Lower Bounds For the Condition Number of a Real Confluent Vandermonde Matrix\textsuperscript{1}

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ABSTRACT

Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix $V_c$ are established in terms of the dimension $n$ or $n$ and the largest absolute value among all nodes that define the confluent Vandermonde matrix or $n$ and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\text{max}}$ (the largest number of equal nodes), $\kappa_p(V_c)$ behaves no smaller than $O_n((1 + \sqrt{2})^n)$, and than $O_n((1 + \sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\text{max}}$.

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Lower Bounds For the Condition Number of a Real Confluent Vandermonde Matrix

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Abstract

Lower bounds on the condition number \( \kappa_p(V_c) \) of a real confluent Vandermonde matrix \( V_c \) are established in terms of the dimension \( n \) or \( n \) and the largest absolute value among all nodes that define the confluent Vandermonde matrix or \( n \) and the interval that contains the nodes. In particular, it is proved that for any modest \( k_{\max} \) (the largest number of equal nodes), \( \kappa_p(V_c) \) behaves no smaller than \( O(n(1+\sqrt{2})^n) \), and than \( O(n((1+\sqrt{2})^2n)) \) if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest \( k_{\max} \). 

1 Introduction

Given \( n \) numbers \( \alpha_1, \alpha_2, \cdots, \alpha_n \) called nodes, the associated Vandermonde Matrix is defined as

\[
V \overset{\text{def}}{=} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1}
\end{pmatrix}.
\]

(1.1)

It, for example, arises from polynomial interpolation and others [4]. \( V \) is invertible if all nodes \( \alpha_j \) are distinct, i.e., \( \alpha_i \neq \alpha_j \) for \( i \neq j \), but it becomes singular whenever \( \alpha_i = \alpha_j \) for some \( i \neq j \). A generalization of \( V \) for nodes not all of which are distinct is the so-called Confluent Vandermonde Matrices, e.g.,

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
\alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\
\alpha_2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\
\alpha_3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\
\alpha_4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\
\alpha_5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4
\end{pmatrix},
\]

where \( \alpha_1 = \alpha_2 = \alpha_3 \) and \( \alpha_5 = \alpha_6 \). The second, third, and sixth columns are obtained by “differentiating” the previous column. Confluent Vandermonde matrices arise in Hermite

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interpolation [3], for example. Adopting the formulation in [8], we define \textit{confluent Vandermonde matrix} $V_c$ as follows. First

$$\{\alpha_j\}_{j=1}^n \text{ are ordered so that equal nodes are contiguous, i.e.,}$$

$$\alpha_i = \alpha_j \ (i < j) \implies \alpha_i = \alpha_{i+1} = \cdots = \alpha_j.$$  \hfill (1.2)

Define

$$V_c = (f_1(\alpha_1) \ f_2(\alpha_2) \cdots \ f_n(\alpha_n)),$$  \hfill (1.3)

where vector function $f_j(t)$ is defined recursively by

$$f_j(t) = \begin{cases} (1 \ t \ \cdots \ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\ \frac{d}{dt} f_{j-1}(t), & \text{otherwise}, \end{cases}$$  \hfill (1.4)

where "\(^T\)" is the transpose of a vector or matrix. As far as defining $V_c$ is concerned, $\alpha_j$ can be real or complex. But in this paper, we shall focus on real $\alpha_j$. In what follows, $\alpha_j$ and $V_c$, as well as $\alpha_{\max} \overset{\text{def}}{=} \max_j |\alpha_j|$, are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his co-author (see [7] and references therein), and more recently by Tyrtyshnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish several lower bounds on the $\ell_p$-condition number $\kappa_p(V_c) \equiv \|V_c\|_p \|V_c^{-1}\|_p$ in terms of $n$ or $n$ and $\alpha_{\max}$ or $n$ and the interval $[\alpha, \beta]$ that contains all nodes. In particular, we will show that for fixed $k_{\max}$ (the largest number of equal nodes), $\kappa_p(V_c)$ behaves no smaller than $O_n((1 + \sqrt{2})^n)$, where notation $a_n = O_n(b_n)$ means $c_1 n^{d_1} \leq a_n/b_n \leq c_2 n^{d_2}$ for some constants $c_1, c_2, d_1,$ and $d_2$. We also obtain a qualitative plot in Figure 1.1 to show how our lower bounds on $\min \alpha_j \kappa_p(V_c)$ and $\min_{\alpha_j \geq 0} \kappa_p(V_c)$ behave qualitatively as functions of $\alpha_{\max}$. What Figure 1.1 says that initially as $\alpha_{\max}$ increases, our lower bound for $\min \alpha_j \kappa_p(V_c)$ and that for $\min_{\alpha_j \geq 0} \kappa_p(V_c)$ decrease until at $\alpha_{\max} = \alpha_{\text{opt}}$ when global minimums of the bounds are reached, and then they start climbing again. Notice $\alpha_{\text{opt}}$ may be different for the two cases, but $\alpha_{\text{opt}} = O(1)$ in both cases. What that is not clear, however, is how sharp our lower bounds are, in contrast to many of those in [10] for Vandermonde matrices that were proved to be asymptotically optimal.

