MATH 2425, Calculus II
Lab 2: L'Hôpital's Rule
Week of 31 August 2009

Study Technique 2. *While working through homework, it is very tempting to use a solutions manual. A solutions manual can be very helpful and very effective, but only if used correctly. The correct way to use a solutions manual is to first attempt a problem. Only when you get stuck, read the first one or two lines of the solution, and then attempt the problem again with the solutions manual closed. If this doesn't help, then read the solution all the way through, but then close the manual and try to reproduce the solution (or at least the main ideas of the solution) without the solutions manual.*

L'Hôpital's Rule helps us to evaluate limits of quotients. One of the results you learned early in Calculus I was

Limit Rule. *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *where* L, M *are finite numbers, and* $M \neq 0$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

With L'ôpital's Rule, we can find limits of **indeterminate forms:**

L'Hôpital's Rule.

(1) *If* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is called an* **indeterminate form of type** $\mathbf{\frac{0}{0}}$. *(Write "type* $\frac{0}{0}$*" when showing work.)*

(2) *If* $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is called an* **indeterminate form of type** $\frac{\infty}{\infty}$. *(Write "type* $\frac{\infty}{\infty}$*" when showing work.)*

In both of the above cases, if

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = K,$$

where K *is a finite number,* $+\infty$, *or* $-\infty$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = K.$$

Example I. We already know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Let's verify this using l'Hôpital's Rule.

(1) Check to see if we have an indeterminate form: $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x = 0$, so $\lim_{x\to 0} \frac{\sin x}{x}$ is indeterminate of type $\frac{0}{0}$.
(2) Differentiate: $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx} x = 1$.
(3) Evaluate the limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1} = \frac{1}{1} = 1.$$

Problem 1. Use l'Hôpital's Rule to find $\lim_{x\to 0} \frac{\cos x + 2x - 1}{3x}$.

Problem 2. Use l'Hôpital's Rule TWICE to find $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$.

Problem 3. We cannot use l'Hôpital's Rule to evaluate $\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2}$. WHY NOT??

Trying to evaluate $\lim_{x\to a} f(x)^{g(x)}$ often results in indeterminate forms of the type 0^0, ∞^0, or 1^∞. In order to use l'Hôpital's Rule to evaluate such limits, we must convert them to be of type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$.

Example II. Evaluate $\lim_{x\to 0^+} x^x$.

This limit is indeterminate of type 0^0. To use l'Hôpital's Rule, we proceed as follows.

(1) Set $\lim_{x\to 0^+} x^x = L$.

(2) Take ln of both sides. Because ln is continuous, we can move the limit inside.

$$\ln(\lim_{x\to 0+} x^x) = \lim_{x\to 0+} \ln(x^x) = \ln L.$$

(3) Use logarithm rules to write $\ln(x^x) = x\ln(x) = \frac{\ln x}{\frac{1}{x}}$.

(4) Now $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ is indeterminate of type $\frac{\infty}{\infty}$.

(5) We compute

$$\ln L = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0+} -x = 0.$$

(6) Thus $\ln L = 0$ and $L = 1$.

Problem 4. Use l'Hôpital's Rule to find $\lim_{x\to\infty}(\ln x)^{1/x}$.

Problem 5. Use l'Hôpital's Rule to find $\lim_{x \to 0^+} (e^x + x)^{1/x}$.