ASYMPTOTICALLY OPTIMAL LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL VANDERMONDE MATRIX

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Abstract. Lower bounds on the condition number \( \min \kappa_p(V) \) of a real Vandermonde matrix \( V \) are established in terms of the dimension \( n \) or \( n \) and the largest absolute value among all nodes that define the Vandermonde matrix. All bounds here are asymptotically sharp, similar to those in Beckermann (Numer. Math., 85 (2000), 553–577). But bounds here are sharper at least for \( p = \infty \), and cover more cases. Also, qualitative behaviors of \( \min \kappa_p(V) \), as well as nearly optimally conditioned real Vandermonde matrices, as functions of the largest absolute value among all nodes are obtained.

The technique used here is extensible to deal with confluent Vandermonde matrices and rectangular Vandermonde matrices.

Key words. Optimal condition number, Vandermonde matrix, confluent Vandermonde matrix, Chebyshev polynomials

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1. Introduction. Given \( n \) numbers \( \alpha_1, \alpha_2, \cdots, \alpha_n \) called nodes, the associated Vandermonde Matrix is defined as

\[
V \overset{\text{def}}{=} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1}
\end{pmatrix}.
\]

(1.1)

It is perhaps one of the best known structural matrices, arising from polynomial interpolation and others [2]. It is invertible if all nodes \( \alpha_j \) are distinct, i.e., \( \alpha_i \neq \alpha_j \) for \( i \neq j \). Vandermonde matrices are notoriously ill-conditioned [14, p.428], [11], i.e., its condition number can become arbitrarily large, even for modest \( n \). This is not surprising because moving one node arbitrarily close to another will make \( V \) arbitrarily close to a singular matrix. Therefore the question of importance about \( V \) is not how bad a Vandermonde matrix \( V \) can be but rather what one can hope for at best from \( V \) as far as its condition number is concerned.

Although \( V \) is well-defined no matter if all or some of \( \alpha_j \) are real or complex, this paper is confined to real Vandermonde matrix \( V \) only, i.e., all \( \alpha_j \) are real, except briefly in Section 8. Throughout this paper, some notation is exclusively reserved for one assignment, including \( V \) and its nodes \( \alpha_j \) and \( \alpha_{\text{max}} \overset{\text{def}}{=} \max |\alpha_j| \), along with many others in Table 1.1. \( V_{\text{sym}} \) is one of those \( V \) whose nodes are real symmetric with respect to 0, i.e., \( \alpha_i + \alpha_{n-i+1} = 0 \).

The major objective of this paper is to bound the \( \ell_p \)-condition number \( \kappa_p(V) = \|V\|_p \|V^{-1}\|_p \) from below, in terms of \( n \) or \( n \) and \( \alpha_{\text{max}} \). Asymptotically optimal bounds have been established. By asymptotically optimal bounds we mean those that will give

\[
\rho \equiv \text{asymptotic speed} \overset{\text{def}}{=} \lim_{n \to \infty} \left[ \min \kappa_p(V) \right]^{1/n}
\]

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exactly, where \( \min \) is taken over some prescribed subset or the entire set of Vandermonde matrices. This is done through establishing bounds like

\[
\alpha \min \kappa_p(V) = O(1)
\]

written for short as \( \min \kappa_p(V) = O_n(\rho^n) \), where \( c_1, c_2, d_1, \) and \( d_2 \) are constants. Particular attention will be given to the case \( p = \infty \). In a sense, considering \( p = \infty \) is sufficient because of the exponential growth of \( \kappa_\infty(V) \) and because (see Section 3)

\[
n^{-2/p} \kappa_p(V) \leq \kappa_\infty(V) \leq n^{2/p} \kappa_p(V),
\]

and thus they all have the same asymptotic speed. Nonetheless, whenever it is possible to establish sharper bounds on \( \kappa_p(V) \) directly instead of indirectly through bounds on \( \kappa_\infty(V) \) combined with (1.4), we shall go for the sharper ones.

In the past, Gautschi and his coauthor had systematically studied the condition number estimation in [6, 7, 8, 9, 12], where various condition number bounds in terms of the nodes \( \alpha_j \) have been established, as well as bounds in terms of dimension \( n \) only. In [12] two lower bounds in terms of \( n \) were obtained for positive nodes \( (\alpha_j \geq 0) \) and real symmetric nodes \( (\alpha_j + \alpha_{n+1-j} = 0) \). A similar bound was obtained by Tyrtyshnikov [21], too, but the proofs there were incomplete. Bounds in [12, 21] are far from asymptotically optimal, however. It is Beckermann [1] in 2000 who obtained asymptotically optimal condition number estimations for real Vandermonde matrices for the first time.

This paper is based on the technical report [18] which was written before the author came across Beckermann’s landmark paper [1]. But we have more detailed and refined analysis and cover more cases, and tighter lower and upper bounds, too. Specifically, major differences are as follows.