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which $V_c$ is lower triangular and thus

$$\kappa_p(V_c) \geq (n-1)! \sim \sqrt{2\pi} n^{n-1/2} e^{-n}$$

by Stirling’s asymptotic formula [1, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to $\infty$ as fast as $(1 + \sqrt{2})^n$ modulo a factor $n^d$ for $d \leq 1 [2, 10]$.

The rest of this paper is organized as follows. Section 2 reviews some preliminary results from [10] in connection to the coefficients of translated Chebyshev polynomials. Two general
Figure 1.1: Qualitative behaviors of our lower bounds on \( \min_{\alpha_j} \kappa_p(V_c) \) and \( \min_{\alpha_j \geq 0} \kappa_p(V_c) \) (subject to fixed \( \alpha_{\text{max}} \)) as \( \alpha_{\text{max}} \) varies for any given \( k_{\text{max}} \).

lower bounds on \( \kappa_p(V_c) \) are established in Section 3, but they are not uniform. Uniform bounds for \( p = \infty \) are obtained in Section 4 for all real \( V_c \) and in Section 5 for \( V_c \) with nonnegative nodes.

2 Preliminaries

Let us start by briefly reviewing relevant notation and results from [10]. Let \([\alpha, \beta]\) be the interval in which lie all \( \alpha_j \). \( T_n(t) = \cos(n \arccos t) \) is the \( n \)th Chebyshev polynomial of the first kind. Define the \( n \)th translated Chebyshev polynomial \( T_n(x; \omega, \tau) \) def = \( T_n(x/\omega + \tau) \), where

\[
\omega = \frac{\beta - \alpha}{2}, \quad \tau = -\frac{\beta + \alpha}{\beta - \alpha}.
\]

Let \( a_{jn} \equiv a_{jn}(\omega, \tau) \) be the coefficient of \( x^j \) in \( T_n(x; \omega, \tau) \), i.e.,

\[
T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,
\] (2.1)

Let \( 1 \leq p \leq \infty \). In [10], \( S_{n,p}(\omega, \tau) \) is defined as

\[
S_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}.
\]

Also explicit formulas were found for \( p = 1 \) and \( -\alpha = \beta \) (for which \( \omega = 0 \)):

\[
S_{n,1}(\omega, 0) = \frac{1}{2} \left[ \left( \frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n + (-1)^n \left( \frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^{-n} \right], \quad (2.2)
\]
\[
\sim \frac{1}{2} \left( \frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n. \tag{2.3}
\]

and for \(\alpha \geq 0\) (for which \(\tau \leq -1\)):

\[
S_{n,1}(\omega, \tau) = \frac{1}{2} \left[ \left( \frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left( \frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^n + \frac{1}{2} \left[ \left( \frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left( \frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^{-n}, \tag{2.4}
\]

\[
\sim \frac{1}{2} \left[ \left( \frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left( \frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^n. \tag{2.5}
\]

For any other \(p\), \(S_{n,p}(\omega, \tau)\) relates to \(S_{n,1}(\omega, \tau)\) by

\[
(n + 1)^{-1/p} S_{n,1}(\omega, \tau) \leq S_{n,p}(\omega, \tau) \leq S_{n,1}(\omega, \tau), \tag{2.6}
\]

\[
[(n + 1)/2]^{-1/p'} S_{n,1}(\omega, 0) \leq S_{n,p}(\omega, 0) \leq S_{n,1}(\omega, 0), \tag{2.7}
\]

where \(1/p + 1/p' = 1\), and \([\xi]\) is the smallest integer that is larger than \(\xi\).

