A New Weighted Metric: the Relative Metric II

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Abstract

In the first part of this investigation, [1], we generalized a weighted distance function of [2] and found necessary and sufficient conditions for it being a metric. In this paper some properties of this so–called M–relative metric are established. Specifically, isometries, quasiconvexity and local convexity results are derived. We also illustrate connections between our approach and generalizations of the hyperbolic metric.

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1. Preliminaries and main results

In this section we introduce the M–relative metric and state the main results. In order to do this, we have to introduce some notation – for a fuller account the reader should consult Section 2 of [1].

A normed space X is called *Ptolemaic* if

$$\|z-w\|\|x-y\| \le \|y-w\|\|x-z\| + \|x-w\|\|z-y\|$$

holds for every $x, y, z, w \in X$ (for background information on Ptolemy's inequality, see e. g. [3, 10.9.2]). Throughout this paper, we will denote by $\mathbb{X}$ a Ptolemaic normed space which is non–degenerate, i.e. $\mathbb{X}$ is non–empty and $\mathbb{X} \neq \{0\}$. By a metric or a norm we understand a function from $\mathbb{X} \times \mathbb{X}$ into $[0, \infty]$ or $\mathbb{X}$ into $[0, \infty]$, respectively.

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An increasing function $f\colon [0,\infty) \to [0,\infty)$ is said to be *moderately increasing* if $f(t)/t$ is decreasing. A function $P\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ of two variables is moderately increasing if both $P(x,\cdot)$ and $P(\cdot,x)$ are moderately increasing for each fixed $x \in [0,\infty)$.

If $P\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies

$$\max\{x^\alpha, y^\alpha\} \geq P(x,y) \geq \min\{x^\alpha, y^\alpha\},$$

for all $x, y \in [0,\infty]$ then it is called an α*–quasimean*. A 1–quasimean is called a *mean*. We define the *trace* of a symmetric quasimean P by $t_P(x) := P(x,1)$ for $x \in [1,\infty]$. We will need the following family of quasimeans

$$S_p(x,y) := (1-p)\frac{x-y}{x^{1-p}-y^{1-p}},\ S_p(x,x) = x^p,\ 0<p<1,$$

$$S_1(x,y) := L(x,y) := \frac{x-y}{\log x - \log y},\ S_1(x,x) = x.$$

Throughout this paper we will denote by M a symmetric function, $M\colon [0,\infty] \times [0,\infty) \to [0,\infty)$. When $M(x,y) = f(x)f(y)$ this means, then, that we assume that $f\colon [0,\infty) \to [0,\infty)$. By the M*–relative distance* (in $\mathbb{X}$) we mean the function

$$\rho_M(x,y) := \frac{\|x-y\|}{M(\|x\|,\|y\|)}$$

where $x, y \in \mathbb{X}$ (here we define 0/0=0). We will use the convention $M(x,y) := M(\|x\|,\|y\|)$ (and $f(x) := f(\|x\|)$, when $M(x,y) = f(x)f(y)$). If ρ_M is a metric, it is called the M*–relative metric*. The main results of the first part of this investigation are summarized in the next theorem.

1.1 **Theorem.** ([1, Sections 1 & 3]) *Let $\mathbb{X}$ denote a non–degenerate Ptolemaic normed space.*

(1) *Assume that M is moderately increasing. Then ρ_M is a metric in $\mathbb{X}$ if and only if it is a metric in $\mathbb{R}$.*

(2) *Let M is an α–quasimean. Then ρ_M is a metric in $\mathbb{R}$ if $M(x,1)/S_\alpha(x,1)$ is increasing in x for $x \geq 1$. If ρ_M is a metric in $\mathbb{R}$ then $M(x,1) \geq S_\alpha(x,1)$ for $x \geq 1$.*

(3) *Assume that $M(x,y) = (x^p + y^p)^{q/p}$ for $p, q > 0$. Then ρ_M is denoted $\rho_{p,q}$ and called the (p, q)–relative distance. It is a metric in $\mathbb{X}$ if and only if $q = 0$ or $0 < q \leq 1$ and $p \geq \max\{1-q, (2-q)/3\}$.*

(4) *Let $M(x,y) = f(x)f(y)$. Then ρ_M is a finite metric (i.e. $\rho_M < \infty$) in $\mathbb{X}$ if and only if f is moderately increasing and convex.*

Like the first part of the investigation, this paper is organized along three threads – one general and two special ones.

In the general case, the moderation assumption also suffices for deriving some results on lipschitz mappings, quasiconvexity and local star–shapedness of the metric (in Sections 2, 4 and 5, respectively).

In the special cases, we can prove a bit more, however we also have to restrict ourselves to the spaces $\mathbb{R}^n$:

1.2 **Theorem.** *Let $\rho_{p,q}$ denote the (p,q)–relative metric as in Theorem 1.1 (3). Then*

(1) *If $n \geq 2$, the (p,q)–relative metric is quasiconvex in $\mathbb{R}^n$ (see Section 4 for the definition) if and only if $q < 1$ in which case it is $c_{p,q}$–quasiconvex, where*

$$\frac{2^{-q/p}}{1-q} \leq c_{p,q} \leq \frac{\max\{2^{q(1-1/p)},1\}}{1-q}.$$

(2) *The (p,q)–relative metric is locally convex (see Section 5 for the definition) if and only if $p < \infty$.*

1.3 **Theorem.** *Let $M(x,y) = f(x)f(y)$. If $n \geq 2$, ρ_M is c–quasiconvex in $\mathbb{X}$ for some $c \leq \sqrt{\pi^2/4+4}$.*

This paper also contains an explicit formula for the α–quasihyperbolic metric in the domain $\mathbb{R}^n \setminus \{0\}$ which might be of independent interest (the α–quasihyperbolic is defined in the begining of Section 4).

1.4 **Theorem.** *For $n \geq 2$ and $0 < \alpha < 1$ we have*

$$k_\alpha(x,y) = \frac{1}{\beta}\sqrt{|x|^{2\beta} + |y|^{2\beta} - 2|x|^\beta|y|^\beta \cos\beta\theta}.$$

Here $\alpha + \beta = 1$ and θ is the angle $\widehat{x0y}$. In particular, as $\alpha \to 1$,

$$k_\alpha(x,y) \to \sqrt{\theta^2 + \log^2(|x|/|y|)},$$

the well–known expression for the quasihyperbolic metric in $\mathbb{R}^n \setminus \{0\}$ ([5, 3.11]).

In the last section we consider how the relative-metric-approach may be applied to extending the hyperbolic metric in $\mathbb{R}^n$ for $n \geq 3$. We illustrate the limitations of the approach by considering a generalization of the hyperbolic metric proposed in [5, 3.25, 3.26] concerning a metric similar to ρ_M and proving the triangle inequality by another method.

2. Bilipschitz mappings and ρ_M

2.1 **Lemma.** *Let M be moderately increasing, ρ_M be a metric in $\mathbb{X}$ and $g\colon \mathbb{X} \to \mathbb{X}$ be L–bilipschitz with respect to the norm $\|\cdot\|$ with $g(0)=0$. Then g is L^3–bilipschitz with respect to the metric ρ_M.*

Proof. Assume first that $x, y \neq 0$. Since M is increasing

$$\rho_M(g(x),g(y)) = \frac{\|g(x)-g(y)\|}{M(g(x),g(y))} \le \frac{L\|x-y\|}{M(x/L,y/L)} \le L^3 \frac{\|x-y\|}{M(x,y)},$$

where the last inequality follows since

$$\frac{M(x/L,y/L)}{xy/L^2} \ge \frac{M(x,y/L)}{xy/L} \ge \frac{M(x,y)}{xy},$$

by the moderation condition. On the other hand if $y=0$ and $M(g(x),0)>0$ then

$$\rho_M(g(x),0) = \frac{\|g(x)\|}{M(g(x),0)} \le \frac{L\|x\|}{M(x/L,0)} \le L^2 \frac{\|x\|}{M(x,0)}.$$

The case $M(g(x),0)=0$ is trivial and so the upper bound is proved. The lower lipschitz bound follows similarly. $\square$

2.2 **Remark.** It is clear that the condition $g(0)=0$ in Lemma 2.1 is essential. For the translation $x \mapsto x+a$ is 1–bilipschitz in the norm $\|\cdot\|$. If, for instance, $M(x,y)=x+y$ then

$$\lim_{\epsilon\to 0} \frac{\rho_M(-\epsilon,\epsilon)}{\rho_M(a-\epsilon,a+\epsilon)} = \infty,$$

hence the translation is not bilipschitz in ρ_M. Note also that the condition $g(0)=0$ can be understood in terms of the generalization of the relative metric presented in Section 6 of [1]: the ρ_M is finite in $\mathbb{X}\setminus\{0\}$ if M is moderately increasing (and $M \not\equiv 0$) and hence the relevant class of mappings are from $\mathbb{X}\setminus\{0\}$ to $\mathbb{X}\setminus\{0\}$, i.e. those with $g(0)=0$.

