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# On some variational problems in the theory of unitarily invariant norms and Hadamard products

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## Abstract

We deal with two recent conjectures of R.-C. Li [Linear Algebra Appl. 278 (1998) 317–326], involving unitarily invariant norms and Hadamard products. In the particular case of the Frobenius norm, the first conjecture is known to be true, whereas the second is still an open problem. In fact, in this paper we show that the Frobenius norm is essentially the only invariant norm which may comply with the two conjectures: more precisely, if a norm satisfies the claim of either conjecture, then it can be controlled from above and from below by the Frobenius norm, uniformly with respect to the dimension. On the other hand, both conjectures remain open in the relevant case of matrices with an upper bound to the rank. As a first partial result in this direction, we prove the first conjecture for matrices of rank 1 and for any unitarily invariant norm. © 2001 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

In a recent paper [3] Li, within the framework of a unifying approach to some spectral stability results in matrix theory, formulated a twofold conjecture (stated below as a Conjecture 1.1) involving unitarily invariant norms and Hadamard products.

Given two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ , we denote  $A \circ B := [a_{ij}b_{ij}]$  their Hadamard product, whereas the symbol  $||| \cdot |||$  denotes a generic unitarily invariant

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(u.i.) norm (we recall that a matrix norm is called u.i. if  $|||A||| = |||UAV|||$  whenever  $U$  and  $V$  are unitary, see [1]). As usual, we assume that every u.i. norm is normalized in such a way that

$$||| \text{diag}(1, 0, \dots, 0) ||| = 1, \tag{1}$$

where  $\text{diag}(\{a_i\})$  denotes the diagonal matrix with entries  $\{a_i\}$ . Finally,  $\|A\|_2$  denotes the spectral norm (i.e. the largest singular value of  $A$ ), whereas  $\|A\|_F$  denotes the Frobenius norm.

The following conjecture was formulated in [3].

**Conjecture 1.1.** *Let  $||| \cdot |||$  be a unitarily invariant norm. Then there exist constants  $c_1, c_2 \geq 1$  depending only on  $||| \cdot |||$  (but not on the dimension) such that, for every square matrix  $G$ , there hold*

$$c_1 \left( \min_{W \text{ nonsingular}} \|W^{-1}\|_2 \cdot |||W \circ G||| \right) \geq \min_{P \text{ permutation}} |||P \circ G|||, \tag{2}$$

$$c_2 \left( \min_{W \text{ nonsingular}} |||W \circ G||| \cdot |||W^{-1} \circ G^T||| \right) \geq \min_{P \text{ permutation}} |||P \circ G|||^2. \tag{3}$$

In [4] the second author has proved that any u.i. norm which satisfies (2) or (3) is necessarily bounded from above (up to a multiplicative constant) by the Frobenius norm, uniformly with respect to the matrix dimension.

In this paper we prove that a similar bound holds also from below, thus concluding that the Frobenius norm is essentially the only norm which may comply with the conjecture. Note that Conjecture 1.1 can be split into two independent claims, namely (2) and (3): in fact, in [3] it was proved that the former holds true (with  $c_1 = 1$ ) if  $||| \cdot |||$  is the Frobenius norm, whereas the validity of the latter for this norm is still an open question.

**Theorem 1.1.** *Let  $||| \cdot |||$  be a unitarily invariant norm, and suppose there exists a constant  $c_1 \geq 1$  such that*

$$c_1 \left( \min_{W \text{ unitary}} |||W \circ G||| \right) \geq \min_{P \text{ permutation}} |||P \circ G||| \tag{4}$$

*holds for every matrix  $G$ . Then  $||| \cdot |||$  is equivalent to the Frobenius norm, uniformly with respect to the dimension. More precisely*

$$c_1^{-1} \|A\|_F \leq |||A||| \leq c_1 \|A\|_F \quad \text{for every matrix } A. \tag{5}$$

*Similarly, if assumption (4) is replaced by*

$$c_2 \left( \min_{W \text{ unitary}} |||W \circ G||| \cdot |||W^* \circ G^T||| \right) \geq \min_{P \text{ permutation}} |||P \circ G|||^2, \tag{6}$$

*then (5) holds true as well, with  $c_1$  replaced by  $\sqrt{c_2}$ .*

Since (2) and (3) imply (4) and (6), respectively, as an immediate consequence of Theorem 1.1 we obtain the following statement.