The \(\ell_p\)-norm of vector \(u = (\mu_1 \mu_2 \cdots \mu_n)^T\) is defined as

\[
\|u\|_p = \left( \sum_{j=1}^{n} |\mu_j|^p \right)^{1/p},
\]

and \(\|u\|_\infty = \lim_{p \to \infty} \|u\|_p = \max_j |\mu_j|\). The associated \(\ell_p\)-operator norm of \(m \times n\) matrix \(A\) is defined as

\[
\|A\|_p = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p}. \tag{2.8}
\]

It can be proved that [9] \(\|A\|_p = \|A^T\|_{p'}\), and [10]

\[
n^{-2/p} \kappa_p(V_c) \leq \kappa_\infty(V_c) \leq n^{2/p} \kappa_p(V_c) \tag{2.9}
\]

which is useful in deriving bounds on \(\kappa_p(V_c)\) from these for \(\kappa_\infty(V_c)\) as in Sections 4 and 5.

### 3 Lower bounds on \(\kappa_p(V_c)\)

For the sake of presentation, we assume, in addition to (1.2),

```
There are \(\ell\) distinct nodes \(\alpha_j\), having multiplicities \(k_1, k_2, \ldots, k_\ell\), respectively, where \(k_1 + k_2 + \cdots + k_\ell = n\). This implies that the first \(k_1\) \(\alpha_j\)'s are equal, the next \(k_2\) \(\alpha_j\)'s are equal, and so on. Define \(k_{\max} = \max_j k_j\). \tag{3.1}
```
Lemma 3.1 Assume (1.2) and (3.1). Then
\[
\|V_c\|_p \geq \max \left\{ \frac{c^1/p'}{\alpha_{\max}^{n-1}} \right\},
\]
\[
\|V_c\|_p \geq \left( \sum_{j=1}^{n} \alpha_{\max}^{(j-1)p} \right)^{1/p}.
\] (3.2) (3.3)

Proof: Let \( e_j \) be the \( j \)-th column of the \( n \times n \) identity matrix \( I_n \) (or simply \( I \) if \( n \) is clear from the context). Use \( \|V_c\|_p \geq \|V_c^T e_1\|_{p'} \) and \( \|V_c\|_p \geq \|V_c^T e_n\|_{p'} \) to get (3.2), and use \( \|V_c\|_p \geq \max_j \|V_c^T e_j\|_p \) to get (3.3).

Lemma 3.2 For \( 0 \leq k \leq n \),
\[
\left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| \leq \frac{[n(n-1)\cdots(n-k+1)]^2}{\omega^k} \text{ for } x \in [\alpha, \beta].
\] (3.4)

Proof: It follows from \( T_n(x; \omega, \tau) = T_n(x/\omega + \tau) \equiv T_n(t) \) that
\[
\frac{d^k}{dx^k} T_n(x; \omega, \tau) = \frac{1}{\omega^k} T_n^{(k)}(t),
\]
where \( t \equiv t(x) = x/\omega + \tau \). It suffices to show that \( |T_n^{(k)}(t)| \leq |n(n-1)\cdots(n-k+1)|^2 \) for \( t \in [-1, 1] \) since \( t(x) \) maps \( x \in [\alpha, \beta] \) to \( t \in [-1, 1] \). By Markov’s Inequality [5, Page 233],
\[
\max_{t \in [-1, 1]} |T_n^{(k)}(t)| \leq (n-k+1)^2 \max_{t \in [-1, 1]} |T_n^{(k-1)}(t)|
\]
\[
\leq \cdots
\]
\[
\leq |n(n-1)\cdots(n-k+1)|^2 \max_{t \in [-1, 1]} |T_n(t)|
\]
\[
= |n(n-1)\cdots(n-k+1)|^2,
\]
as expected.