- We obtain a qualitative plot in Figure 1.1 which shows how \( \min_{\alpha_j} \kappa_p(V) \) and \( \min_{\alpha_j \geq 0} \kappa_p(V) \) subject to a fixed \( \alpha_{\text{max}} \) behave qualitatively as functions of \( \alpha_{\text{max}} \). What Figure 1.1 says that initially as \( \alpha_{\text{max}} \) increases, both \( \min_{\alpha_j} \kappa_p(V) \) and \( \min_{\alpha_j \geq 0} \kappa_p(V) \) decrease until at \( \alpha_{\text{max}} = \alpha_{\text{opt}} \) when global minimums of \( \kappa_p(V) \) are reached, and then they start climbing again. Notice \( \alpha_{\text{opt}} \) may be different for the two cases, but \( \alpha_{\text{opt}} = O(1) \) in both cases.
• Our lower and upper bounds are tighter: we have \(d_2 - d_1 = 1\) always in (1.3) (see Remark 6.1 and 7.1). Although Theorem 4.1 in [1] is for \(p = 2\), but it was remarked that bounds for the \(\ell_p\)-condition number can also be gotten similarly but with\(^1\) \(d_2 - d_1 = 2 - 1/p\). Our bounds for \(p = \infty\) can be even much tighter. In fact, for \(p = \infty\) the approach by Beckermann [1] would give \(d_2 - d_1 = 2\), while our best results in later sections give \(d_2 - d_1 = \sqrt{2}/4\), and therefore smaller upper over low bound ratios for large \(n\). See Tables 6.1 and 7.1.

• Also for \(p = \infty\), we have results that give \(d_1 = d_2 = 0\) for \(\alpha_{\text{max}} \leq \delta\) or for \(\alpha_{\text{max}} \leq \delta\) and all \(\alpha_i \geq 0\), where \(\delta \leq 1\) is given, while no results as such were presented in [1]. See Tables 6.1 and 7.1. Both in [1] and as well as here it is obtained exactly

\[
\rho = 1 + \sqrt{2} \quad \text{for} \min_{\alpha_i} \kappa_p(V), \quad \rho = (1 + \sqrt{2})^2 \quad \text{for} \min_{\alpha_i \geq 0} \kappa_p(V),
\]

but we also get, e.g., \(\rho = 2 + \sqrt{3}\) for \(\min_{\alpha_{\text{max}} \leq 1/2} \kappa_p(V)\) which was not considered in [1]. See Theorems 6.4 and 7.5.

It is worth mentioning that despite of its notoriously ill-conditioning, there is a way to compute its singular value decompositions to a highly relative accuracy [5, 16].

Although our study here does not yield optimally conditioned \(V\), i.e., \(V\) that achieves \(\min \kappa_p(V)\) under various circumstances, it does, however, conclude that a nearly optimally conditioned \(V\) with nodes in \([\alpha_{\text{max}}, \alpha_{\text{max}}]\) (i.e., with \(\alpha_{\text{max}}\) fixed, \(\alpha_j\)’s vary within the interval) is the one defined with the translated Chebyshev nodes in a slightly larger interval (so that the two exterior nodes are \(-\alpha_{\text{max}}\) and \(\alpha_{\text{max}}\), respectively) and similarly for \(V\) with nodes in \([\alpha, \beta]\), where \(0 \leq \alpha < \beta = \alpha_{\text{max}}\). If all \(\alpha_j\) are allowed to vary freely along the entire real line, a nearly optimally conditioned \(V\) is the one defined with Chebyshev nodes (for which \(\alpha_{\text{max}} = \cos \frac{\pi}{2n} \approx 1\); If all \(\alpha_j\) are forced nonnegative but otherwise free, a nearly optimally conditioned \(V\) is the one defined with the translated Chebyshev nodes in the interval \([\alpha, \beta] = [0, 1]\). However, those nearly optimally conditioned \(V\) are truely by the word “nearly”, that is to say they are just nearly optimal but not optimal, according to those few optimally conditioned \(V\) computed by [8] under the condition that the optimal \(V\) is unique (for any fixed \(n\)). Beckermann [1, Theorem 4.1] also implies the same nearly optimal conditioned \(V\) for the case \(\alpha_i \in \mathbb{R}\) and the case \(\alpha_i \geq 0\) but without any constraint on \(\alpha_{\text{max}}\).

The rest of this paper is organized as follows. A cornerstone of our study is the use of the absolute sums of the coefficients of the translated Chebyshev polynomials of the first kind. They are defined and computed for a symmetric interval or a nonnegative interval in Section 2. Section 3 reviews the \(\ell_p\)-vector and \(\ell_p\)-operator norms, and norm equivalence relations that yields (1.4). Our upper bounds on \(\min \kappa_p(V)\) are obtained by the computations for \(V\) with the translated Chebyshev nodes. This is done in Section 4. Section 5 proves a general bound on \(V\) with nodes restricted to a given interval \([\alpha, \beta]\). Section 6 derives various asymptotically optimal bounds with or without fixing \(\alpha_{\text{max}}\). Section 7 considers the case when all \(\alpha_j \geq 0\). Finally Section 9 draws a few concluding remarks.