2.3 **Lemma.** *Let M be moderately increasing with $M \not\equiv 0$ and let $\mathbb{X} = \mathbb{R}^n$. If ρ_M is a metric and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is L–bilipschitz with respect to the metric ρ_M then g is quasiconformal in $\mathbb{R}^n$ with linear dilatation coefficient less than or equal to L^2.*

Proof. We will first prove that g is continuous in $\mathbb{R}^n \setminus \{0\}$. Since M is moderately continuous and $M \not\equiv 0$ it follows that $M(x,y)>0$ unless $xy=0$.

Fix a point $x \in \mathbb{R}^n$ such that $g(x) \neq 0$. Since ρ_M is a metric and g is bilipschitz with respect to $|rho_M$ it follows that g is injective. Hence there exists a neighborhood U_x of

x such that $0 \notin U_x$ and $g(y) \neq 0$ for $y \in U_x$. For $y \in U$ $\|g(y)\|$ has an upper bound independent of y. For if $\|g(y)\| \geq \|g(x)\|$ then the inequality

$$\|g(y)\| - \|g(x)\| \leq \|g(x) - g(y)\| \leq L\frac{\|g(y)\|}{\|g(x)\|} M(g(x), g(x))\rho_M(x, y)$$

implies that

$$\|g(y)\| \leq \|g(x)\| \left(1 - \frac{L}{\|g(x)\|} M(g(x), g(x))\rho_M(x, y)\right)^{-1}.$$

It follows that

$$\|g(x) - g(y)\| \leq \frac{LM(g(x), g(x))}{1 - LM(g(x), g(x))\rho_M(x, y)/\|g(x)\|}\rho_M(x, y).$$

From this we easily see that $g(y) \to g(x)$ as $y \to x$. Hence g is continuous in $\mathbb{R}^n \setminus \{g^{-1}(0)\}$.

Let $z \notin \{0, g^{-1}(0)\}$, $x, y \in U_z$ and $\|x - z\| = \|y - z\| = r$. Then

$$\frac{\|g(x) - g(z)\|}{\|g(y) - g(z)\|} \leq L^2 \frac{M(g(x), g(z))M(y, z)}{M(g(y), g(z))M(x, z)}.$$

Since M is moderately increasing it is continuous in $\mathbb{R}^n \setminus \{0\}$ by [1, Lemma 2.3]. By the continuity of M and g the right hand side tends to L^2 as $r \to 0$. Hence we have proved that g is quasiconformal in $\mathbb{R}^n \setminus \{0, g^{-1}(0)\}$. But then g is quasiconformal in $\mathbb{R}^n$ by well-known continuation results (see e.g. [9]). $\square$.

2.4 **Remark.** If M and g are as in the previous lemma and additionally $M(x, 0) = 0$ for every $x > 0$ then $g(0) = 0$. For the bilipschitz condition

$$\frac{1}{L}\frac{\|x - y\|}{M(x, y)} \leq \frac{\|g(x) - g(y)\|}{M(g(x), g(y))} \leq L\frac{\|x - y\|}{M(x, y)}$$

implies that $M(x, y)$ and $M(g(x), g(y))$ are simultaneously 0. Therefore $\|x\|\|y\| = 0$ iff $M(x, y) = 0$ iff $M(g(x), g(y)) = 0$ iff $\|g(x)\|\|g(y)\| = 0$, which implies $g(0) = 0$.

2.5 **Corollary.** *If M is moderately increasing with $M \not\equiv 0$ and $g\colon \mathbb{X} \to \mathbb{X}$ is a ρ_M–isometry then g is conformal.* $\square$

2.6 **Remark.** The mapping $g(x) = |x|x$ is 2–bilipschitz in the $\rho_{\infty,1}$ metric ($=\rho_M$ with $M(x, y) = \max\{x, y\}$) but is not lipschitz with respect to the Euclidean metric ($=\rho_M$ with $M \equiv 1$). The spherical metric, q ($=\rho_M$ with $M(x, y) = \sqrt{1 + x^2}\sqrt{1 + y^2}$) and the inversion $x \mapsto x/\|x\|^2$ is a q–isometry. However, this inversion is certainly not lipschitz with respect to the Euclidean metric. These examples show that the class of ρ_M–lipschitz mappings depends on M in a non–trivial way.

3. α–quasihyperbolic metrics

The length of a (rectifiable) path $\gamma\colon[0,l]\to\mathbb{X}$ in the metric ρ_M with continuous M is defined by

$$\ell_M(\gamma) := \lim_{n\to\infty}\sum_{i=0}^{n}\rho_M(\gamma(t_i),\gamma(t_{i+1})),$$

where $t_i<t_{i+1}$, $t_0=0$, $t_n=l$ and $\max\{t_{i+1}-t_i\}\to 0$. If γ is any path connecting x and y in $\mathbb{X}$ then $\rho_M(x,y)\le\ell_M(\gamma)$ by the triangle inequality.

Let M be an α–quasimean ($0<\alpha\le 1$). By taking the infimum over all rectifiable paths joining x and y we conclude that

$$\rho_M(x,y)\le\inf_\gamma\ell_M(\gamma)=\inf_\gamma\int\frac{ds}{\|\gamma(s)\|^\alpha}=:k_\alpha(x,y),$$

since $M(x,x+\epsilon)\ge x^\alpha$ for $\epsilon>0$. Here k_α stands for the α–quasihyperbolic metric, which was introduced in [7]. More precisely, it is the α–quasihyperbolic metric in the domain $G=\mathbb{R}^n\setminus\{0\}$. In this section we will derive an explicit expression for $k_\alpha(x,y)$, which will be used to study quasiconvexity in the next section.

3.1 **Proof of Theorem 1.4.** It is clearly sufficient to limit ourselves to the case $\mathbb{X}=\mathbb{R}^2$ in this proof. It is also clear that the geodesic can be parameterized by $(r(\theta),\theta)$ in polar coordinates. The kernel of the integral then becomes $r^{-\alpha}\sqrt{(r')^2+r^2}$, where $r'=dr/d\theta$. Then the Euler equation (cf. [8, p. 36 (5)]) tells us that the geodesic satisfies the differential equation

$$r^{-\alpha}\sqrt{(r')^2+r^2}-\frac{r^{-\alpha}(r')^2}{\sqrt{(r')^2+r^2}}=c_1.$$

Since c_1 is independent of r, one easily sees that $c_1\ne 0$. Then the equation is equivalent to $r^\beta/c_1=\sqrt{((\log r)')^2+1}$.