**Corollary 1.1.** *Let  $||| \cdot |||$  be a unitarily invariant norm, which satisfies (2) for every matrix  $G$  and for some universal constant  $c_1 \geq 1$ . Then (5) necessarily holds true. Similarly, if (3) is satisfied, then (5) holds true with  $c_1$  replaced by  $\sqrt{c_2}$ .*

The main idea underlying the proof of Theorem 1.1 consists in finding matrices  $G$ , of arbitrarily large dimension, such that the left-hand side of (4) is much smaller than the corresponding right-hand side. This sort of counterexamples can be constructed with some degrees of freedom with respect to some parameters, and the possibility of letting these parameters vary allows one to obtain (5). We just mention that the matrices  $G$  needed to obtain the first inequality in (5) are quite different from those used in [4] to prove the second inequality (in fact, the two constructions are in some sense dual to each other).

Despite of the negative result of Theorem 1.1, the interest for Conjecture 1.1 is far from being exhausted, since a closer look at Li’s [3] motivations for his conjecture reveals that it would be of particular interest to prove it (or disprove it) when  $G$  has some special structure, for instance  $G_{ij} = \lambda_i - \mu_j$  or  $G_{ij} = (\lambda_i - \mu_j) / \sqrt{\lambda_i \mu_j}$ . Note that, in both cases, the rank of  $G$  is at most 2, whereas the matrices we construct in our counterexamples have larger and larger rank as the matrix dimension grows.

In fact, in the particular case where  $G$  has rank 1, we can prove that (2) (as well as a weaker form of (3)) is satisfied by any u.i. norm.

**Theorem 1.2.** *Let  $G$  be a square matrix such that  $\text{rank}(G) = 1$ . Then*

$$\min_{W \text{ nonsingular}} \|W^{-1}\|_2 \cdot |||W \circ G||| = \min_{P \text{ permutation}} |||P \circ G||| \tag{7}$$

and

$$\min_{W \text{ unitary}} |||W \circ G||| \cdot |||W^{-1} \circ G^T||| = \min_{P \text{ permutation}} |||P \circ G|||^2, \tag{8}$$

for every unitarily invariant norm  $||| \cdot |||$ .

The proof is based on the following matrix inequality, which seems to be of some interest in itself.

**Proposition 1.1.** *Let  $X, Y, W$  be square matrices of the same size, such that  $W$  is invertible. Then for every u.i. norm*

$$\|W^{-1}\|_2 \cdot |||XWY||| \geq |||\text{diag}(s^\downarrow(X)) \text{diag}(s^\uparrow(Y))|||, \tag{9}$$

where  $\text{diag}(s^\downarrow(X))$  (respectively,  $\text{diag}(s^\uparrow(Y))$ ) denotes the diagonal matrix with the singular values of  $X$  (respectively, of  $Y$ ) along the diagonal, arranged in non increasing (respectively, non-decreasing) order.

The paper is organized as follows. In Section 2 we prove Theorem 1.1, whereas in Section 3 we prove Proposition 1.1 and Theorem 1.2. Finally, in Section 4 we prove some auxiliary statement which might be useful in further investigations of these kinds of problems.

## 2. Proof of the main result

This section is entirely devoted to the proof of Theorem 1.1.

We introduce two sequences of matrices  $\{U_k\}$  and  $\{V_k\}$  of order  $2^k$ , such that each  $U_k$  is orthogonal, defined as follows:

$$V_0 := [1], \quad V_{k+1} := \begin{bmatrix} V_k & -V_k \\ V_k & V_k \end{bmatrix}, \quad U_k := 2^{-k/2} V_k, \quad k \geq 0. \quad (10)$$

The second inequality in (5) was first established in [4]. Here we shortly recall how it can be obtained.