Notation. We shall stick to the global assignments in Table 1.1, unless otherwise explicitly stated. \(1 \leq p \leq +\infty\) and \(1/p + 1/p’ = 1\). Besides those, \(\mathbb{R}^{m \times n}\) is the set

\(^1V\) in [1] is \(V^T\) here.
of all \( m \times n \) real matrices, \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \) and \( \mathbb{R} = \mathbb{R}^1 \). \( \text{sign}(\xi) \) is the sign of \( \xi \in \mathbb{R} \). \( \lfloor \xi \rfloor \) is the largest integer that is smaller than \( \xi \); while \( \lceil \xi \rceil \) is the smallest integer that is larger than \( \xi \). For two sequences of numbers \( a_n \) and \( b_n \): \( a_n \sim b_n \) means \( a_n / b_n \to 1 \) as \( n \to +\infty \); \( a_n = \mathcal{O}(b_n) \) means \( c_1 \leq a_n / b_n \leq c_2 \) for constants \( c_1 \) and \( c_2 \); \( a_n = \mathcal{O}_n(b_n) \) means \( c_1 n^{d_1} \leq a_n / b_n \leq c_2 n^{d_2} \) for constants \( c_1, c_2, d_1, \) and \( d_2 \). In this paper, both \( a_n \) and \( b_n \) grow exponentially in \( n \), and thus the hidden factors \( n^{d_i} \) in \( a_n = \mathcal{O}_n(b_n) \) are less significant, compared to the exponential growth.

2. Coefficients of Chebyshev polynomials. The \( n \)th Chebyshev polynomial of the first kind on the interval \([-1, 1]\) is \( T_n(t) = \cos(n \arccos t) \), and thus

\[
T_n(-t) = \cos(n(\pi - \arccos t)) = (-1)^n T_n(t).
\]

Other useful formulas are [3, Page 30]

\[
T_n(t) = \frac{1}{2} \left( t + \sqrt{t^2 - 1} \right)^n + \frac{1}{2} \left( t - \sqrt{t^2 - 1} \right)^n,
\]

\[
= \frac{n}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{(n-j-1)!}{j!(n-2j)!} (2t)^{n-2j}.
\]

The \( n \)th Translated Chebyshev Polynomial is defined by

\[
T_n(x; \omega, \tau) \overset{\text{def}}{=} T_n(x/\omega + \tau),
\]

obtained upon applying the transformation

\[
t(x) = x/\omega + \tau.
\]

Here and in the rest of this paper \( T_n \) is overloaded with distinctions according to its argument(s). It can be seen that \( T_n(x; \omega, \tau) \) is a polynomial of degree \( n \) in \( x \). Write

\[
T_n(x; \omega, \tau) = a_{nn} x^n + a_{n-1} n x^{n-1} + \cdots + a_1 n x + a_0 n,
\]

where \( a_{jn} \equiv a_{jn}(\omega, \tau) \) are functions of \( \omega \) and \( \tau \) which, wherever referenced, should be either clear from the context or will be explicitly stated. Define

\[
S_{n,p}(\omega, \tau) \overset{\text{def}}{=} \left( \sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p},
\]
a function of $\omega$ and $\tau$, too. Successful computation of $S_{n,p}(\omega, \tau)$ is crucial to our later development. But in its generality, an explicit formula for $S_{n,p}(\omega, \tau)$ is hard to find. Nevertheless we still manage to find formulas for $S_{n,1}(\omega, \tau)$ for two different cases $\tau = 0$ or $|\tau| \geq 1$. Notice $S_{n,p}(\omega, \tau)$ and $S_{n,1}(\omega, \tau)$ are related by the following inequalities

\begin{equation}
(n + 1)^{-1/p'} S_{n,1}(\omega, \tau) \leq S_{n,p}(\omega, \tau) \leq S_{n,1}(\omega, \tau).
\end{equation}

The first inequality can be improved when $\tau = 0$ for which, by Theorem 2.2 below, only $[(n + 1)/2]$ of all $a_{jn}$ are nonzero. Therefore,

\begin{equation}
[(n + 1)/2]^{-1/p'} S_{n,1}(\omega, 0) \leq S_{n,p}(\omega, 0) \leq S_{n,1}(\omega, 0).
\end{equation}

Both (2.8) and (2.9) can be proved by using Hölder inequality

\begin{equation}
\sum_{j=1}^{m} |\xi_j \zeta_j| \leq \left( \sum_{j=1}^{m} |\xi_j|^p \right)^{1/p} \left( \sum_{j=1}^{m} |\zeta_j|^{p'} \right)^{1/p'}
\end{equation}

and the fact that \( \left( \sum_{j=1}^{m} |\xi_j|^p \right)^{1/p} \) is decreasing in $p$ [13, Lemma 1.1].