Lemma 3.3 Assume (1.2) and (3.1). Then
\[
\|V_c^{-1}\|_p \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^{2} \omega^{k-1} \times \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}.
\] (3.5)

Proof: Let \( v \) be the vector of the coefficients of the translated Chebyshev polynomial \( T_{n-1}(x; \omega, \tau) \), i.e., \( v = (a_{0_{n-1}} a_{1_{n-1}} \cdots a_{n-1_{n-1}})^T \). Then
\[
V_c^T v = (T_{n-1}(\alpha_1; \omega, \tau) \ T_{n-1}(\alpha_1; \omega) \ \cdots \ T_{n-1}^{(k_1-1)}(\alpha_1; \omega) \ \cdots )^T
\]
which yields, by Lemma 3.2, for \( 1 \leq p' < \infty \)
\[
\|V_c^T v\|_{p'} \leq \sum_{j=1}^{t} \left( 1^{p'} + \left[ \frac{(n-1)!}{\omega} \right]^{p'} + \cdots + \left[ \frac{(n-1)(n-2)\cdots(n-k_j+1)!}{\omega^{k_j-1}} \right]^{p'} \right)
\] (3.6)
Assume follows from (3.3) and (3.5) that \[ \| \text{estimating} \] of the lower bounds for \( H \) by Hölder inequality. This yields (3.10).

**Proof:** Inequality (3.9) is a consequence of Lemmas 3.1 and 3.3. We now prove (3.10).

Let \( \omega = \eta \alpha_{\text{max}} \). Then

\[ \kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left( \frac{(n-k)!}{(n-1)!} \right)^2 \omega^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\text{max}} n \} S_{n-1,p'}(\omega, \tau) n^{1/p'} . \]  

(3.9)

Let \( \omega = \eta \alpha_{\text{max}} \). Then

\[ \kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left( \frac{(n-k)!}{(n-1)!} \right)^2 \omega^{k-1} \times \frac{S_{n-1,1}(\eta, \tau)}{n^{1/p'}} . \]  

(3.10)

**Proof:** Inequality (3.9) is a consequence of Lemmas 3.1 and 3.3. We now prove (3.10). Since \( a_{jn}(\omega, \tau) = \omega^{-j} a_{jn}(1, \tau) \) [10], \( S_{n-1,p'} = \left( \sum_{j=0}^{n-1} |\omega|^{-jp'} |a_{jn-1}(1, \tau)|^{p'} \right)^{1/p'} \). Therefore it follows from (3.3) and (3.5) that

\[ n^{1/p'} \max_{1 \leq k \leq k_{\text{max}}} \left( \frac{(n-1)!}{(n-1)!} \right)^2 \omega^{k-1} \times \kappa_p(V_c) \]

\[ \geq \left( \sum_{j=0}^{n-1} \alpha_{\text{max}}^{jp'} \right)^{1/p'} \left( \sum_{j=0}^{n-1} |\omega|^{-jp'} |a_{jn-1}(1, \tau)|^{p'} \right)^{1/p'} \]

\[ \geq \sum_{j=0}^{n-1} |\eta|^{-j} |a_{jn-1}(1, \tau)| \]

\[ = S_{n-1,1}(\eta, \tau) , \]

by Hölder inequality. This yields (3.10).

For \( k_{\text{max}} = 1 \), i.e., \( \ell = n \) and \( k_1 = s = k_n = 1 \) (and thus \( V_c = V \)), (3.9) becomes one of the lower bounds for \( \kappa_p(V) \) in [10]. In general, we may also use (3.6), instead of (3.7), in estimating \( \| V_c^{-1} \|_p \). Doing so will lead to a more complicated lower bound on \( \kappa_p(V_c) \).
Corollary 3.1 Assume (1.2) and (3.1).

\[ \kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \alpha_{\text{max}}^{k-1} \frac{S_{n-1,1}(1,0)}{n^{1/p'}}. \]  

(3.11)

If, in addition, all \( \alpha_j \geq 0 \), then

\[ \kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \frac{\alpha_{\text{max}}^{k-1}}{2} \frac{S_{n-1,1}(1/2,1)}{n^{1/p'}}. \]  

(3.12)

Proof: Apply (3.10) to \([\alpha, \beta] = [-\alpha_{\text{max}}, \alpha_{\text{max}}]\) (and thus \( \eta = 1, \tau = 0, \) and \( \omega = \alpha_{\text{max}} \)) to get (3.11). Apply (3.10) to \([\alpha, \beta] = [0, \alpha_{\text{max}}]\) (and thus \( \eta = 1/2, \tau = -1, \) and \( \omega = \alpha_{\text{max}}/2 \)) to get (3.12).