To solve this equation, we change variables by substituting $y:=\log r$. The equation then becomes $e^{\beta y}=c_1\sqrt{(y')^2+1}$, where $y'=dy/d\theta$. We introduce an auxiliary parameter, t, by $\sinh t=y'$. Then $e^{\beta y}=c_1\cosh t$ and

$$d\theta=\frac{dy/dt}{dy/d\theta}dt=\frac{dt}{\beta\cosh t}.$$

Solving this equation gives $\tan((\beta\theta+c_2)/2)=e^t$, hence

$$r(\theta)^\beta=\frac{c_1}{2}\left(\tan((\beta\theta+c_2)/2)+\frac{1}{\tan((\beta\theta+c_2)/2)}\right)=\frac{c_1}{\sin(\beta\theta+c_2)}.$$

Let us now calculate the distance in the k_α metric between 1 and $re^{i\theta_1}$, where $r\ge 1$ and $0\le\theta_1\le\pi$, using the formula for the geodesic (denoted by γ):

$$k_\alpha(1,re^{i\theta_1})=\int_\gamma\frac{\sqrt{(r')^2+r^2}}{r^\alpha}d\theta=\int_0^{\theta_1}\frac{c_1}{\sin^2(\beta\theta+c_2)}d\theta=\frac{c_1}{\beta}\left(\cot c_2-\cot(\beta\theta_1+c_2)\right).$$

It remains to express c_1 and c_2 in terms of the boundary values:

$$\sin c_2 = c_1,\ r^\beta \sin(\beta\theta_1 + c_2) = c_1.$$

These equations imply that

$$c_1 = \frac{r^\beta \sin\beta\theta_1}{\sqrt{1 + r^{2\beta} - 2r^\beta \cos\beta\theta_1}},$$

from which it follows that

$$k_\alpha(1, re^{i\theta_1}) = \frac{1}{\beta}\left(\sqrt{r^{2\beta} - c_1^2} \pm \sqrt{1 - c_1^2}\right) = \frac{r^\beta|r^\beta - \cos\beta\theta_1| \pm |r^\beta\cos\beta\theta_1 - 1|}{\beta\sqrt{1 + r^{2\beta} - 2r^\beta\cos\beta\theta_1}},$$

where $\pm$ is a plus when c_2 is greater than $\pi/2$ and a minus when it is not. This means that effectively the absolute value is disregarded and the $\pm$ sign is a minus sign since c_2 is greater than $\pi/2$ exactly when $r^\beta\cos\beta\theta_1 \geq 1$.

Then

$$r^\beta|r^\beta - \cos\beta\theta_1| \pm |r^\beta\cos\beta\theta_1 - 1| = r^\beta(r^\beta - \cos\beta\theta_1) - (r^\beta\cos\beta\theta_1 - 1) =$$

$$= 1 + r^{2\beta} - 2r^\beta\cos\beta\theta_1$$

from which the claim follows. □

3.2 **Remark.** To get a picture of what k_α looks like we consider how the distance between points changes as α changes. Since k_α is β-homogeneous and spherically symmetric, we assume that $y = 1$. Consider first the case when x is a real number greater than one. Then $k_\alpha(x, 1) = (x^\beta - 1)/\beta$. This is an increasing function with respect to β. Consider now another point $z \in S^{n-1}(0, 1)$. Then $k_\alpha(z, 1) = \sqrt{2(1 - \cos\beta\theta)}/\beta$. This is decreasing in β. Hence, intuitively speaking, increasing α increases angular distance but decreases radial distance. Note that these considerations imply, in particular, that k_α is not monotone in α.

3.3 **Corollary.** *Let $\alpha + \beta = 1$ with $0 \leq \alpha < 1$. Then $k_\alpha(x, y) \leq (|x|^\beta + |y|^\beta)/\beta$.* □

3.4 **Lemma.** *Let $\alpha + \beta = 1$ with $0 \leq \alpha < 1$. Then*

$$\frac{|x|^\beta - |y|^\beta}{\beta(|x| - |y|)} \leq \frac{k_\alpha(x, y)}{|x - y|} \leq \frac{k_\alpha(-|x|, |y|)}{|x| + |y|} \leq \frac{|x|^\beta + |y|^\beta}{\beta(|x| + |y|)} \leq \frac{2^\alpha}{\beta}|x - y|^{-\alpha}.$$

Let $t \in \mathbb{R}^+$. There is equality in the first inequality for $x = ty$, in the second for $x = -ty$, in the third for $x = -ty$ and $\beta = 1$ and in the fourth for $x = -y$.

Proof. It suffices to show that

$$\frac{k_\alpha(re^{i\theta},1)}{|re^{i\theta}-1|}$$

is increasing in θ for $r \geq 1$ and $0 \leq \theta \leq \pi$. Using the explicit formula for k_α from Theorem 1.4 we need to show that

$$\frac{1+r^{2\beta}-2r^\beta\cos\beta\theta}{\beta(1+r^2-2r\cos\theta)}$$

is increasing in θ. We differentiate the equation with respect to θ and see that this follows if we show that

$$(1+r^{2\beta}-2r^\beta\cos\beta\theta)/(\beta r^\beta\sin\beta\theta)$$

is increasing in β.

When we differentiate this equation with respect to β, we see that it suffices to show that

(3.5) $$(s-1/s)\log s\sin x+2x+\sin 2x \geq (s+1/s)(x\cos x+\sin x),$$

where we have denoted $s := r^\beta \geq 1$ and $0 \leq x := \beta\theta \leq \pi$. The inequality holds in (3.5) for $s = 1$ since $x - \sin x \geq \cos x(x-\sin x)$ for $x \geq 0$. Differentiating (3.5) with respect to s leads to

$$(s^2+1)\log s+s^2-1 \geq (s^2-1)(x/\tan x+1).$$

Since $x/\tan x \leq 1$ for $0 \leq x \leq \pi$, it suffices to show that $\log s \geq 1-2/(s^2+1)$, which follows since $2/(s^2+1)+\log s$ is increasing in s. □

3.6 **Remark.** It would be interesting to see how the above estimates for k_α generalize to other domains than $\mathbb{R}^n \setminus \{0\}$.

4. Quasiconvexity

In this section, we will assume that $n \geq 2$ and consider the space $\mathbb{R}^n$. The length of a curve was defined at the beginning of the previous section. Following [10], we define a metric ρ_M (actually a metric space, $(\mathbb{X},\ \rho_M)$) to be c–quasiconvex if $\inf_\gamma \ell_M(\gamma) \leq c\rho_M(x,y)$, where that infimum is taken over all rectifiable paths γ joining x and y. For instance if $G \subset \mathbb{R}^n$ is convex then $(G,\ |\cdot|)$ is 1–quasiconvex, whereas $(D,\ |\cdot|)$ is not quasiconvex for $D := B^n \setminus [0,1)$, since we need a path of lenght ≥ 1 to connect $x := (1-t)e_1+te_2$ with $x := (1-t)e_1-te_2$ (e_1 and e_2 are basis vectors of $\mathbb{R}^n$).

4.1 **Theorem.** *Let M be an α–quasimean such that ρ_M is a metric. Then ρ_M is quasiconvex if and only if*

$$c_M := \sup_{x\geq 0, y>0} \frac{k_\alpha(x,-y)}{x+y} M(x,y) < \infty,$$

in which case it is c_M–quasiconvex.

Proof. The claim follows directly from the second inequality in Lemma 3.4, since $\inf_\gamma \ell_M(\gamma) = k_\alpha(x,y)$ by definition. □.

4.2 **Corollary.** *Let M be α–homogeneous with $M(1,1) = 1$ such that ρ_M is a metric. Then ρ_M is quasiconvex if and only if*

$$c_M := \sup_{r\geq 1} \frac{k_\alpha(r,-1)}{r+1} M(r,1) < \infty,$$

in which case it is c_M–quasiconvex. □

4.3 **Corollary.** *Let M be α–homogeneous with $M(1,1) = 1$ such that ρ_M is a metric. If $\alpha < 1$ then ρ_M is $2^\alpha/(1-\alpha)$–quasiconvex. If $M \leq A_p^\alpha$ then ρ_M is $c_{p,\alpha}$–quasiconvex, where*

$$c_{p,\alpha} := \frac{\max\{2^{\alpha(1-1/p)}, 1\}}{1-\alpha}.$$

Proof. Let us consider $M = A_1^\alpha$. Then, by Corollary 4.2 and Lemma 3.4,

$$c_M \leq \sup_{r\geq 1} \frac{r^{1-\alpha}+1}{(1-\alpha)(r+1)} \left(\frac{r+1}{2}\right)^\alpha = \frac{1}{2^\alpha(1-\alpha)} \sup_{r\geq 1} \frac{r^{1-\alpha}+1}{(r+1)^{1-\alpha}} \leq 1/(1-\alpha),$$

since $(r^{1-\alpha}+1)(r+1)^{\alpha-1}$ is decreasing.