Choose an integer  $k \geq 1$  and a diagonal matrix  $D$  of order  $2^k$ , and let

$$G := V_k D.$$

Since every entry of  $V_k$  is either 1 or  $-1$  and  $D$  is diagonal, for every permutation  $P$  the singular values of  $P \circ (V_k D)$  coincide with those of  $D$ . Hence

$$\min_{P \text{ permutation}} |||P \circ G||| = |||D||| \quad \text{for every u.i. norm.} \quad (11)$$

On the other hand, since  $D$  is diagonal we have

$$U_k \circ G = U_k \circ (V_k D) = (U_k \circ V_k) D$$

and, since  $U_k \circ V_k$  is the matrix with every entry equal to  $2^{-k/2}$ , the matrix  $(U_k \circ V_k) D$  has rank 1 and its non-trivial singular value equals the Frobenius norm of  $D$ . Recalling (1), we find

$$|||U_k \circ G||| = \|D\|_F \quad \text{for every u.i. norm.} \quad (12)$$

If assumption (4) is satisfied by some u.i. norm, then in particular

$$c_1 |||U_k \circ G||| \geq \min_{P \text{ permutation}} |||P \circ G|||,$$

and from (11) and (12) we obtain  $c_1 \|D\|_F \geq |||D|||$ . Then the second inequality in (5) is established, since  $k$  is arbitrary and  $D$  is an arbitrary diagonal matrix of order  $2^k$  (given a matrix  $A$  of any order, it suffices to take as  $D$  the diagonal matrix with the singular values of  $A$  along the diagonal, and fill it out with zeros until the dimension is a power of 2).

Since  $U_k$  is real hence  $U_k^* = U_k^T$ , the same computations can be repeated with  $U_k^* \circ G^T$  in place of  $U_k \circ G$ . Then assuming (6) instead of (4) and arguing in the same way, one obtains the second inequality in (5), with  $c_1$  replaced by  $\sqrt{c_2}$ .

We now turn our attention to the first inequality in (5). Basic to our construction is the following simple lemma of linear algebra.

**Lemma 2.1.** *Let  $n \geq 2$  be an integer. Then there exist  $n$  vectors in  $\mathbb{R}^{n-1}$ , denoted by  $v_i, i = 1, \dots, n$ , such that*

$$\langle v_i, v_j \rangle = \delta_{ij} - \frac{1}{n}, \quad 1 \leq i, j \leq n. \tag{13}$$

**Proof.** Let  $e_1, \dots, e_n$  be orthonormal vectors in  $\mathbb{R}^n$ , and let

$$b := \frac{1}{n} \sum_{i=1}^n e_i$$

denote their barycentre. Then set  $u_i = e_i - b, i = 1, \dots, n$ . By a straightforward computation, one can check that

$$\langle u_i, u_j \rangle = \delta_{ij} - \frac{1}{n}, \quad 1 \leq i, j \leq n.$$

Note that  $u_i \in \mathbb{R}^n$ . However, since the vectors  $\{u_i\}$  are linearly dependent (indeed, their sum is the null vector), they are contained in some subspace of  $\mathbb{R}^n$  of dimension  $n - 1$ , and this proves our claim due to the invariance of the scalar product.  $\square$

Choose an integer  $k \geq 1$ , let  $n := 2^k$  and let  $G$  be the matrix of order  $2n - 1$  partitioned into blocks as

$$G := \begin{bmatrix} V_k & O_{n,n-1} \\ O_{n-1,n} & O_{n-1,n-1} \end{bmatrix} \in \mathbb{R}^{(2n-1) \times (2n-1)}, \quad n = 2^k, \tag{14}$$

where  $O_{i,j}$  is the null matrix of order  $i \times j$  and  $V_k$  is given by (10). Note that every entry of the block  $V_k$  is  $\pm 1$ ; a simple argument then reveals that, if  $P$  is any permutation matrix of order  $2n - 1$ , then

$$\| \| P \circ G \| \| \geq 1 \quad \text{for every u.i. norm.}$$

Indeed, any permutation matrix  $P$  of order  $2n - 1$  has at least one entry equal to 1 in the upper left block of order  $n$ . Moreover, if  $P$  is the flip matrix of order  $2n - 1$ , then  $P \circ G$  has just one nonzero entry equal to 1, thus we obtain

$$\min_{P \text{ permutation}} \| \| P \circ G \| \| = 1 \quad \text{for every u.i. norm.} \tag{15}$$

