A \rightarrow -1^+} \left. \frac{1}{-\frac{3}{4} + 1} (x+1)^{-\frac{3}{4} + 1} \right|_A^1 \\ &= \lim_{A \rightarrow -1^+} 4 \left[2^{1/4} - (A+1)^{1/4} \right] \\ &= 4 \cdot 2^{1/4}. \end{aligned}$$

Problem 13. Use the Trapezoidal Rule with $n = 4$ to estimate $\int_0^4 x^4 dx$.

Solution: $a = 0$, $b = 4$, $n = 4$, $\Delta x = \frac{b-a}{n} = 1$, $x_i = a + i(\Delta x)$ for $i = 0, 1, 2, 3, 4$, and $y_i = f(x_i) = x_i^4$:

i	x_i	$y_i = x_i^4$
0	0	0^4
1	1	1^4
2	2	$2^4 = 16$
3	3	$3^4 = 81$
4	4	$4^4 = 256$

The estimate by the Trapezoidal Rule is

$$\frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = \frac{1}{2} (0 + 2 + 32 + 162 + 256) = 226.$$

Problem 14. Evaluate the integral $\int \frac{x^2 + x}{(x-1)(x^2+1)} dx$.

Solution: Use its partial fractional expansion. Notice

- there is no common factors in the numerator and the denominator;
- the degree of the numerator is less than the degree of the denominator.

We have

$$\frac{x^2 + x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+1}$$

for some constant A , B , and D . Multiply both sides by $(x-1)(x^2+1)$ to get

$$x^2 + x = A(x^2 + 1) + (x-1)(Bx + D).$$

Set $x = 1$ to obtain immediately $A = 1$. Therefore

$$x^2 + x = x^2 + 1 + Bx^2 + (D - B)x - D = (B + 1)x^2 + (D - B)x + (1 - D)$$

which yields

$$1 = B + 1, 1 = D - B, 0 = 1 - D.$$

Solve these equations to get $B = 0$, $D = 1$. Therefore

$$\frac{x^2 + x}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{1}{x^2+1},$$

and

$$\int \frac{x^2 + x}{(x-1)(x^2+1)} dx = \int \frac{1}{x-1} dx + \int \frac{1}{x^2+1} dx = \ln|x-1| + \arctan x + C.$$

Problem 15. Evaluate the integral $\int \frac{x^2}{\sqrt{1-x^2}} dx$.

SOLUTION 1: Perform substitution $x = \sin \theta$ and then $\sqrt{1-x^2} = \cos \theta$ and $dx = \cos \theta d\theta$ to get

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-x^2}} dx &= \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta = \int \sin^2 \theta d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta + C = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta + C \\ &= \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C. \end{aligned}$$

SOLUTION 2: Perform substitution $x = \cos \theta$ and then $\sqrt{1-x^2} = \sin \theta$ and $dx = -\sin \theta d\theta$ to get

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-x^2}} dx &= -\int \frac{\cos^2 \theta}{\sin \theta} \sin \theta d\theta = -\int \cos^2 \theta d\theta = -\int \frac{1 + \cos 2\theta}{2} d\theta \\ &= -\frac{1}{2}\theta - \frac{1}{4} \sin 2\theta + C = -\frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta + C \\ &= -\frac{1}{2} \arccos x - \frac{1}{2} x \sqrt{1-x^2} + C. \end{aligned}$$