Since $A_p \leq \max\{2^{1-1/p}, 1\} A_1$ the second claim is proved. Since

$$M(x,1) \leq A_\infty^\alpha(x,1) \leq \{2A_1(x,1)\}^\alpha$$

for every α–homogeneous M, the first claim also follows. □.

4.4 **Proof of Theorem 1.2(1).** The upper bound follows from Corollary 4.3. For the lower bound let $r \to \infty$ in Corollary 4.2. □

4.5 **Corollary.** *$\rho_{p,1/2}$ is $\max\{\sqrt{2}, 2^{1-1/(2p)}\}$–quasiconvex, where the constant is the smallest possible.*

Proof. Setting $\alpha = 1/2$ in Corollary 4.2 yields

$$c_M = \sup_{r\geq 1} 2^{1-1/(2p)}\sqrt{(r^p+1)^{1/p}/(r+1)},$$

from which the claim follows since $(r^p+1)^{1/p}/(r+1)$ is increasing for $p \geq 1$ and decreasing for $p \leq 1$. □

4.6 **Proof of Theorem 1.3.** We will handle the cases $f(0) = 0$ and $f(0) > 0$ separately. In the first case $f(x) = cx$ for some c, as was shown in [1, Remark 5.1]. Denote by x' the image of x under the inversion $x \mapsto x/|x|^2$. Then $\rho_M(x, y) = |x' - y'|$ and hence the line from x' to y' is mapped onto a curve γ (actually a segment or an arc of a circle) with $\ell_M(\gamma) = \rho_M(x, y)$, hence ρ_M is 1–quasiconvex in this case.

In the second case we may assume without loss of generality that $f(0) = 1$.

Let us fix the points x and y with $\|x\| \geq \|y\| > 0$. Denote by γ_1 the path which is radial from x to $(\|y\|/\|x\|)x$ and then circular (with radius $\|y\|$) about the origin to y and by γ_2 the path which is first circular (with radius $\|x\|$) and then radial from $(\|x\|/\|y\|)y$ to y.

In what follows we will denote $\|x\|$ by x and similarly for y and z, since there is no danger of confusion. We derive estimates for the lengths of the γ_i:

$$\min\{\ell_M(\gamma_1), \ell_M(\gamma_2)) \leq \theta \min\left(\frac{x}{f(x)^2}, \frac{y}{f(y)^2}\right) + \int_{\|y\|}^{\|x\|} \frac{dz}{f(z)^2},$$

where θ is the angle $\widehat{x0y}$. Since f is moderately increasing and convex we find that $f(z) \geq \max\{1 + z(f(y) - 1)/y, zf(x)/x\}$ for $z \in [y, x]$. Let $z_0 \in Rp$ be such that $1 + z_0(f(y) - 1)/y = z_0 f(x)/x$. Then

$$\int_{\|y\|}^{\|x\|} \frac{dz}{f(z)^2} \leq \int_{\|y\|}^{z_0} \frac{dz}{\{1 + z(f(y) - 1)/y\}^2} + \int_{z_0}^{\|x\|} \frac{dz}{\{zf(x)/x\}^2} \leq$$
$$\leq \frac{2\|x\|}{f(x)} - \frac{y}{f(y)} - \frac{\|x\|}{f(x)^2}\left(\frac{x}{y}(f(y) - 1) + 1\right) \leq \frac{2(\|x\| - y)}{f(x)f(y)}.$$

To see that the last inequality holds, multiply by $f(x)^2 f(y)$ and rearrange:

$$2(\|x\| - y)f(x) - 2\|x\|f(y)f(x) + yf(x)^2 \geq \left(\frac{\|x\|}{y}(f(y) - 1) + 1\right)\|x\|f(y).$$

Notice that the right hand side is independent of $f(x)$ whereas the left hand side is increasing in $f(x)$ since

$$y(f(x) - 1) = (y - 0)(f(x) - f(0)) \geq (\|x\| - 0)(f(y) - f(0)) = \|x\|(f(x) - 1),$$

which follows from the convexity of f. The inequality then follows, when we insert the minimum value for $f(x)$, that is $\|x\|(f(y) - 1)/y + 1$ and use $y(f(x) - 1) \geq \|x\|(f(y) - 1)$ again.

In the case $y = 0$ which was excluded above one easily derives the estimate

$$\ell_M(\gamma_1) \leq \frac{2f(x)-1}{f(x)^2}x \leq \frac{2x}{f(x)}.$$

Now c–quasiconvexity follows, if we show that

$$\theta \min\left(x\frac{f(y)}{f(x)}, y\frac{f(x)}{f(y)}\right) + 2(x-y) \leq c\sqrt{x^2+y^2-2xy\cos\theta}.$$

For fixed x and y, $\min\{xf(y)/f(x), yf(x)/f(y)\} \leq \sqrt{xy}$. Hence it suffices to show that

$$\theta^2 xy + 4\theta(x-y)\sqrt{xy} + 4(x-y)^2 + 2c^2 xy\cos\theta \leq c^2(x^2+y^2).$$

Since the case $y = 0$ is clear we set $s := x/y \geq 1$ and divide through by xy, obtaining:

$$\theta^2 + 4(\sqrt{s} - \sqrt{1/s})\theta + 4(\sqrt{s} - \sqrt{1/s})^2 + 2c^2\cos\theta - c^2(s+1/s) \leq 0.$$

The derivative of the left hand side with respect to s is positive when

$$2\theta(s+1) \geq (c^2-4)\sqrt{s}(s-1/s)$$

or, equivalently, when $\sqrt{s} - \sqrt{1/s} \leq 8\theta/\pi^2$. Hence the only zero of the derivative is a maximum, and we have

$$\theta^2 + 4(\sqrt{s} - \sqrt{1/s})\theta + 4(\sqrt{s} - \sqrt{1/s})^2 + 2c^2\cos\theta - c^2(s+1/s) \leq$$

$$\leq (1+16\pi^{-2})^2\theta^2 + 2c^2\cos\theta - 2c^2(32\theta^2/\pi^4+1).$$

To see that the last expression in the inequality is less than zero, we use the expression $\pi^2/4+4$ for c^2:

$$(1+16\pi^{-2})^2\theta^2 + 2(\pi^2/4+4)(\cos\theta - 32\theta^2\pi^4 - 1) \leq 0.$$

When we divide by $1+16\pi^{-2}$, we see that this is equivalent to $\theta^2 \leq \pi^2(1-\cos\theta)/2$, which concludes the proof. □

4.7 **Remark.** The first part of the proof of the previous theorem shows that for the universal constant c for which every ρ_M with $M(x,y) = f(x)f(y)$ is c–quasiconvex is at least 2. For if x and y are on the same ray emanating from the origin the clearly the segment of the ray between x and y is the geodesic. Moreover the above derivation up to

$$\int_{\|y\|}^{\|x\|} \frac{dz}{f(z)^2} \leq \frac{2(x-y)}{f(x)f(y)}$$

is sharp. Hence $c \geq 2$, as claimed.

Metrics that are 1–quasiconvex are particularly interesting, since in these metric spaces any two points can be connected with a a path γ with $\ell_M(\gamma) = d(x,y)$, where d is the metric, which is to say that the metric equals its own inner metric. The next lemma shows that, except for the Euclidean distance and its "reciprocal", there are no 1–quasiconvex M–relative metrics in $\mathbb{R}^n$ with $n \geq 2$.