Let  $E_k$  denote the matrix of order  $n = 2^k$  with all entries equal to 1, and let  $D$  be an arbitrary diagonal matrix of the same size, different from the null matrix. We claim the existence of an orthogonal matrix of order  $2n - 1$ , whose upper left block of order  $n$  coincides with

$$E_k D / (\sqrt{n} \| D \|_F). \tag{16}$$

To prove this claim, note that the above matrix has equal rows, and that the scalar product of two of them is equal to  $1/n$ : since by Lemma 2.1 there exist  $n$  vectors in

$\mathbb{R}^{n-1}$  satisfying (13), we can use these vectors to complete the rows of block (16), thus obtaining  $n$  orthonormal vectors in  $\mathbb{R}^{2n-1}$ . Finally, the  $n$  rows thus constructed can be completed to an orthonormal basis of  $\mathbb{R}^{2n-1}$ , and our claim follows.

Therefore, let  $U$  denote an orthogonal matrix of order  $2n - 1$ , having the matrix in (16) as upper left block of order  $n$  (note that this block matches the corresponding one in (14)). Hence, recalling (10), we have for every u.i.norm

$$\begin{aligned} |||U \circ G||| &= \|D\|_F^{-1} |||(E_k D/\sqrt{n}) \circ V_k||| = \|D\|_F^{-1} |||(E_k \circ V_k/\sqrt{n}) D||| \\ &= \|D\|_F^{-1} |||(2^{-k/2} V_k) D||| = \|D\|_F^{-1} |||U_k D||| \\ &= \|D\|_F^{-1} |||D|||, \end{aligned}$$

since  $U_k$  is unitary. On taking the minimum, we obtain

$$\min_{U \text{ unitary}} |||U \circ G||| \leq \|D\|_F^{-1} |||D|||.$$

Now, if (4) is satisfied, then from (15) and the last inequality we obtain

$$c_1 |||D||| \geq \|D\|_F$$

and the first inequality in (5) is established, since  $D$  is an arbitrary diagonal matrix of order a power of 2.

Similarly, if (6) holds instead, then the first inequality in (5) follows in the same way (replacing, of course,  $c_1$  with  $\sqrt{c_2}$ ).

### 3. The case of rank-one matrices

Before proving Theorem 1.2, let us introduce some notation. Given a matrix  $A$  of order  $n$ , we let

$$s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$$

denote its singular values arranged in non-increasing order, and we define

$$\|A\|_{(k)} := \sum_{i=1}^k s_i(A), \quad 1 \leq k \leq n,$$

the Ky Fan norm of order  $k$ . Note that  $\|\cdot\|_{(1)}$  coincides with the spectral norm  $\|\cdot\|_2$  (although we use both notations, no confusion should arise).

Given a vector  $x$ , we let  $x^\uparrow$  (respectively,  $x^\downarrow$ ) denote the vector obtained from  $x$  by rearranging its entries in non-decreasing (respectively, non-increasing) order. Finally,  $\text{diag}(x)$  denotes the diagonal matrix with the entries of  $x$  along the main diagonal.

Let  $x, y \in \mathbb{R}^n$ . According to a standard notation (see [1]), we say that  $x$  is weakly submajorized by  $y$ , in symbols  $x \prec_w y$ , if

$$\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad 1 \leq k \leq n.$$

We also write  $y \succ_w x$  when  $x \prec_w y$ .