Neither bounds in (3.11) and (3.12) are uniform, because both depend on \( \alpha_{\text{max}} \). They do not yield useful lower bounds on \( \min_{\alpha_j} \kappa_p(V_c) \) or \( \min_{\alpha_j \geq 0} \kappa_p(V_c) \). In fact, the minimums of both right-hand sides of (3.11) and (3.12) over either all \( \alpha_j \in \mathbb{R} \) or all \( \alpha_j \geq 0 \) are zero! In the next two sections, we shall establish two uniform bounds using (3.9) for all real \( V_c \) and for those with nonnegative nodes.

Remark 3.1 Lemma 3.3 is made possible by Lemma 3.2 which was proved with the help of Markov’s Inequality. Another classical inequality for the same purpose is Bernstein’s Inequality [5, Page 233], using which we can obtain the following: For \( 0 \leq k \leq n \), if \( \alpha < a \overset{\text{def}}{=} \min_j \alpha_j < b \overset{\text{def}}{=} \max_j \alpha_j < \beta \), then

\[ \left| \frac{d}{dx} T_n(x; \omega, \tau) \right| \leq \frac{n(n-1)s(n-k+1)}{\left[ \omega \sqrt{1 - \left( \frac{\max \{\beta-b, a-\alpha\}}{\omega} \right)^2} \right]^k} \quad \text{for} \quad x \in [\alpha, \beta]. \]  

(3.13)

This inequality improves (3.4) in the numerator part, but has complications in the denominator, and also it requires the interval \([\alpha, \beta]\) be (slightly) larger than the smallest interval containing all nodes, as follows.

This can be bad because larger \([\alpha, \beta]\) will weaken the effectiveness of \( S_{n,p'}(\omega, \tau) \) in the later bounds on \( \kappa_p(V_c) \), for example \( S_{n,p'}(\omega, \tau) \) is decreasing in \( \omega \) [10].

4 Uniform bounds for all real \( V_c \)

We’ll restrict ourselves to \( p = \infty \) because the availability of formulas for \( S_{n,1} \) in Section 2 that allow us to do analysis below. Equivalent relation (2.9) makes it possible to derive uniform lower bounds on \( \kappa_p(V_c) \) for \( p \neq \infty \).
Let $\Phi$ be the right-hand side of (3.9) with $-\alpha = \beta = \alpha_{\max}$ for $p = \infty$ (and thus $p' = 1$):
\[
\Phi = \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n - k)!}{(n - 1)!} \right]^2 \alpha_{\max}^{k-1} \times \max\{\ell, \alpha_{\max}^{n-1} S_{n-1,1}(\alpha_{\max}, 0) / n\} \equiv \max\{\Phi_1, \Phi_2\},
\]
where
\[
\Phi_1 = \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n - k)!}{(n - 1)!} \right]^2 \alpha_{\max}^{k-1} \times \frac{\ell S_{n-1,1}(\alpha_{\max}, 0)}{n}, \tag{4.1}
\]
\[
\Phi_2 = \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n - k)!}{(n - 1)!} \right]^2 \alpha_{\max}^{k-1} \times \frac{\alpha_{\max}^{n-1} S_{n-1,1}(\alpha_{\max}, 0)}{n}. \tag{4.2}
\]