4.8 **Lemma.** *Let M be moderately increasing. Then ρ_M is a 1–quasiconvex metric in $\overline{\mathbb{R}^n}$ if and only if $M \equiv c > 0$ or $M(x,y) = cxy$.*

Proof. In this proof we will write r for re_1 etc. If $M \equiv c > 0$ or $M(x,y) = xy$ then clearly ρ_M is 1–quasiconvex (the latter claim was shown in the Proof of Theorem 1.3). Assume conversely that ρ_M is 1–quasiconvex. Consider the 1–quasiconvex path γ, connecting $-r$ and r, where $r > 0$.

Now either $\infty \in \gamma$ or γ crosses the e_2-axis. In the latter case let $b \in [0,\infty)$ be such that γ crosses the e_2-axis in be_2. Then, by the triangle (in)equality,

$$\frac{2r}{M(r,r)} = \frac{2\sqrt{r^2+b^2}}{M(r,b)}$$

or, equivalently, $M(r,b) = \sqrt{1+(b/r)^2}M(r,r)$. Suppose that $b \neq 0$. Then $M(r,b) > M(r,r)$ and $b > r$ since M is increasing and hence $(b/r)M(r,r) \geq M(r,b)$ since M is moderately increasing. It follows that

$$\frac{b}{r}M(r,r) \geq M(r,b) = \sqrt{1+(b/r)^2}M(r,r)$$

from which it follows that $b/r = \sqrt{1+(b/r)^2}$, which is impossible, hence $b = 0$.

It then follows that the path connecting $-r$ and r is the segment $[-r,r]$. By considering the triangle equality for a point a, with $a < r$, on the path we find that $M(r,a) = M(r,r)$. We then consider again three distinct points y, z and x on $[0,r)$ in this order. The triangle equality becomes

$$\frac{|x-y|}{M(x,x)} = \frac{|x-z|}{M(x,x)} + \frac{|z-y|}{M(z,z)},$$

hence $M(x,x) = M(z,z)$. But then $M(x,y) = M(x,x) = M(z,z) = M(z,w)$ (assuming $x \geq y$ and

$z \geq w$, similarly otherwise) and we conclude $M(x,y) = c$ for $x, y \leq r$.

Hence γ does not cross e_2-axis, and we have $\infty \in \gamma$. This means that the path is the segment $[-\infty,-r] \cup [r,\infty]$. Now we may choose any point b, with $b \geq r$ on the path and get $2r/M(r,r) = 2b/M(r,b)$, hence $M(r,b) = (b/r)M(r,r)$ for all $b \geq r$. Then consider three arbitrary distinct points y, z and x on (r,∞) in this order. The triangle equality becomes

$$\frac{y-y^2/x}{M(y,y)} = \frac{x-y}{M(x,y)} = \frac{x-z}{M(x,z)} + \frac{z-y}{M(z,y)} = \frac{z-z^2/x}{M(z,z)} + \frac{y-y^2/z}{M(y,y)}.$$

This leads to

$$\frac{y^2/z - y^2/x}{M(y,y)} = \frac{z^2/z - z^2/x}{M(z,z)},$$

hence, since $1/z - 1/y \neq 0$, $M(y,y)/y^2 = M(z,z)/z^2$ for $y < z$. It then follows that $M(r,b) = (b/r)M(r,r) = brM(1,1)$, i.e. M is of the form $M(x,y) = cxy$ for all $x, y \geq r$.

We have seen that there are two possible cases, either $M(x,y) = c$ for every $x, y \in B^2(0,r)$ or $M(x,y) = cxy$ for every $x, y \notin B^2(0,r)$. If there is a path from $-r$ to r trough 0 then the same path will connect $-r'$ with r' for $r' < r$ as well. Similarly for paths through ∞ and $r' > r$. Hence there exists an r_0 such that $M(x,y) = c$ for $|x|, |y| \leq r$ and $M(x,y) = cxy/r^2$ for $|x|, |y| \geq r$. If $r_0 = 0$ or $r_0 = \infty$ then everything M equals cxy or c in the whole space.

Assume then that $0 < r_0 < \infty$. We may assume without loss of generality that $c = r_0 = 1$. Consider then the points 1/2 and 2. The 1–quasiconvex path connecting these points goes through 1, hence

$$\rho_M(1/2, 2) = \rho_M(1/2, 1) + \rho_M(1, 2) = 1/2 + 1/2 = 1$$

and $M(1/2, 2) = (3/2)\rho_M(1/2, 2) = 3/2$. The 1–quasiconvex path connecting $-1/2$ with 2 crosses $S^{n-1}(0,1)$ at some point z. If $\theta = \widehat{20z}$ then

$$\rho_M(2, z) = \sqrt{5/4 - \cos\theta},\ \rho_M(-1/2, z) = \sqrt{5/4 - \cos\theta},$$

so that $\rho_M(2,z) + \rho_M(-1/2, z) \geq 2 > 5/3 = (5/2)/M(1/2, 2) = \rho_M(-1/2, 2)$, contrary to the assumtion that z lies on a 1–quasiconvex path. This contradiction shows that this mixed case cannot occur. $\square$

4.9 **Remark.** Note that the question of when a generalized relative metric, of the type introduced in Section 6 of [1] are quasiconvex is not directly answered by the results in this section. However since the quasiconvexity of either the j_G metric or Seittenranta's metric, which are both generalized relative metrics, characterize uniform domains this question is clearly of interest. (See [6, 4.3-4.5].)

5. Local convexity

In this section we consider how the relative metric grows in different directions. We will denote by $B_d(x,r) := \{y \in \mathbb{X} \colon d(x,y) < r\}$ denote the open ball in the metric space $(\mathbb{X}, d)$ and by $B^n(x,r)$ the Euclidean open ball of radius r centered at x. Also $S_d(x,r) = \partial B_d(x,r)$ and $S^{n-1}(x,r) = \partial B^n(x,r)$. We will use the abbreviation $B_{\rho_M} =: B_M$ and $S_{\rho_M} =: S_M$.

5.1 **Definition.**

(i) We say that a metric d is *isotropic* if

$$\lim_{r\to 0} \inf_{|x-z|=r} d(x,z) = \lim_{r\to 0} \sup_{|x-z|=r} d(x,z)$$

for every x.

(ii) The metric d is called *locally star–shaped* if for every $x \in \mathbb{X}$ there exists an $r_0 > 0$ such that $B_d(x,r)$ is star–shaped with respect to the center of the ball, x, for every $r < r_0$. (A set K is star–shaped with respect to x if every ray emanating at x intersects ∂D exactly once.)

(iii) The metric d is called *locally convex* if for every $x \in \mathbb{X}$ there exists an $r_0 > 0$ such that $B_d(x,r)$ is convex for every $r < r_0$.

5.2 **Lemma.** *If $f_x(y) := M(x,y)$ is continuous at x for every $x \in (0,\infty)$ then ρ_M is isotropic.*

Proof. Fix a point $x \in \mathbb{X}$. If $x = 0$ then $\rho_M(x,z) = \rho_M(x,y)$ for every $|z| = |y|$. Let then $x \neq 0$. If $f_x(x) = 0$ then $\lim_{r\to 0} inf_{|x-z|=r} \rho_M(x,z) = \infty$ and ρ_M is isotropic at x. Let then $c := f_x(x) > 0$. For every $0 < \epsilon < f_x(x)/2$ there exists a neighborhood U of x such that $|f_x(y) - f_x(z)| \leq \epsilon$. Then

$$\sup_{|x-z|=r} \rho_M(x,z) - inf_{|x-z|=r} \rho_M(x,z) \leq \frac{r}{c-\epsilon} - \frac{r}{c+\epsilon} \leq 2r\epsilon/c$$

for every $0 < r < d(\partial U, x)$ (here d refers to the Euclidean distance). □

5.3 **Remark.** It is possible that ρ_M is isotropic even when M is not continuous. For instance if $M(x,y) = xy$ for $x + y > 0$ and $M(0,0) = 1$ then ρ_M is an isotropic metric, but clearly M is not continuous at the origin. This example is due to Pentti Järvi.