**Theorem 2.1.**
1. $a_{jn}(\omega, \tau) = -\xi_j a_{jn}(1, \tau)$, and $S_{n,p}(\omega, \tau) = S_{n,p}(|\omega|, |\tau|)$;
2. In $|\omega|$, $S_{n,p}(\omega, \tau)$ is decreasing; while $|\omega|^n S_{n,p}(\omega, \tau)$ is increasing;
3. If $|\tau| \geq 1$, $S_{n,p}(\omega, \tau)$ is increasing in $|\tau|$.

**Proof.** Because $T_n(t)$ is either even or odd, $T_n^{(j)}(t)$ is either even or odd. Thus $|T_n^{(j)}(t)| = |T_n^{(j)}(|t|)|$. With (2.5), it can be seen that

\[ \frac{d}{dx} T_n(x; \tau) = \frac{d}{dx} T_n(x/\omega + \tau) = -\xi_j \frac{d}{dx} T_n(t) \]

and thus

\begin{equation}
a_{jn}(\omega, \tau) = \frac{1}{\xi_j} \frac{d}{dx} T_n(x; \omega, \tau) \bigg|_{x=0} = -\xi_j \frac{1}{\xi_j} T_n^{(j)}(\tau)
\end{equation}

from which it follows $a_{jn}(\omega, \tau) = -\xi_j a_{jn}(1, \tau)$ and that

\begin{equation}
S_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} \omega^{-j} \frac{1}{\xi_j} T_n^{(j)}(\tau) \right)^{1/p} = S_{n,p}(|\omega|, |\tau|),
\end{equation}

and that as $|\omega|$ increases, $S_{n,p}(\omega, \tau)$ decreases and $|\omega|^n S_{n,p}(\omega, \tau)$ increases. This proves Items 1 and 2. For Item 3, we claim that $T_n^{(j)}(t)$ increases as $t$ does for $t \geq 1$, this is because the zeros of $T_n^{(j)}(t)$ are all within $(-1, 1)$, and thus $\text{sign}(T_n^{(j)}(t)) = \text{sign}(t^{n-j}) = 1$ for $t \geq 1$ and therefore for $t \geq 1$, $T_n^{(j)}(t) = T_n^{(j)}(t)$ is increasing. 

**Theorem 2.2.** For $T_n(x; \omega, 0) \equiv T_n(x/\omega)$, i.e., $\tau = 0$ in (2.5), we have
1. $a_{n-1,n} = a_{n-3,n} = \cdots = 0$, and $\text{sign}(a_{n-2,n}) = (-1)^{n-1} \text{sign}(\omega^n)$.
2. $S_{n,1}(\omega, 0) = |T_n(t/\omega)|$, where $t = \sqrt{-1}$ is the imaginary unit. Thus

\[ S_{n,1}(\omega, 0) = |T_n(t/\omega)| \sim \frac{1}{2} \left( \frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n. \]
Proof. It can be seen that $a_{jn}(\omega, 0) \equiv a_{jn}(1, 0)/\omega^j$. So Item 1 is a consequence of (2.3). Now, we have $S_{n,1}(\omega, 0) = |an - an-2 + an-4 - \cdots|$, and thus $S_{n,1}(\omega, 0) = |T_n(t/\omega)|$. □

**Theorem 2.3.** For $\omega \neq 0$ and $|\tau| \geq 1$, we have
1. If $\tau \geq 1$, then $\operatorname{sign}(a_{jn}) = [\operatorname{sign}(\omega)]^j$, and thus
   $$S_{n,1}(\omega, \tau) = \begin{cases} T_n(-1; \omega, \tau), & \text{if } \omega > 0, \\ (-1)^n T_n(-1; \omega, \tau), & \text{if } \omega < 0; \end{cases}$$
2. If $\tau \leq -1$, then $\operatorname{sign}(a_{jn}) = (-1)^{n-1}[\operatorname{sign}(\omega)]^j$, and thus
   $$S_{n,1}(\omega, \tau) = \begin{cases} (-1)^n T_n(-1; \omega, \tau), & \text{if } \omega > 0, \\ T_n(-1; \omega, \tau), & \text{if } \omega < 0; \end{cases}$$
3. Thus for $|\tau| \geq 1$,
   $$S_{n,1}(\omega, \tau) = T_n\left(\frac{1}{|\omega|} + |\tau|\right) \sim \frac{1}{2} \left[\left(\frac{1}{|\omega|} + |\tau|\right) + \sqrt{\left(\frac{1}{|\omega|} + |\tau|\right)^2 - 1}\right]^n.$$