$\Phi$ is $\Phi_1$ for $\alpha_{\max} \leq \ell^{1/(n-1)}$ and $\Phi_2$ for $\alpha_{\max} \geq \ell^{1/(n-1)}$. $\Phi_2$ is increasing in $\alpha_{\max}$ for $\alpha_{\max} > 0$ because $\alpha_{\max}^{n-1} S_{n-1,1}(\alpha_{\max}, 0)$ is increasing in $\alpha_{\max}$ for $\alpha_{\max} > 0$ [10]. Lemma 4.1 shows that $\Phi_1$ is decreasing in $\alpha_{\max}$ for $\alpha_{\max} \leq 1$ for $k_{\max}$ approximately no bigger than $1 + (n - 1)/\sqrt{2}$. Between $1 \leq \alpha_{\max} \leq \ell^{1/(n-1)}$, $\Phi = \Phi_1 \geq \Phi_2$ and $\Phi_1 = \mathcal{O}_n(\Phi_2)$ because for $1 \leq \alpha_{\max} \leq \ell^{1/(n-1)},$
\[
\left[ \frac{(n - k_{\max})!}{(n - 1)!} \right]^2 \frac{\alpha_{\max}^{k_{\max}}}{n} \leq \Phi_1 \leq \frac{n S_{n-1,1}(\alpha_{\max}, 0)}{n},
\]
\[
\left[ \frac{(n - k_{\max})!}{(n - 1)!} \right]^2 \frac{\alpha_{\max}^{k_{\max}}}{n} \leq \Phi_2 \leq \frac{n S_{n-1,1}(\alpha_{\max}, 0)}{n},
\]
noting that $\Phi_2$ is increasing in $\alpha_{\max}$ and that $S_{n-1,1}(\alpha_{\max}, 0)$ is decreasing in $\alpha_{\max}$. These inequalities, together with,
\[
n^{1/(n-1)} = 1 + \frac{\ln(n-1)}{n-1} + \frac{2 + \ln^2(n-1)}{2(n-1)^2} + \cdots,
\]
\[
S_{n-1,1}(n^{1/(n-1)}, 0) \sim \frac{(1 + \sqrt{2})^{n-1}}{2(n-1)^{1/\sqrt{2}}},
\]
imply $\Phi_1 = \mathcal{O}_n(\Phi_2)$ for $1 \leq \alpha_{\max} \leq \ell^{1/(n-1)}$. Therefore we have the qualitative plot in Figure 1.1 for $\Phi$.

**Lemma 4.1** Let $k \geq 0$. $\rho^k S_{n-1,1}(\rho, 0)$ is decreasing in $\rho$ for $0 \leq \rho \leq 1$ if
\[
k \leq \frac{n - 1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2n+2} \right] \sim \frac{n - 1}{\sqrt{2}}. \tag{4.3}
\]

**Proof:** We claim that under (4.3) $\frac{d}{d\rho} \rho^k S_{n-1,1}(\rho, 0) \leq 0$ for $0 \leq \rho \leq 1$. To this end, we notice
\[
\frac{d}{d\rho} \rho^k S_{n-1,1}(\rho, 0) = k \rho^{k-1} S_{n-1,1}(\rho, 0) + \rho^k \frac{d}{d\rho} S_{n-1,1}(\rho, 0).
\]

Now for $0 \leq \rho \leq 1$, by (2.2), we have
\[
S_{n-1,1}(\rho, 0) \leq \frac{1}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{n-1} \left[ 1 + e^{-2n+2} \right],
\]
\[
- \frac{d}{d\rho} S_{n-1,1}(\rho, 0) \geq \frac{n - 1}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{n-2} \left[ 1 - \delta^{-2n+2} \right]
\]
\[
\times \left[ \frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1 + \rho^2}} \right].
\]
where $\epsilon = 1 + \sqrt{2}$ and $\delta = 0$ for even $n - 1$, and $\epsilon = 0$ and $\delta = 1 + \sqrt{2}$ for odd $n - 1$. Therefore for $\rho \leq 1$

$$
\frac{d}{d\rho} \rho^k S_{n-1,1}(\rho, 0) \leq \frac{k}{n-1} + \rho \frac{d}{d\rho} S_{n-1,1}(\rho, 0) \\
\leq \frac{k}{n-1} - \frac{\rho}{1 + \sqrt{1 + \frac{1}{\rho^2}}} \left[ 1 - \delta^{-2n+2} \right] \\
\leq \frac{k}{n-1} - \frac{\rho}{\sqrt{1 + \rho^2}} \left[ 1 - \delta^{-2n+2} \right] \\
\leq \frac{k}{n-1} - \frac{1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2n+2} \right] \\
= 0
$$

upon using (4.3).

Theorem 4.1 If (4.3) with $k = k_{\text{max}} - 1$ holds, then

$$
\kappa_{\infty}(V_c) \geq \left[ \frac{(n - k_{\text{max}})!}{(n-1)!} \right]^2 \left[ \frac{\alpha_{\text{max}}}{2} \right]^{k-1} \max \{\ell, \alpha_{\text{max}}^{-1} S_{n-1,1}(\alpha_{\text{max}}/2, 1) / n\} = \max \{\Psi_1, \Psi_2\},
$$

Proof: It can be verified that $\Phi \geq \Phi_2|_{\alpha_{\text{max}}=1}$.