5.4 **Lemma.** *Let $\mathbb{X}$ be an inner product space. If M is moderately increasing and ρ_M is a metric then it is locally star–shaped.*

Proof. Let us consider balls centered at z. Since the case $z = 0$ is trivial, we assume $z \neq 0$. The case $M \equiv 0$ is also trivial and then, since M is moderately increasing, $M(x,y) > 0$ for every $xy > 0$.

Let r be a unit vector. Now if $\rho_M(z, z+sr) = s/M(z, z+sr)$ is increasing in $s > 0$ for some range independent of the direction of r then we are done. If $\|z + sr\|$ is decreasing in s, then ρ_M is the product of two positive increasing factors, s and $1/M(z, z + sr)$, and is hence itself increasing.

If $\|z + sr\|$ is increasing in s, we write

$$\rho_M(z, z + sr) = \frac{s}{M(z, z + sr)} = \frac{\|z + sr\|}{M(z, z + sr)} \frac{s}{\sqrt{\|z\|^2 + s^2 + 2s(r, z)}},$$

where (r, z) denotes the inner product of r and z. The first factor is increasing by the moderation part of the moderately increasing condition of M. The second factor is increasing provided $\|z\|^2 \geq -s^2(r, z)$. Since $(r, z) \geq -\|z\|$ $\rho_M(z, z + sr)$ is increasing for $s \leq \sqrt{\|z\|}$.

Since M is moderately increasing $M(x, y)$ is bounded from above in $B^n(z, s)$, say by c_z. Then $B_M(z, s/c_z) \subset B^n(z, s)$ and hence $B_M(z, s/c_z)$ is star–shaped. □

The local star–shaped condition says that that the metric increases locally when we move away from the point (in the Euclidean metric), the isotropy condition says that it does so equally fast in every direction. Both of these facts follow from the convexity result that we prove next.

5.5 **Lemma.** *Let $\mathbb{X} = \mathbb{R}^n$, M be moderately increasing and ρ_M be a metric. Assume also that $M(x, \cdot) \in C^2(\mathbb{R}^+)$. Then ρ_M is locally convex.*

Proof. Without loss of generality we may assume that $\mathbb{X} = \mathbb{R}^2$ since $B_M(z, r)$ is formed by rotating a two dimensional disk $B_M(z, r) \cap \mathbb{R}^2$ about the axis tz. Let us consider disks about ze_1, in particular, the locus of points (x, y) with $\rho_M((x, y), z) = r > 0$, i.e. points for which the following equation holds:

(5.6) $$\frac{\sqrt{(x - z)^2 + y^2}}{M(\sqrt{x^2 + y^2}, 1)} = r.$$

We will first show that if $y > 0$ then $d^2y/dx^2 < 0$. Let us denote $M(\sqrt{x^2 + y^2}, z)$ by M, $dM(w, z)/dw$ by $M'(w)$ and $d^2M(w, z)/dw^2$ by $M''(w)$. We multiply (5.6) with M and square it, then we differentiate with respect to x:

$$yy' + x - z = r^2MM'(x + yy')(x^2 + y^2)^{-1/2}.$$

From this it follows that

(5.7) $$yy' = \left(1 - \frac{r^2MM'}{\sqrt{x^2 + y^2}}\right)^{-1} - x.$$

Differentiating again gives

$$(y')^2 + yy'' = r^2 \left(1 - \frac{r^2MM'}{\sqrt{x^2 + y^2}}\right)^{-2} \left(\frac{(M')^2 + MM''}{\sqrt{x^2 + y^2}} - \frac{MM'(x^2 + y^2)^{-1}}{\sqrt{x^2 + y^2 - r^2MM'}}\right) - 1.$$

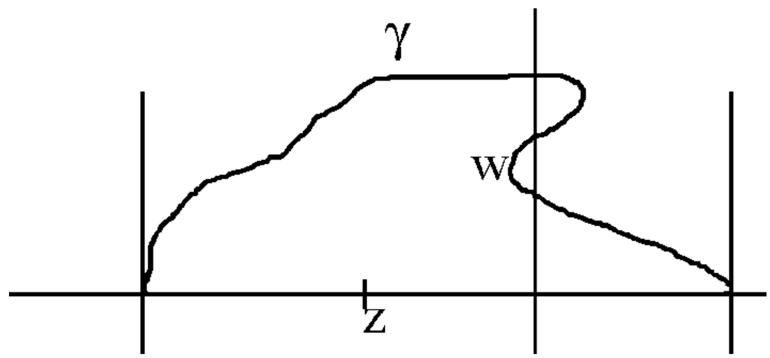

Figure 1: Proof of Lemma 5.5

By choosing r sufficiently small, we may assume that $(x, y) \in B^2(1, \delta)$ for arbitrary given $\delta > 0$. Then $\sqrt{x^2 + y^2} \in [1 - \delta, 1 + \delta]$ and there exists a constant c such that $M(z), M'(z), M''(z) \leq c$ for $z \in [1 - \delta, 1 + \delta]$ since $M \in C^2(\mathbb{R}^+)$. If we choose $\delta \geq 1/2$ we also have $M(z)M'(z)/z \leq 2c^2$ so that

$$yy'' \leq 4r^2c^2(1 - 2r^2c^2)^{-2} - 1 - (y')^2.$$

Since c is a constant it follows that $y'' < 0$ for sufficiently small $r > 0$.

Call the curve formed by the points which satisfy (5.6) γ. If γ could be parameterized by $(x, y(x))$ in Euclidean coordinates then the fact that $d^2y/dx^2 < 0$ would imply that it is convex. Suppose that γ can not be parameterized by $(x, y(x))$ (as shown in Figure 1). Then some half-line $K := \{(x, t) : t \in \mathbb{R}^+\}$ intersects γ at least twice. It follows that $dy/dx = \infty$ for some point w in the upper half-plane. However, we see from (5.7) that this is only possible for $y = 0$, provided r is small enough. Hence w is in the e_1-axis, which is impossible. It follows that that γ can be parameterized by $(x, y(x))$ and that the area under the curve is convex. Since $B_M(ze_1, r)$ is symmetric with respect to the e_1-axis it is convex, as well. $\square$

5.8 **Proof of Theorem 1.2(2).** It is immediately clear that $A_p(x, 1)^q \in C^2(\mathbb{R}^+)$ if $p < \infty$.

For $p = \infty$ we have $\rho_{\infty,q}(x, 1) = |x - 1|/\max\{1, |x|^q\}$. Let us write $S(1, r)$ in polar coordinates about 1. Then $s(\theta) = r$ for $\cos\theta \geq r/2$ and

$$s^2 = r^2(s^2 + 1 - 2s\cos\theta)^q$$

for $\cos\theta \leq r/2$. It follows that for $\cos\theta < r/2$ we have

$$ss' = r^2q(s^2 + 1 - 2s\cos\theta)^{q-1}(ss' - s'\cos\theta + s\sin\theta).$$

Denote $\theta_0 = \arccos(r/2)$. Since $s \to r$ as $\theta \to \theta_0^+$ (θ approachesθ_0 from above), we have

$$\lim_{\theta \to \theta_0^+} s' = r^2 q\sqrt{4-r^2}/(2-r^2q) > 0.$$

Since $\lim_{\theta \to \theta_0^-} s' = 0$, the point $(2\cos\theta_0, \theta_0)$, will be an inner corner of $S(1,r)$ for every $r > 0$, which means that $S(1,r)$ is not convex. $\square$

5.9 **Remark.** If a metric d is locally star–shaped, isotropic or locally convex then so are $\log(1+d)$, $\operatorname{arsh} d$ and $\operatorname{arch}(1+d)$. Moreover, provided that M is continuous these properties are also carried over to the generalized relative metrics considered in Section 6 of [1].