**Proof of Proposition 1.1.** Given two matrices  $A, B$  it is known (see [1, p. 72]) that

$$\prod_{j=1}^k s_j(AB) \geq \prod_{j=1}^k s_{i_j}(A) s_{n+1-i_j}(B)$$

for all  $1 \leq i_1 < \dots < i_k \leq n$ . Choosing, for every  $1 \leq k \leq n$ , the  $k$ -tuple  $i_1, \dots, i_k$  which maximizes the right-hand side, we obtain using the Hadamard product

$$\prod_{j=1}^k s_j(AB) \geq \prod_{j=1}^k \left( s^\downarrow(A) \circ s^\uparrow(B) \right)_j^\downarrow, \quad 1 \leq k \leq n,$$

which, in its turn, implies (see [1, Example II.3.5]) that

$$s^\downarrow(AB) \succ_w s^\downarrow(A) \circ s^\uparrow(B). \tag{17}$$

Now choose matrices  $X, Y, W$  with  $W$  invertible. Since

$$\|W^{-1}\|_2 \|WYv\| \geq \|Yv\|, \quad v \in \mathbb{C}^n,$$

from the minimax principle for singular values we obtain

$$s_k(WY) \geq \|W^{-1}\|_2^{-1} s_k(Y), \quad 1 \leq k \leq n.$$

Using (17) with the choice  $A := X$  and  $B := WY$  and combining it with the last inequality, one obtains

$$s^\downarrow(XWY) \succ_w \|W^{-1}\|_2^{-1} s^\downarrow(X) \circ s^\uparrow(Y).$$

Writing  $\tilde{W} = W \|W^{-1}\|_2$ , this can be rewritten as

$$s^\downarrow(X\tilde{W}Y) \succ_w s^\downarrow(X) \circ s^\uparrow(Y)$$

and, observing that  $s^\downarrow(X) \circ s^\uparrow(Y)$  is the vector of the singular values of the diagonal matrix  $\text{diag}(s^\downarrow(X)) \text{diag}(s^\uparrow(Y))$ , using Ky Fan norms we can rewrite the last weak submajorization as

$$\|X\tilde{W}Y\|_{(k)} \geq \|\text{diag}(s^\downarrow(X)) \text{diag}(s^\uparrow(Y))\|_{(k)}, \quad 1 \leq k \leq n.$$

Due to the Fan Dominance theorem [1, p. 93] the last inequality is valid for every u.i. norm, and (9) is established.  $\square$

**Proof of Theorem 1.2.** Since  $\text{rank}(G) = 1$ , there exist vectors  $x, y \in \mathbb{C}^n$  such that  $G = [x_i y_j]$ . Now observe that

$$W \circ [x_i y_j] = \text{diag}(x) W \text{diag}(y).$$

Letting  $X = \text{diag}(x)$  and  $Y = \text{diag}(y)$ , inequality (9) yields for every u.i. norm

$$\min_{W \text{ nonsingular}} \|W^{-1}\|_2 \cdot \|W \circ [x_i \ y_j]\| \geq \| \text{diag}(|x|^\downarrow) \ \text{diag}(|y|^\uparrow) \|, \tag{18}$$

where  $|x|$  and  $|y|$  denote the vectors with entries  $\{|x_i|\}$  and  $\{|y_i|\}$ . Observe that

$$\| \text{diag}(|x|^\downarrow) \ \text{diag}(|y|^\uparrow) \| \geq \min_{P \text{ permutation}} \|P \circ [x_i \ y_j]\|, \tag{19}$$

since the left-hand side is achieved when  $P$  is a suitable permutation matrix. On combining (18) and (19), we see that  $\geq$  occurs in (7), hence (7) is established since the opposite inequality is obvious.