**Proof.** These are consequences of (2.11) and the fact that $\operatorname{sign}(T_n^{(j)}(t)) = \operatorname{sign}(t^{(n-j)}) = (-1)^{n-j}$ for $|t| \geq 1$. □

Theorems 2.2 and 2.3 established $S_{n,1}(\omega, \tau)$ exactly in terms of values of $T_n(x; \omega, \tau)$ at $\pm 1$ or $\pm 1$. The case for $|\tau| < 1$, except when $\tau = 0$, is still left open. This is caused by the irregularities in the signs of $a_{jn}$ for $|\tau| < 1$ and $\tau \neq 0$.

**Theorem 2.4.** Let $\omega > 0$ and $n \geq 2$
1. $\omega S_{n,1}(\omega, 0)$ is decreasing in $\omega$ if $\omega \leq \max\{\sqrt{n-1}, \sqrt{2}\}$ or $n$ is odd;
2. $\omega S_{n,1}(\omega, 1)$ is decreasing in $\omega$ if $\omega \leq \max\{n-1, \sqrt{2}\}$;
3. $\frac{2\omega}{\sqrt{n^2-1}} S_{n,1}(\omega, 1)$ is decreasing if $\omega \leq \sqrt{n-1}(\sqrt{n-1} + \sqrt{n})$.

**Proof.** A proof can be found in [18]. □

**3. $\ell_p$-vector and $\ell_p$-operator norms.** Let $1 \leq p \leq \infty$. Given $u = (\mu_1, \mu_2, \cdots, \mu_n)^T \in \mathbb{R}^n$, its $\ell_p$-norm is defined as
   $$\|u\|_p = \left(\sum_{j=1}^{n} |\mu_j|^p\right)^{1/p},$$
and $\|u\|_\infty = \lim_{p \to \infty} \|u\|_p = \max_j |\mu_j|$. The associated $\ell_p$-operator norm of $A \in \mathbb{R}^{m \times n}$ is defined as
   $$(3.1) \qquad \|A\|_p = \max_{u \neq 0 \in \mathbb{R}^n} \frac{\|Au\|_p}{\|u\|_p}.$$ 

It is proved that [17] $\|A\|_p = \|A^T\|_{p'}$. In particular, $\|A\|_\infty = \|A^T\|_1$. All matrix norms are equivalent, and in particular, we have for $1 \leq p, q \leq \infty$ and $m = n$ [18]
   $$(3.2) \quad n^{-1/p-1/q} \|A\|_q \leq \|A\|_p \leq n^{1/p-1/q} \|A\|_q.$$ 

Thus $\kappa_p(V)$ and $\kappa_q(V)$ differ by a factor $n^d$ for some $|d| \leq 2$. In particular, we have (1.4). Another useful inequality due to Kato [15, Page 29] is
   $$(3.3) \quad \|A\|_p \leq \|A\|_1^{1/p} \|A\|_\infty^{1/p'}.$$
4. Vandermonde matrices with translated Chebyshev nodes. The zeros of $T_n(t)$ are called

\begin{equation} \tag{4.1} \text{Chebyshev Nodes: } t_j = \cos \theta_j, \quad \theta_j = \frac{2j-1}{2n} \pi \quad (1 \leq j \leq n), \end{equation}

and the zeros of its translated Chebyshev polynomial $T_n(x; \omega, \tau)$ as in (2.4) are called

\begin{equation} \tag{4.2} \text{Translated Chebyshev Nodes: } x_j = \omega(t_j - \tau) \quad (1 \leq j \leq n). \end{equation}

Given interval $[\alpha, \beta]$, set

\begin{equation} \tag{4.3} \omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = -\frac{\beta + \alpha}{\beta - \alpha}. \end{equation}

Then the linear transformation $t = x/\omega + \tau$ defined in (2.5) maps $x \in [\alpha, \beta]$ one-to-one and onto $t \in [-1, 1]$. The inverse transformation is $x = \omega(t - \tau)$. This section, inspired by Gautschi [7], computes $\kappa_\infty(V)$ for $V$ with the translated Chebyshev nodes for the case $-\alpha = \beta$ and the case $0 \leq \alpha < \beta$. But we are still unsure how to deal with the general case $\alpha < 0 < \beta$ but $-\alpha \neq \beta$.