6. The hyperbolic metric and limitations of our approach

In this section, we will introduce the hyperbolic metric, show how our method can be used to generalize the hyperbolic metric in one setting but not in another. We use a separate method to deal with the latter case, thus solving a problem from [5, Remark 3.29].

The hyperbolic metric can be defined in several different ways, for a fuller account the reader is referred to an introductory work on hyperbolic geometry, for instance [5, Section 2]. One possible definition of the hyperbolic metric, ρ, is

$$\rho(x,y) := 2\operatorname{arsh}\left(\frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}\right) \tag{6.1}$$

for $x, y \in B^n$. An important property of the hyperbolic metric is that it is invariant under Möbius mappings of B^n. The groups formed by these Möbius mappings is denoted by $GM(B^n)$.

6.2 **Lemma.** *Let $M(x,y) = f(x)f(y)$ with $f(0) = 1$ be such that ρ_M is a metric. Then ρ_M is invariant under all mappings in $GM(B^n)$ if and only if $f(x) = \sqrt{1-x^2}$.*

6.3 **Remark.** Note that here $f(x)$ is defined only for $x \in [0,1)$. Therefore ρ_M is not exactly an M–relative metric in the sense defined in Section 1. The interpretation is nevertheless clear; strictly speaking we could extend f by setting $f(x) = 0$ when it was not previously defined and relying on the conventions regarding ∞.

Proof. The "if" part say essentially that the hyperbolic metric is Möbius invariant, as is seen from (6.1). and is hence clear, see e.g. [5, 2.49]. Assume, conversely, that ρ_M is invariant under all mappings in $GM(B^n)$.

Fix $0 < r < 1$ and set $d := r\sqrt{1-r^2}$. Then $d < 2r$ and we may choose points $x, y \in B^n$ with $|x| = |y| = r$ and $|x - y| = d$. Let g be a Möbius mapping in $GM(B^n)$ which maps y onto the origin. It follows from [5, 2.47], that $|g(x)| = r$. Hence by Möbius invariance,

$$\frac{d}{f(r)^2} = \frac{|x-y|}{f(|x|)^2} = \frac{|g(x)-0|}{f(|g(x)|)f(0)} = \frac{r}{f(r)}$$

hence $f(r) = d/r = \sqrt{1-r^2}$. $\square$

The classical definition of the hyperbolic metric makes sense only in the unit ball and domains Möbius equivalent to it (for $n \geq 3$). There are however various generalizations of the hyperbolic metric to other domains. The best known of these is probably the quasihyperbolic metric that we met in Section 4. The quasihyperbolic metric is within a factor of 2 from the hyperbolic metric in the domain B^n ([5, Remark 3.3]).

Seittenranta's cross ratio metric is another generalization of the hyperbolic metric, with the advantage, that it equals the hyperbolic metric in B^n. The reader may recall that we showed in [1], Corollary 6.5, that Seittenranta's metric can be interpreted as $\delta_G^{-\infty}$ in the one–parameter family δ_G^p,

$$\delta_G^p(x,y) := \log\{1 + \rho'_{M,G}(x,y)\}$$

with $M = \max\{1, 2^{-1/p}\}A_p$, where A_p is the power-mean,

$$A_p(x,y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}$$

for $p \in (-\infty, 0) \cup (0, \infty)$ and

$$A_{-\infty}(x,y) = \min\{x,y\},\ \ A_0(x,y) := \sqrt{xy} \text{ and } A_\infty(x,y) = \max\{x,y\}$$

defined for $x, y \in \mathbb{R}^+$. Here

$$\rho'_{M,G}(x,y) = \sup_{a,b\in\partial G} \frac{1}{M(|x,y,a,b|, |x,y,b,a|)},$$

where

$$|a,b,c,d| := \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}$$

denotes the cross–ratio of the points $a, b, c, d \in \overline{\mathbb{R}}^n$.

Seittenranta's metric is the generalization of the logarithmic expression for the hyperbolic metric given in [5, Lemma 8.39]. We now move on to study a generalization starting from the expression based the hyperbolic cosine ([5, Lemma 3.26]):

$$\rho_G(x,y) := \operatorname{arch}\{1 + \sup_{a,b\in\partial G} |a,x,b,y||a,y,b,x|/2\}. \tag{6.4}$$

This can be expressed as

$$\rho_G(x,y) := \operatorname{arch}\{1 + (\rho'_{A_0,G}(x,y))^2/2\},$$

with $A_0(x,y) := \sqrt{xy}$.

We note that by [1, Corollary 6.5] we know that

$$\log\{1 + \rho'_{A_0,G}(x,y)\}$$

is a metric provided $\operatorname{card}\partial G \geq 2$. Hence by [1, Remark 3.7] we already know that

$$\operatorname{arch}\{1 + \rho'_{A_0,G}(x,y)\}$$

is a metric when $\operatorname{card}\partial G \geq 2$. Hence one might speculate that the area hyperbolic cosine representation of the hyperbolic metric could be generalized to the one–parameter family

$$\rho^p_G(x,y) := \operatorname{arch}\{1 + (\rho'_{A_0,G}(x,y))^p/p\}.$$

In what follows we will however restrict our attention to the case $p = 2$.

Since this quantity has previously attracted some interest, we state some of its basic properties and give an independent proof that it is in fact metric in most domains:

6.5 **Theorem.** ([5, 3.25 & 3.26])

(i) ρ_G *is Möbius invariant.*

(ii) ρ_G *is monotone in* G*, that is, if* $G \subset G'$ *then* $\rho_{G'}(x,y) \leq \rho_G(x,y)$ *for all* $x, y \in G$*.*

(iii) $\rho_G(x,y) \geq \cosh\{(q(\partial G)q(x,y))^2\} - 1$.

(iv) For $G = B^n$ *and* $G = H^{n+}$ *(the upper half-plane),* ρ_G *equals the hyperbolic metric.*

Note that ρ_G is almost a generalized relative metric, indeed, we have

$$\rho_{\mathbb{R}^n\setminus\{0\}}(x,y) := \operatorname{arch}\left(1 + \frac{|x-y|^2}{2|x||y|}\right).$$

(Note that here $\mathbb{R}^n \setminus \{0\}$ has the boundary points 0 and ∞ in $\overline{\mathbb{R}^n}$.) This expression differs from a generalized relative metric (essentially) only by the exponent 2 of $|x-y|$. However, because of this difference the question of whether it is a metric does not lend itself to the generalized metric approach of Section 6, [1].

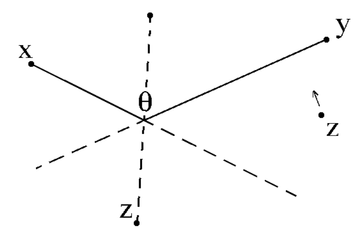

Figure 2: The point z is between x and y

6.6 **Theorem.** *The quantity ρ_G defined in (6.4) is a metric for every open $G \subset \overline{\mathbb{R}}^n$ with* card $\partial G \geq 2$.

Proof. It is clear that ρ_G is symmetric in its arguments. That (x, x) are the only zeros of ρ_G is also evident. Moreover, as card $\partial G \geq 2$, ρ_G is finite. It remains to check that it satisfies the triangle inequality.

