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THE p-RELATIVE DISTANCE IS A METRIC*

ANDERS BARRLUND[†]

Abstract. The conjecture that the p-relative distance, $\varrho_p(\alpha,\tilde\alpha) = |\alpha-\tilde\alpha|/\sqrt[p]{|\alpha|^p+|\tilde\alpha|^p}$, is a metric is proved.

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1. Introduction.

The *p-relative distance* between α and $\tilde\alpha \in \mathbb{C}$ is defined as

$$\varrho_p(\alpha,\tilde\alpha) = \frac{|\alpha-\tilde\alpha|}{\sqrt[p]{|\alpha|^p+|\tilde\alpha|^p}} \quad \text{for } 1 \le p \le \infty. \tag{1.1}$$

For convenience, we define $0/0 := 0$.

Li [2] presented the conjecture that the p-relative distance is a metric. It is trivial that

$$\begin{aligned}&\varrho_p(\alpha,\beta) \ge 0,\\ &\varrho_p(\alpha,\beta) = 0 \quad \text{if and only if } \alpha=\beta,\\ &\varrho_p(\alpha,\beta) = \varrho_p(\beta,\alpha).\end{aligned}$$

However, it has been an open question whether

$$\varrho_p(\alpha,\beta) \le \varrho_p(\alpha,\gamma) + \varrho_p(\gamma,\beta). \tag{1.2}$$

Li [2] proved (1.2) for $\alpha,\beta,\gamma \in \mathbb{R}$. Day [1] proved (1.2) for $p=\infty$. In this paper we prove (1.2) for $\alpha,\beta,\gamma\in\mathbb{C}$.

The paper is outlined as follows: In section 2 we prove a simple lemma which is used to prove (1.2) in section 3. We will let $\|x\|_2$ denote the Euclidean vector norm.

2. A simple lemma.

LEMMA 1. *Let* $1 \le p \le \infty$. *If* $x \ge 0$ *and* $y > 0$, *then*

$$\sqrt[p]{\frac{1+x^p}{1+y^p}} \ge \min(x, 1/y). \tag{2.1}$$

If $x \ge y \ge 1$, *then also*

$$\sqrt[p]{\frac{1+x^p}{1+y^p}} > \sqrt{x/y}. \tag{2.2}$$

Proof. If $x \le 1/y$, then $x^py^p \le 1$, which implies that $1+x^p \ge x^py^p + x^p = (1+y^p)x^p$, from which (2.1) follows. If $x > 1/y$, then $x^py^p > 1$, which implies that $1+y^p < x^py^p + y^p = (1+x^p)y^p$, from which (2.1) follows.

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[†] Department of Computing Science, University of Umeå, S-90187 Umeå, Sweden (abrr@cs.umu.se).

If $u \geq v \geq 1$, then $(uv-1)(u-v) \geq 0$, which implies that $v + u^2 v \geq u + uv^2$ and

$$\frac{1+u^2}{1+v^2} \geq \frac{u}{v}.$$

This inequality with $u = x^{p/2}$ and $v = y^{p/2}$ gives (2.2). ∎

3. The proof.

THEOREM 1. *Let* $\alpha, \beta, \gamma \in \mathbb{C}$ *and* $1 \leq p \leq \infty$. *Then*

$$\varrho_p(\alpha, \beta) \leq \varrho_p(\alpha, \gamma) + \varrho_p(\gamma, \beta), \tag{3.1}$$

where ϱ_p *is defined by* (1.1).

Proof. Without loss of generality, we prove (3.1) in the cases with $|\alpha| \leq |\beta|$. The cases with $|\beta| < |\alpha|$ are proved by swapping α and β in the rest of the proof.