First we compute $\|V\|_\infty$ for $V$ with $\alpha_j = x_j = \omega(\cos \theta_j - \tau)$. This is relatively easy. By [8, Theorem 2.1],

\begin{equation} \tag{4.4} \|V\|_\infty = \max \left\{ n, \sum_{j=1}^{n} |\alpha_j|^{n-1} \right\} = \max \left\{ n, \omega^{n-1} \Lambda_n(\tau) \right\}, \end{equation}

where

\begin{equation} \tag{4.5} \Lambda_n(\tau) \overset{\text{def}}{=} \sum_{j=1}^{n} |\cos \theta_j - \tau|^{n-1}. \end{equation}

It can be seen that $\Lambda_n(\tau) = \Lambda_n(-\tau)$. In [18, Appendix B], the following asymptotical behaviors

\begin{equation} \tag{4.6} \Lambda_n(0) \sim \sqrt{\frac{2n}{\pi}}, \quad \Lambda_n(1) \sim \sqrt{\frac{n}{\pi}} 2^{n-1}. \end{equation}

were obtained. With (4.6), we have the following theorem.

**Theorem 4.1.** Let $\alpha_j = x_j$ (1 $\leq j \leq n$) as in (4.2) with (4.3).

1. If $-\alpha = \beta > 0$ (and thus $\omega = \beta$), then

\[ \|V\|_\infty \sim \max \left\{ n, \sqrt{\frac{2n}{\pi}} \omega^{n-1} \right\} \sim \max \left\{ n, \sqrt{\frac{2n}{\pi}} \alpha_{\max}^{n-1} \right\}. \]

2. If $0 = \alpha < \beta$, then

\[ \|V\|_\infty \sim \max \left\{ n, \sqrt{\frac{n}{\pi}} b^{n-1} \right\} \sim \max \left\{ n, \sqrt{\frac{n}{\pi}} \alpha_{\max}^{n-1} \right\}. \]

In both cases $-\alpha = \beta$ or $0 = \alpha < \beta$, $\sum_{j=1}^{n} |x_j|^{n-1} = O(\sqrt{n} \alpha_{\max}^{n-1})$. But will this be also true for arbitrary interval $[\alpha, \beta]$? We do not know.
We now estimate \( \|V^{-1}\|_\infty \) with translated Chebyshev nodes. It is made possible by Gautschi’s formulas for \( \|V^{-1}\|_\infty \) for \( V \) with symmetric nodes or with nonnegative nodes. Let \( f(x) = \prod_{j=1}^{n} (x - \alpha_j) \). Gautschi [7] showed that

\[
\|V^{-1}\|_\infty = \min_{\alpha_j \geq 0} \left\{ \left( 1 + \frac{\alpha_j}{\omega_j} \right) |f'(\alpha_j)| \right\}, \quad \text{for symmetric nodes},
\]

\[
\|V^{-1}\|_\infty = \min_{1 \leq j \leq n} \left\{ \left( 1 + \frac{\alpha_j}{\omega_j} \right) |f'(\alpha_j)| \right\}, \quad \text{for nonnegative nodes}.
\]

With the help of these two formulas, Li [18] obtained the following two theorems whose proofs can be found there.

**Theorem 4.2.** Suppose \(-\alpha = \beta > 0\), and let \( \alpha_j = x_j \) (1 \( \leq j \leq n \)) as in (4.2) with (4.3). Then \( V \) is a \( V_{sym} \), and

\[
\omega \min \left\{ 1 + \frac{\omega}{1 + \omega^2} \right\} \frac{1}{n} \leq \|V^{-1}\|_\infty \leq \omega \max \left\{ 1 + \frac{\omega}{1 + \omega^2} \right\} \frac{3^{3/4}}{2n},
\]

where the first inequality is valid for \( n \geq 3 \) only.

**Theorem 4.3.** Suppose \( 0 \leq \alpha < \beta \), and let \( \alpha_j = x_j \) (1 \( \leq j \leq n \)) as in (4.2) with (4.3). Then

\[
\frac{\beta - \alpha}{n \left( 1 + \frac{\beta + \alpha}{2} \right)} \leq \|V^{-1}\|_\infty \leq \frac{\beta - \alpha}{2n \sqrt{(1 + \beta)(1 + \alpha)}}.
\]

Theorems 4.2 and 4.3 say that for \( V \) with translated Chebyshev nodes on \([\alpha, \beta]\) if \(-\alpha = \beta\) or \( 0 \leq \alpha < \beta \) or \( \alpha < \beta \leq 0 \), then

\[
n\|V^{-1}\|_\infty \frac{1}{S_{n,1}(\omega, \tau)} = O(1).
\]

(The case \([\alpha, \beta]\) for \( \alpha < \beta \leq 0 \) can be turned into \([-\beta, -\alpha]\), a case that is covered by Theorem 4.3.) But what happens when \( \alpha < 0 < \beta \) and \(-\alpha \neq \beta\)? Is (4.11) still true? We conjecture it would be, but do not have any proof for now.