Since the supremum in the definition (6.4) is over a compact set (in $\overline{\mathbb{R}^n}$) it is actually a maximum. Fix x, y and z in G. Let $a, b \in \partial G$ be points such that

$$\cosh \rho_G(x, y) = 1 + |a, x, b, y||a, y, b, x|/2.$$

Define $s(a, x, y, b) := |a, x, b, y||a, y, b, x|/2$. Now

$$\text{arch}(1 + s(a, x, z, b)) \leq \rho_G(x, z),\ \text{arch}(1 + s(a, z, y, b)) \leq \rho_G(z, y).$$

Hence it suffices to prove

$$\text{arch}(1 + s(a, x, y, b)) \leq \text{arch}(1 + s(a, x, z, b)) + \text{arch}(1 + s(a, z, y, b)). \tag{6.7}$$

Since s is conformally invariant, we may assume that $a = 0$ and $b = \infty$. Denote

$$s := s(0, x, z, \infty)/2,\ t := s(0, z, y, \infty)/2,\ u := s(0, x, y, \infty)/2.$$

It follows that

$$s = \frac{|x - z|^2}{2|x||z|},\ t = \frac{|z - y|^2}{2|z||y|},\ u = \frac{|x - y|^2}{2|x||y|}. \tag{6.8}$$

For fixed x and y it is clear that we can move the point z so that both s and t get smaller if $|z| \leq \min\{|x|, |y|\}$ (since $s = (x/z) + (z/x) - 2\cos\theta$ is increasing in z for $z \leq x$, and similarly for t). Hence we may assume that $|z| \geq \min\{|x|, |y|\}$. Similarly, if $|z| > \max\{|x|, |y|\}$ we can decrease s, t for fixed x and y, hence we may also assume that $|z| \leq \max\{|x|, |y|\}$. Unless $\widehat{x0y} = \pi$ we may also assume that z lies within this angle. Otherwise we may apply the transformations shown in Figure 2 (keeping x, y and $|z|$ fixed and rotating or mirroring z according to where it started.)

Since cosh is increasing, we apply it to both sides of (6.7) and use

$$\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$$

to conclude that (6.7) is equivalent to

$$u \leq s + t + st + \sqrt{s^2+2s}\sqrt{t^2+2t}. \tag{6.9}$$

Getting rid of the square–root, this equation is implied by

$$s^2 + t^2 + u^2 \leq 2(st + su + tu + stu)$$

which is equivalent to

$$(u - s - t)^2 \leq (4 + 2u)st. \tag{6.10}$$

Let us assume without loss of generality that $z = 1$. Assume, for the time being, that 0, x, y and 1 are co–linear and that $x > 1 > y > 0$. Then

$$s = \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2,\ t = \frac{1}{2}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2,\ u = \frac{1}{2}\left(\sqrt{\frac{y}{x}} - \sqrt{\frac{x}{y}}\right)^2. \tag{6.11}$$

Inserting these into (6.10) gives

$$|x + y + \frac{1}{x} + \frac{1}{y} - \frac{y}{x} - \frac{x}{y} - 2| \leq \left(\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}}\right)\left(\sqrt{x} - \sqrt{\frac{1}{x}}\right)\left(\sqrt{\frac{1}{y}} - \sqrt{y}\right),$$

which is actually an equality.

Let us now consider the general case in which 0, x, y and 1 are no longer necessarily co–linear. Denote s, t and u from (6.11) by s_0, t_0 and u_0, respectively and let s, t and u be as in (6.8). Denote

$$\delta_s := s - s_0 = (1 - \cos\theta),\ \delta_t := t - t_0 = (1 - \cos\phi),\ \delta_u := u - u_0 = (1 - \cos(\theta + \phi)),$$

where $\theta := \widehat{x01}$ and $\phi := \widehat{10y}$. Inserting $s = s_0 + \delta_s$ etc. into (6.10) and canceling the equality $(s_0 + t_0 - u_0)^2 = 2(2 + u_0)s_0t_0$ leads to

$$2(s_0 + t_0 - u_0)(\delta_s + \delta_t - \delta_u) + (\delta_s + \delta_t - \delta_u)^2 \leq 2\delta_u st + 2(2 + u_0)(t_0\delta_s + s_0\delta_t + \delta_s\delta_t)$$

which is equivalent to

$$(2s_0 + \delta_s)(\delta_s - \delta_t - \delta_u) + (2t_0 + \delta_t)(\delta_t - \delta_s - \delta_u) + (2u_0 + \delta_u)(\delta_u - \delta_s - \delta_t) \leq$$

$$\leq 2(stu - s_0t_0u_0). \tag{6.12}$$

We will first show that

$$\delta_s(\delta_s - \delta_t - \delta_u) + \delta_t(\delta_t - \delta_s - \delta_u) + \delta_u(\delta_u - \delta_s - \delta_t) \leq 0. \tag{6.13}$$

Note first that $\delta_s \geq 0$, $\delta_t \geq 0$ and $\delta_u \geq 0$. Now either all the parenthesis are negative or $\delta_u - \delta_s - \delta_t \geq 0$, since $\delta_u \geq \delta_s, \delta_t$. In the latter case the left hand side of the inequality is increasing in δ_u. Since δ_s, δ_t and δ_u are squares of the sides of a triangle we see that

$$\delta_u \leq \delta_s + \delta_t + 2\sqrt{\delta_s \delta_t}.$$

Hence it suffices to check (6.13) for the maximal δ_u, in which case it is an equality.

Let us then continue from (6.12), using (6.13), rearranging and dividing by 2:

$$\delta_s(s_0 - t_0 - u_0) + \delta_t(t_0 - s_0 - u_0) + \delta_u(u_0 - s_0 - t_0) \leq stu - s_0 t_0 u_0.$$

Since $\delta_s, \delta_t \geq 0$ it follows that $stu - s_0 t_0 u_0 \geq s_0 t_0 \delta_u$. We will then complete the proof by showing that

$$\delta_s(s_0 - t_0 - u_0) + \delta_t(t_0 - s_0 - u_0) + \delta_u(u_0 - s_0 - t_0 - s_0 t_0) \leq 0.$$

We may assume that (6.9) holds with equality, hence

$$u_0 = s_0 + t_0 + s_0 t_0 + \sqrt{s_0^2 + 2s_0}\sqrt{t_0^2 + 2t_0}.$$

Then it suffices to show that

$$(\delta_u - \delta_s - \delta_t)\sqrt{s_0^2 + 2s_0}\sqrt{t_0^2 + 2t_0} \leq 2(t_0\delta_s + s_0\delta_t) + (\delta_s + \delta_t)s_0 t_0. \tag{6.14}$$

By the formula for the cosine of a sum we have, from the definition,

$$\delta_u = \delta_s + \delta_t - \delta_s\delta_t + \sqrt{2\delta_s - \delta_s^2}\sqrt{2\delta_t - \delta_t^2} \geq \delta_s + \delta_t + \sqrt{2\delta_s - \delta_s^2}\sqrt{2\delta_t - \delta_t^2}.$$

Then (6.14) follows if we can show that

$$\sqrt{2\delta_s - \delta_s^2}\sqrt{2\delta_t - \delta_t^2}\sqrt{s_0^2 + 2s_0}\sqrt{t_0^2 + 2t_0} \leq 2(t_0\delta_s + s_0\delta_t) + (\delta_s + \delta_t)s_0 t_0.$$

Let us square this equation and subtract $2\delta_s\delta_t st(2+s)(2+t)$ from both sides:

$$(2 - 2(\delta_s + \delta_t) + \delta_s\delta_t)\delta_s\delta_t(2 + s_0)(2 + t_0)s_0 t_0 \leq \delta_s^2 t_0^2 (2 + s_0)^2 + \delta_t^2 s_0^2 (2 + t_0)^2.$$

Divide both sides by $\delta_s\delta_t(2 + s_0)(2 + t_0)s_0 t_0$:

$$2 - 2(\delta_s + \delta_t) + \delta_s\delta_t \leq a + 1/a,$$

where

$$a := \frac{\delta_s(2 + s_0)t_0}{\delta_t s_0(2 + t_0)}$$

(this is OK, since the cases where $\delta_t = 0$ or $s_0 = 0$ are trivial.) Now then $a + 1/a \geq 2$ (by the arithmetic-geometric inequality, for instance) so it suffices to show that $\delta_s\delta_t \leq 2(\delta_s + \delta_t)$ or equivalently,

$$\frac{1}{2} \leq \frac{1}{\delta_s} + \frac{1}{\delta_t}.$$

But since $\delta_s, \delta_t \leq 2$ directly from the definition, this is clear. $\square$

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