In cases with $|\gamma| \leq |\alpha| \leq |\beta|$, we get

$$\begin{aligned} \varrho_p(\alpha, \beta) = \frac{|\alpha - \beta|}{\sqrt[p]{|\alpha|^p + |\beta|^p}} &\leq \frac{|\alpha - \gamma|}{\sqrt[p]{|\alpha|^p + |\beta|^p}} + \frac{|\gamma - \beta|}{\sqrt[p]{|\alpha|^p + |\beta|^p}} \\ &\leq \frac{|\alpha - \gamma|}{\sqrt[p]{|\alpha|^p + |\gamma|^p}} + \frac{|\gamma - \beta|}{\sqrt[p]{|\gamma|^p + |\beta|^p}} = \varrho_p(\alpha, \gamma) + \varrho_p(\gamma, \beta). \end{aligned}$$

In cases with $\alpha = 0$, the inequality (3.1) is trivial.

Next we consider cases with $0 < |\alpha| \leq |\beta| \leq |\gamma|$. In these cases (2.1) gives

$$\frac{\sqrt[p]{|\alpha|^p + |\beta|^p}}{\sqrt[p]{|\alpha|^p + |\gamma|^p}} = \left|\frac{\beta}{\alpha}\right| \frac{\sqrt[p]{1 + |\alpha/\beta|^p}}{\sqrt[p]{1 + |\gamma/\alpha|^p}} \geq \left|\frac{\beta}{\alpha}\right| \min(|\alpha/\beta|, |\alpha/\gamma|) = \left|\frac{\beta}{\gamma}\right|, \tag{3.2}$$

$$\frac{\sqrt[p]{|\alpha|^p + |\beta|^p}}{\sqrt[p]{|\beta|^p + |\gamma|^p}} = \left|\frac{\alpha}{\beta}\right| \frac{\sqrt[p]{1 + |\beta/\alpha|^p}}{\sqrt[p]{1 + |\gamma/\beta|^p}} \geq \left|\frac{\alpha}{\beta}\right| \min(|\beta/\alpha|, |\beta/\gamma|) = \left|\frac{\alpha}{\gamma}\right|.$$

We also have

$$\left|\frac{\beta}{\gamma}\right| |\alpha - \gamma| + \left|\frac{\alpha}{\gamma}\right| |\beta - \gamma| = \left|\frac{\beta\alpha}{\gamma} - \beta\right| + \left|\frac{\beta\alpha}{\gamma} - \alpha\right| \geq |\alpha - \beta|. \tag{3.3}$$

Combining (3.2) and (3.3) gives (3.1).

Finally, we consider cases with $0 < |\alpha| < |\gamma| < |\beta|$. Let $r_1 e^{i\theta_1} = \beta/\alpha$ and $r_2 e^{i\theta_2} = \gamma/\alpha$ be the polar decompositions of β/α and γ/α, respectively. Then (3.1) can be rewritten as

$$\frac{\sqrt{1 + r_1^2 - 2r_1 \cos\theta_1}}{\sqrt[p]{1 + r_1^p}} \leq \frac{\sqrt{1 + r_2^2 - 2r_2 \cos\theta_2}}{\sqrt[p]{1 + r_2^p}} + \frac{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}}{\sqrt[p]{r_1^p + r_2^p}}. \tag{3.4}$$

We now derive some inequalities which can be combined to a proof of (3.4). Since $1 < r_2 < r_1$ and $1 \leq p \leq \infty$, we have

$$\begin{aligned} \frac{\sqrt[p]{r_2^p + r_1^p} - \sqrt[p]{1 + r_1^p}}{r_2 - 1} &= \frac{1}{r_2 - 1} \int_1^{r_2} \frac{d}{dx} \sqrt[p]{x^p + r_1^p}\, dx = \frac{1}{r_2 - 1} \int_1^{r_2} \left(\frac{x^p}{x^p + r_2^p}\right)^{1-1/p} dx \\ &< \frac{1}{r_1 - r_2} \int_{r_2}^{r_1} \left(\frac{x^p}{x^p + 1}\right)^{1-1/p} dx = \frac{\sqrt[p]{r_1^p + 1} - \sqrt[p]{r_2^p + 1}}{r_1 - r_2}. \end{aligned} \tag{3.5}$$