To prove (8), note that if  $W$  is unitary, then (18) implies

$$\begin{aligned} \|W \circ [x_i \ y_j]\| &\geq \| \text{diag}(|x|^\downarrow) \ \text{diag}(|y|^\uparrow) \|, \\ \|W^{-1} \circ [y_i \ x_j]\| &\geq \| \text{diag}(|y|^\downarrow) \ \text{diag}(|x|^\uparrow) \|. \end{aligned}$$

Since the two right-hand sides are equal (indeed,  $\text{diag}(|x|^\downarrow) \ \text{diag}(|y|^\uparrow)$  and  $\text{diag}(|y|^\downarrow) \ \text{diag}(|x|^\uparrow)$  are permutationally equivalent), using (19) and repeating the above argument, one obtains (8).  $\square$

**Remark 3.1.** We were not able to prove the stronger statement that

$$\min_{W \text{ nonsingular}} \|W \circ G\| \cdot \|W^{-1} \circ G^T\| = \min_{P \text{ permutation}} \|P \circ G\|^2 \tag{20}$$

for every u.i. norm when  $G$  has rank 1. We point out that (20) is equivalent to the following matrix inequality:

$$\|XWY\| \cdot \|Y^*W^{-1}X^*\| \geq \| \text{diag}(s^\downarrow(X)) \ \text{diag}(s^\uparrow(Y)) \|^2 \tag{21}$$

for every u.i. norm and arbitrary matrices  $X, W, Y$  such that  $W$  is invertible. In fact, invoking the singular value decomposition of  $X$  and  $Y$  and using unitarily invariance, we lose no generality if we assume that  $X, Y$  are real diagonal matrices. Then the technique employed in the proof of Theorem 1.2 reveals that (20) and (21) are indeed equivalent.

In view of this fact, a deeper investigation of the validity of (21) would be desirable.

#### 4. Some further results

In the light of Theorem 1.2 we have a partial solution to the conjecture, in the special case where  $G$  has rank 1. However, the proof techniques of the last section are quite special to the rank-one case, and they do not seem to be useful to shed some light on the more general case where  $\text{rank}(G)$  is bounded from above by some constant (as we have already mentioned in Section 1, the case where  $\text{rank}(G) \leq 2$  would be particularly relevant).

In this section we prove two lemmas which, hopefully, might be of some use to further investigate the conjecture. Finally, we will discuss a particular case where the minimum over unitary matrices is achieved by a permutation.

If  $I, J \subseteq \{1, \dots, n\}$  and  $G$  is a matrix order  $n$ ,  $G_{I,J}$  denotes the matrix of order  $|I| \times |J|$  obtained from  $G$ , cancelling all the rows  $\{r_i\}$  such that  $i \notin I$  and all the columns  $\{c_j\}$  such that  $j \notin J$ .

**Lemma 4.1.** *Let  $G$  be a matrix of order  $n > 1$ . Then there exist two subsets  $I, J \subseteq \{1, \dots, n\}$  such that  $|I| + |J| = n + 1$  and*

$$\min_{P \text{ permutation}} \|P \circ G\|_2 = \min_{\substack{i \in I \\ j \in J}} |g_{ij}|, \tag{22}$$

where we have set  $G = [g_{ij}]$ .

**Proof.** Let  $\mu$  denote the left-hand side of (22). Then, for every permutation  $\sigma$  of order  $n$ , there exists  $i \in \{1, \dots, n\}$  such that  $\mu \leq |g_{i,\sigma_i}|$ . Now define the matrix  $A = [a_{ij}]$  as follows:  $a_{ij} = 0$  if  $\mu \leq |g_{ij}|$ , and  $a_{ij} = 1$  otherwise. Then the König–Frobenius theorem (see [1, p. 37]) yields the existence of a  $k \times l$  submatrix of  $A$  with all entries 0, for some  $k, l$  such that  $k + l > n$ . In particular, this implies the existence of a submatrix  $G_{I,J}$  of  $G$ , with  $|I| + |J| = n + 1$ , such that  $\mu \leq |g_{ij}|$  whenever  $i \in I$  and  $j \in J$ , and hence  $\leq$  occurs in (22) for this choice of  $I$  and  $J$ .