5 Uniform bounds for $V_c$ with $\alpha_i \in [\alpha, \beta]$ and $0 = \alpha < \beta$

Let $\Psi$ be the right-hand side of (3.9) with $0 = \alpha < \beta = \alpha_{\text{max}}$ for $p = \infty$ (and thus $p' = 1$):

$$
\Psi = \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n - k)!}{(n-1)!} \right]^2 \left[ \frac{\alpha_{\text{max}}}{2} \right]^{k-1} \max \{\ell, \alpha_{\text{max}}^{-1} S_{n-1,1}(\alpha_{\text{max}}/2, 1) / n\} = \max \{\Psi_1, \Psi_2\},
$$

where

$$
\Psi_1 = \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n - k)!}{(n-1)!} \right]^2 \left[ \frac{\alpha_{\text{max}}}{2} \right]^{k-1} \frac{\ell S_{n-1,1}(\alpha_{\text{max}}/2, 1)}{n},
$$

$$
\Psi_2 = \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n - k)!}{(n-1)!} \right]^2 \left[ \frac{\alpha_{\text{max}}}{2} \right]^{k-1} \frac{\alpha_{\text{max}}^{n-1} S_{n-1,1}(\alpha_{\text{max}}/2, 1)}{n}.
$$

$\Psi$ is $\Psi_1$ for $\alpha_{\text{max}} \leq \ell^{1/(n-1)}$ and $\Psi_2$ for $\alpha_{\text{max}} \geq \ell^{1/(n-1)}$. $\Psi_2$ is increasing in $\alpha_{\text{max}}$ for $\alpha_{\text{max}} > 0$ because $\alpha_{\text{max}}^{n-1} S_{n-1,1}(\alpha_{\text{max}}/2, 1)$ is increasing in $\alpha_{\text{max}}$ for $\alpha_{\text{max}} > 0$ [10]. Lemma 5.1 shows that $\Psi_1$ is decreasing in $\alpha_{\text{max}}$ for $\alpha_{\text{max}} \leq 1$ for $k_{\text{max}}$ approximately no bigger than
1 + (n − 1)/√2. Between 1 ≤ α_{max} ≤ μ_{1/n−1}, Ψ = Ψ_{1} ≥ Ψ_{2} and Ψ_{1} = O_{n}(Ψ_{2}) because for
1 ≤ α_{max} ≤ μ_{1/n−1},
\[ \left(\frac{n - k_{max}}{(n - 1)!}\right)^{2} \frac{1}{2^{k_{max}-1}} \frac{S_{n-1,1}(n^{1/(n-1)}/2,1)}{n} \leq \Psi_{1} \leq \frac{n S_{n-1,1}(1/2,1)}{n}, \]
\[ \left(\frac{n - k_{max}}{(n - 1)!}\right)^{2} \frac{1}{2^{k_{max}-1}} \frac{S_{n-1,1}(1/2,1)}{n} \leq \Psi_{2} \leq \frac{n S_{n-1,1}(n^{1/(n-1)}/2,1)}{n}, \]
noting that Ψ_{2} is increasing in α_{max} and that S_{n-1,1}(α_{max}/2,1) is decreasing in α_{max}. These
inequalities, together with
\[ n^{1/(n-1)} = 1 + \frac{\ln(n - 1)}{n - 1} + \frac{2 + \ln^{2}(n - 1)}{2(n - 1)^{2}} + \cdots, \]
\[ S_{n-1,1}(n^{1/(n-1)}/2,1) \sim \frac{(3 + 2\sqrt{2})^{n-1}}{2(n - 1)^{1/\sqrt{2}}}, \]
imply Ψ_{1} = O_{n}(Ψ_{2}) for 1 ≤ α_{max} ≤ μ_{1/n−1}. Therefore we have the qualitative plot in
Figure 1.1 for Ψ.