Combining (3.5) and $\sqrt[p]{r_1^p+r_2^p} > \sqrt[p]{1+r_2^p}$ gives

$$\sqrt[p]{r_1^p+r_2^p}\Big(\sqrt[p]{1+r_1^p}-\sqrt[p]{1+r_2^p}\Big)(r_2-1) > \sqrt[p]{1+r_2^p}\Big(\sqrt[p]{r_1^p+r_2^p}-\sqrt[p]{1+r_1^p}\Big)(r_1-r_2),$$

which can be rewritten as

$$\frac{r_1-1}{\sqrt[p]{1+r_1^p}} < \frac{r_2-1}{\sqrt[p]{1+r_2^p}} + \frac{r_1-r_2}{\sqrt[p]{r_1^p+r_2^p}}. \tag{3.6}$$

The second inequality (2.2) in Lemma 1 gives

$$\sqrt[p]{\frac{1+r_1^p}{1+r_2^p}} > \sqrt{r_1/r_2} \tag{3.7}$$

and

$$\sqrt[p]{\frac{1+r_1^p}{r_1^p+r_2^p}} = \frac{1}{r_2}\sqrt[p]{\frac{1+r_1^p}{1+(r_1/r_2)^p}} > \frac{1}{r_2}\sqrt{r_2} = 1/\sqrt{r_2}. \tag{3.8}$$

From the trigonometric identity $\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$, we get

$$|\sin(x+y)| \le |\sin(x)||\cos(y)| + |\sin(y)||\cos(x)| \le |\sin(x)| + |\sin(y)|,$$

which implies that

$$|\sin(\theta_1/2)| \le |\sin(\theta_2/2)| + |\sin((\theta_2-\theta_1)/2)|. \tag{3.9}$$

From the trigonometric identity $\sin^2(x/2) = \frac{1-\cos(x)}{2}$, we get

$$\sqrt{1+r_2^2-2r_2\cos\theta_2} = \left\| \begin{bmatrix} r_2-1 \\ 2\sqrt{r_2}\sin(\theta_2/2) \end{bmatrix} \right\|_2 \tag{3.10}$$

and

$$\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)} = \left\| \begin{bmatrix} r_1-r_2 \\ 2\sqrt{r_1r_2}\sin((\theta_1-\theta_2)/2) \end{bmatrix} \right\|_2. \tag{3.11}$$

Combining (3.6), (3.7), (3.8), (3.9), (3.10), and (3.11) gives

$$\begin{aligned}
&\sqrt[p]{1+r_1^p}\cdot\left(\frac{\sqrt{1+r_2^2-2r_2\cos\theta_2}}{\sqrt[p]{1+r_2^p}} + \frac{\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}}{\sqrt[p]{r_1^p+r_2^p}}\right)\\
&\qquad= \left\| \begin{bmatrix} \frac{\sqrt[p]{1+r_1^p}}{\sqrt[p]{1+r_2^p}}(r_2-1) \\ 2\frac{\sqrt[p]{1+r_1^p}}{\sqrt[p]{1+r_2^p}}\sqrt{r_2}\sin(\theta_2/2) \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} \frac{\sqrt[p]{1+r_1^p}}{\sqrt[p]{r_1^p+r_2^p}}(r_1-r_2) \\ 2\frac{\sqrt[p]{1+r_1^p}}{\sqrt[p]{r_1^p+r_2^p}}\sqrt{r_1r_2}\sin((\theta_1-\theta_2)/2) \end{bmatrix} \right\|_2\\
&\qquad\ge \left\| \begin{bmatrix} r_1-1 \\ 2\sqrt{r_1}\sin(\theta_1/2) \end{bmatrix} \right\|_2 = \sqrt{1+r_1^2-2r_1\cos\theta_1}.
\end{aligned}$$

That is, we have proved (3.4) and hence also (3.1). □

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