To prove the opposite inequality, let  $\sigma$  be the permutation which yields the minimum of the left-hand side of (22). This means that

$$\mu = \max_{1 \leq i \leq n} |g_{i,\sigma_i}|. \tag{23}$$

Now consider the set of values  $\{\sigma_i\}_{i \in I}$ : these are  $|I|$  pairwise distinct natural numbers in the range  $1, \dots, n$  and, recalling that  $|I| + |J| = n + 1$ , it necessarily holds

$$\{\sigma_i\}_{i \in I} \cap J \neq \emptyset.$$

Choosing  $k \in I$  such that  $\sigma_k \in J$ , from (23) we have

$$\mu \geq |g_{k,\sigma_k}| \geq \min_{\substack{i \in I \\ j \in J}} |g_{ij}|,$$

and hence also  $\geq$  occurs in (22).  $\square$

**Lemma 4.2.** *Suppose  $I, J \subseteq \{1, \dots, n\}$ , and let  $X$  be an invertible matrix order  $n$ . If  $|I| + |J| \geq n + k$  for some  $k \in \{1, \dots, n\}$ , then*

$$\|X^{-1}\|_2 s_k(X_{I,J}) \geq 1, \tag{24}$$

where  $s_k(X_{I,J})$  is the  $k$ th largest singular value of  $X_{I,J}$ . In particular,

$$\|X^{-1}\|_2 \|X_{I,J}\|_{(k)} \geq k, \tag{25}$$

where  $\|\cdot\|_{(k)}$  is the Ky Fan norm of order  $k$ .

**Proof.** By a suitable rearrangement of the rows and columns of  $X$ , we can assume that  $X_{I,J}$  is an upper left block of  $X$ , i.e., that

$$X = \begin{bmatrix} X_{I,J} & A \\ B & C \end{bmatrix}$$

for suitable matrices  $A, B, C$ . Note that  $B$  has  $n - |I|$  rows and  $|J|$  columns. Hence  $\dim \ker(B) \geq k$  due to our assumption, and we can find a subspace  $\mathcal{M} \subseteq \ker(B)$  such that  $\dim(\mathcal{M}) = k$ . If we pick an arbitrary vector  $v \in \mathcal{M}$ , we have  $Bv = 0$  and hence

$$\|X_{I,J}v\| = \left\| X \begin{pmatrix} v \\ 0 \end{pmatrix} \right\|.$$

On the other hand, we have

$$\|v\| = \left\| \begin{pmatrix} v \\ 0 \end{pmatrix} \right\| = \left\| X^{-1}X \begin{pmatrix} v \\ 0 \end{pmatrix} \right\| \leq \|X^{-1}\|_2 \left\| X \begin{pmatrix} v \\ 0 \end{pmatrix} \right\|.$$

On combining the last two lines we obtain from the arbitrariness of  $v \in \mathcal{M}$

$$\inf_{\substack{v \in \mathcal{M} \\ v \neq 0}} \frac{\|X_{I,J}v\|}{\|v\|} \geq \|X^{-1}\|_2^{-1},$$

from which (24) follows, since  $s_k(X_{I,J})$  is equal to the supremum of the left-hand side of the last inequality, over all subspaces  $\mathcal{M}$  of dimension  $k$  (this is the minimax characterization of the  $k$ th largest singular value of a matrix, see [1]).  $\square$

**Remark 4.1.** Inequalities (24) and (25) are sharp. Indeed, if  $X$  is the flip matrix of order  $n$  and  $X_{I,J}$  is an upper left block, then  $X_{I,J}$  has  $\max\{|I| + |J| - n, 0\}$  singular values equal to 1, and the remaining are equal to 0.

We point out that the last two lemmas yield a direct proof of (7), in the case of the spectral norm.