**Theorem 4.4.** Let \( \alpha_j = x_j \) (1 \( \leq j \leq n \)) as in (4.2) with (4.3).

1. If \(-\alpha = \beta > 0\), then

\[
\min_{\alpha = \beta} \kappa_\infty (V) \leq \frac{3^{3/4}}{2} \beta_{opt} S_{n,1}(\beta_{opt}, 0) \sim \frac{3^{3/4}}{2} \left( \frac{2}{\pi} \right)^{\sqrt{2/4}} (1 + \sqrt{2})^n \frac{1}{2n \sqrt{2/4}},
\]

where \( \beta_{opt} \equiv \omega_{opt} = (n/\Lambda_n(0))^{1/(n-1)} \).

2. If \( 0 = \alpha < \beta \), then

\[
\min_{0 = \alpha < \beta} \kappa_\infty (V) \leq \frac{\beta_{opt}^+}{2 \sqrt{1 + \beta_{opt}^+}} S_{n,1}(\beta_{opt}^+/2, 1) \sim \sqrt{2} (1 + \sqrt{2})^2n \frac{1}{4(n\pi) \sqrt{2/4}},
\]

where \( \beta_{opt}^+/2 \equiv \omega_{opt}^+ = (n/\Lambda_n(1))^{1/(n-1)} \).

**Proof.** A proof can be found in [18]. \( \square \)
5. Condition numbers of Vandermonde matrices – a general theorem.

We shall start by establishing a general theorem on $\kappa_p(V)$ for

$$\alpha \leq \min_j \alpha_j \leq \max_j \alpha_j \leq \beta.$$  

The case $\alpha = \beta$ is of no interest, because then $V$ is of rank 1 and thus $\kappa_p(V) = +\infty$ (unless $n = 1$). There are lots of ways to realize (5.1), and it is tempting to always let $\alpha = \min_j \alpha_j$ and $\beta = \max_j \alpha_j$; but that may not be always possible for theorems that require $-\alpha = \beta$. Recall $\omega$ and $\tau$ defined by (4.3), and $\alpha_{\text{max}} \overset{\text{def}}{=} \max_j |\alpha_j|$.

**Lemma 5.1.**

$$\max\{n, n\alpha_{\text{max}}^{n-1}\} \geq \|V\|_1 = \sum_{j=1}^{n} \alpha_{\text{max}}^{j-1} \geq \max\{1, \alpha_{\text{max}}^{n-1}\},$$  

$$\max\{n, n\alpha_{\text{max}}^{n-1}\} \geq \|V\|_{\infty} = \max\{n, \sum_{j=1}^{n} |\alpha_{\text{max}}|^{n-1}\} \geq \max\{n, \alpha_{\text{max}}^{-n}\},$$  

$$\max\{n, n\alpha_{\text{max}}^{n-1}\} \geq \|V\|_{p} \geq \max\{n^{1/p'}, \alpha_{\text{max}}^{-n}\},$$  

$$\|V\|_{p} \geq \left(\sum_{j=1}^{n} \alpha_{\text{max}}^{(j-1)p}\right)^{1/p},$$  

$$\|V^{-1}\|_{p} \geq \frac{S_{n-1,p'}(\omega, \tau)}{n^{-1/p'}} \geq \frac{S_{n-1,1}(\omega, \tau)}{n}.$$  

*Proof.* The middle equation in (5.2) is due to the known formula for $\| \cdot \|_1$ [4, Page 22] and the middle equation in (5.3) is due to [8, Theorem 2.1]. Inequalities in (5.2) and (5.3) are their consequences. Let $e_j$ be the $j$th column of the $n \times n$ identity matrix. Then

$$\|V\|_p = \|V^T\|_{p'} \geq \begin{cases} \|V^T e_1\|_{p'} = n^{1/p'}, \\ \|V^T e_n\|_{p'} \geq \alpha_{\text{max}}^{-n}. \end{cases}$$  

This yields the second inequality in (5.4). Use (3.3), (5.2), and (5.3) to arrive at the first inequality there. Inequality (5.5) is gotten by noticing $\|V\|_p \geq \max_j \|V e_j\|_p$.

We now show (5.6). Let $v$ be the vector of the coefficients of $T_{n-1}(x; \omega, \tau) \equiv T_{n-1}(x/\omega + \tau)$, i.e., $v = (a_0, a_{n-1} a_1, a_{n-2} \cdots a_{n-1}, a_{n-1})^T$. Then

$$V^T v = (T_{n-1}(a_1/\omega + \tau) T_{n-1}(a_2/\omega + \tau) \cdots T_{n-1}(a_n/\omega + \tau))^T$$  

which yields $\|V^T v\|_{p'} \leq n^{1/p'}$ because $|T_{n-1}(x/\omega + \tau)| \leq 1$ for $x \in [\alpha, \beta]$. We therefore have

$$\|V^{-1}\|_p = \|V^{-T}\|_{p'} \geq \frac{\|v\|_{p'}}{\|V^T v\|_{p'}} \geq \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}} \geq \frac{S_{n-1,1}(\omega, \tau)}{n},$$

by (2.8). \(\square\)