**Lemma 5.1** Let k ≥ 0. ρ^{k}S_{n-1,1}(ρ/2,1) is decreasing in ρ for 0 ≤ ρ ≤ 1 if
\[ k \leq \frac{n - 1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-4(n-1)} \right]^{-1} \sim \frac{n - 1}{\sqrt{2}}. \]  
(5.3)

**Proof:** We claim that under (5.3) \( \frac{d}{d\rho} \rho^{k}S_{n-1,1}(\rho/2,1) \leq 0 \) for 0 ≤ ρ ≤ 1. To this end, we
notice
\[ \frac{d}{d\rho} \rho^{k}S_{n-1,1}(\rho/2,1) = k\rho^{k-1}S_{n-1,1}(\rho/2,1) + \rho^{k} \frac{d}{d\rho} S_{n-1,1}(\rho/2,1). \]
Now for 0 ≤ ρ ≤ 1, by (2.4), we have
\[ S_{n-1,1}(\rho/2,1) \leq \frac{1}{2} \left[ \frac{2}{\rho} + 1 + \sqrt{\left( \frac{2}{\rho} + 1 \right)^{2}} - 1 \right]^{n-1} \left[ 1 + (3 + 2\sqrt{2})^{-2n+2} \right], \]
\[ -\frac{d}{d\rho} S_{n-1,1}(\rho/2,1) \geq \frac{n - 1}{2} \left[ \frac{2}{\rho} + 1 + \sqrt{\left( \frac{2}{\rho} + 1 \right)^{2}} - 1 \right]^{n-2} \times \frac{2}{\rho^{2}} \left[ 1 + \frac{2 + \rho}{2\sqrt{1 + \rho}} \right]. \]
Therefore for ρ ≤ 1
\[ \frac{d}{d\rho} \rho^{k}S_{n-1,1}(\rho/2,1) \leq \frac{k}{(n - 1)} + \frac{\rho \frac{d}{d\rho} S_{n-1,1}(\rho/2,1)}{(n - 1)S_{n-1,1}(\rho/2,1)} \]
\[ \leq \frac{k}{(n - 1)} - \frac{\rho^{2} \frac{\rho}{\rho^{2}} \left[ 1 + \frac{2 + \rho}{2\sqrt{1 + \rho}} \right]}{\frac{2}{\rho} + 1 + \sqrt{\left( \frac{2}{\rho} + 1 \right)^{2}} - 1} \left[ 1 + (3 + 2\sqrt{2})^{-2n+2} \right] \]
\[
\begin{align*}
\frac{k}{n-1} - \frac{1}{\sqrt{1 + \rho}} \left[ 1 + (3 + 2\sqrt{2})^{-2n+2} \right]^{-1} \\
\leq \frac{k}{n-1} - \frac{1}{\sqrt{2}} \left[ 1 + (3 + 2\sqrt{2})^{-2n+2} \right]^{-1} \\
\leq 0
\end{align*}
\]

upon using (5.3). □

**Theorem 5.1** If (5.3) with \( k = k_{\text{max}} - 1 \) holds and all \( \alpha_i \geq 0 \), then

\[
\kappa_{\infty}(V_c) \geq \left[ \frac{(n-k_{\text{max}})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} S_{n-1,1}(1/2,1) \frac{S_{n-1,1}(1/2,1)}{n} \]

\[
\sim \left[ \frac{(n-k_{\text{max}})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{1 + \sqrt{2}}{2(n-1)} \frac{1 + \sqrt{2}}{n}.
\]

**Proof:** It can be verified that \( \Psi \geq \Psi_2|_{\alpha_{\text{max}}=1} \). □

### 6 Conclusions

We have obtained several lower bounds on the condition number \( \kappa_p(V_c) \) of a real confluent Vandermonde matrix \( V_c \). Two of them are uniform in the sense that they depend on \( n \), the dimension of \( V_c \) only, while the others are either functions of \( n \) and \( \alpha_{\text{max}} \) or \( n \) and the interval \([\alpha, \beta]\) that contains all \( \alpha_j \). These bounds grow exponentially for any fixed \( k_{\text{max}} \), much as expected. Qualitative behaviors of our general lower bound (3.9) for \(-\alpha = \beta = \alpha_{\text{max}}\) and for \( 0 = \alpha < \beta = \alpha_{\text{max}}\) are plotted in Figure 1.1. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if \( k_{\text{max}} \) also grows, e.g., linearly in \( n \). This is illustrated by the extreme example \( k_{\text{max}} = n \), as we commented in Section 1.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an tuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [6, 11]. We plan to investigate this issue in future work.
References