To see this, take  $G$  with  $\text{rank}(G) = 1$ , and let  $I, J \subset \{1, \dots, n\}$  be according to Lemma 4.1. Then we are reduced to proving that

$$\min_{W \text{ nonsingular}} \|W^{-1}\|_2 \cdot \|W \circ G\|_2 \geq \min_{\substack{i \in I \\ j \in J}} |g_{ij}|.$$

Since  $G_{I,J}$  is a submatrix of  $G$ ,  $\text{rank} G_{I,J} \leq 1$  and we can find numbers  $\{x_i\}, \{y_j\}$  such that  $G_{I,J} = [x_i y_j]$ . Since for every  $W$  we have  $\|W \circ G\|_2 \geq \|W_{I,J} \circ G_{I,J}\|_2$ , hence to prove the last inequality it suffices to check that

$$\|W^{-1}\|_2 \cdot \|W_{I,J} \circ G_{I,J}\|_2 \geq \left( \min_{1 \leq i \leq |I|} |x_i| \right) \left( \min_{1 \leq j \leq |J|} |y_j| \right) \quad (26)$$

for every invertible matrix  $W$ . Writing  $W_{I,J} \circ G_{I,J} = \text{diag}(x) W_{I,J} \text{diag}(y)$  (and assuming that  $\text{diag}(x), \text{diag}(y)$  are invertible, otherwise the right-hand side of (26) is 0 and the inequality is trivial), we have from (25) with  $k = 1$

$$\begin{aligned}
 1 &\leq \|W^{-1}\|_2 \cdot \|W_{I,J}\|_2 = \|W^{-1}\|_2 \cdot \|\text{diag}(x)^{-1}(W_{I,J} \circ G_{I,J}) \text{diag}(y)^{-1}\|_2 \\
 &\leq \|W^{-1}\|_2 \cdot \|W_{I,J} \circ G_{I,J}\|_2 \|\text{diag}(x)^{-1}\|_2 \|\text{diag}(y)^{-1}\|_2 \\
 &= \|W^{-1}\|_2 \cdot \|W_{I,J} \circ G_{I,J}\|_2 \left(\min_{1 \leq i \leq |I|} |x_i|\right)^{-1} \left(\min_{1 \leq j \leq |J|} |y_j|\right)^{-1}
 \end{aligned}$$

and (26) follows.

An intermediate step in the proof of equations such as (7) and (8) consists in establishing the weaker statements obtained by replacing “invertible” with “unitary” in the left-hand side. It turns out that this can be done when  $G$  has non-negative entries, at least in the case of the the Ky Fan norm of highest order (also known as the trace norm).

**Theorem 4.1.** *Assume that  $G$  is of the form  $G = \Lambda A \Theta$ , where  $A = [a_{ij}]_{i,j=1}^n$  has non-negative entries and  $\Lambda, \Theta$  are diagonal unitary matrices. Then*

$$\min_{W \text{ unitary}} \|W \circ G\|_{(n)} = \min_{P \text{ permutation}} \|P \circ G\|_{(n)}. \tag{27}$$

**Proof.** Since for every matrix  $X$

$$X \circ G = X \circ (\Lambda A \Theta) = \Lambda (X \circ A) \Theta$$

(which follows from the fact that  $\Lambda$  and  $\Theta$  are diagonal matrices), by unitarily invariance we lose no generality if we assume that  $G = A$  (i.e., that  $G$  has non-negative entries).

Since the spectral norm is dual to the Ky Fan norm of order  $n$  (see [1]), we have

$$\|W \circ G\|_{(n)} = \|W \circ A\|_{(n)} = \sup_{\|X\|_2 \leq 1} |\text{tr}(W \circ A)X| = \sup_{\|X\|_2 \leq 1} \left| \sum_{i,j} w_{ij} a_{ij} x_{ji} \right|.$$

When  $W$  is unitary, choosing  $X = W^*$  one obtains

$$\|W \circ G\|_{(n)} \geq \left| \sum_{i,j} w_{ij} a_{ij} \overline{w_{ij}} \right| = \sum_{i,j} |w_{ij}|^2 a_{ij}.$$

Observing that  $[|w_{ij}|^2]$  is doubly stochastic, by a well-known argument due to Hoffman and Wielandt [2,3] one obtains the desired result.  $\square$

**Remark 4.2.** It is not clear to us whether the above argument can be adapted to handle any Ky Fan norm (and hence any u.i. norm), and whether the result remains valid if the minimum on the left-hand side is extended over all invertible matrices (including, of course, the normalization factor  $\|W^{-1}\|_2$ ).

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