**Remark 5.1.** The last inequality in (5.6) can be improved when $-\alpha = \beta$ (and thus $\tau = 0$) because of (2.9): if $-\alpha = \beta$, then

$$\|V^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega, 0)}{n^{1/p'}} \geq \left(\frac{n}{|n/2|}\right)^{1/p} \frac{S_{n-1,1}(\omega, 0)}{n}.$$
An interesting observation also for the case $-\alpha = \beta$ (and thus $\tau = 0$) is as follows. For even $n$, $T_{n-1}(x; \omega, 0)$ is odd, i.e., $a_{0,n-1} = a_{2,n-1} = \cdots = 0$, and thus only the odd rows of $V$ got picked up by $V^T v$, completely discarding all the even rows; while for odd $n$, for the same reason only the even rows of $V$ got picked up by $V^T v$, completely discarding all the odd rows. Therefore, we conclude that if $-\alpha = \beta$, then

\begin{equation}
\|\hat{V}^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega, 0)}{n^{1/p'}} \geq \left( \frac{n}{\lfloor n/2 \rfloor} \right)^{1/p} \frac{S_{n-1,1}(\omega, 0)}{n}
\end{equation}

for any $\hat{V}$ that has the same odd rows as $V$ if $n$ is even or the same even rows as $V$ if $n$ is odd. This remark applies to some of the later lemmas and theorems, too.

**Theorem 5.2.**

\begin{align*}
\kappa_p(V) &\geq \max \left\{ S_{n-1,p'}(\omega, \tau), \frac{n^{n-1}S_{n-1,1}(\omega, \tau)}{n^{1/p'}} \right\} \quad \text{(5.9)} \\
&\geq \max \left\{ \frac{S_{n-1,1}(\omega, \tau)}{n^{1/p'}}, \frac{n^{n-1}S_{n-1,1}(\omega, \tau)}{n} \right\} \quad \text{(5.10)}
\end{align*}

**Proof.** Inequalities (5.9) and (5.10) are immediate consequences of (5.4) and (5.6).

1. Take $\alpha = \min_j \alpha_j$ and $\beta = \max_j \alpha_j$ and then compute the right-hand side of (5.9) or (5.10). But unless $\alpha \geq 0$ or $-\alpha = \beta$ by coincidence, we may have to compute $S_{n-1,1}(\omega, \tau)$ in a brutal force way because no explicit formula has been found yet. In this case, both $\alpha$ and $\beta$ are nodes of $V$.

2. Take $-\alpha = \beta = \alpha_{\max}$ (and thus $\omega = \alpha_{\max}$ and $\tau = 0$) and then use the explicit formula for $S_{n-1,1}(\alpha_{\max}, 0)$ to compute the right-hand side of (5.10). In this case, one of $\alpha$ and $\beta$ is guaranteed to be a node for $V$.

**Remark 5.2.** The lower bounds in [1] were essentially obtained as follows. Let $\omega = \eta \alpha_{\max}$. It follows from (5.5) and (5.6) that

\begin{equation}
n^{1/p'} \kappa_p(V) \geq \left( \sum_{j=0}^{n-1} \alpha_{\max}^{p} \right)^{1/p} S_{n-1,p'}(\omega, \tau).
\end{equation}

But by Theorem 2.1, $S_{n-1,p'}(\omega, \tau) = \left( \sum_j |\omega^{-j}a_{j,n-1}(1, \tau)|^{p'} \right)^{1/p'}$. By H"older inequality (2.10), we have

\begin{equation}
n^{1/p'} \kappa_p(V) \geq \sum_j \eta^{-j} |a_{j,n-1}(1, \tau)| = S_{n-1,1}(\eta, \tau)
\end{equation}

which gives

\begin{equation}
\kappa_p(V) \geq \frac{S_{n-1,1}(\eta, \tau)}{n^{1/p'}}.
\end{equation}

In the case of [1], $p = p' = 2$, either $\eta = 1$ and $\tau = 0$ or $\eta = 1/2$ and $\tau = -1$. This is a pretty decent bound, but it partially collapses the interval information, unlike (5.9)
and (5.10) which form the basis for us to eventually arrive at the qualitative behaviors in Figure 1.1.

**Remark 5.3.** Another consequence of Lemma 5.1 is

\[
\kappa_p(V) \geq \begin{cases} 
S_{n-1,p'}(\omega, \tau), & \text{if } \alpha_{\max} \leq 1, \\
\alpha_{\max}^{-1}S_{n-1,p'}(\omega, \tau)/n^{1/p'}, & \text{if } \alpha_{\max} > 1 
\end{cases}
\]

upon noticing that \( \|V\|_p \geq n^{1/p'} \) if \( \alpha_{\max} \leq 1 \), and \( \|V\|_p \geq \alpha_{\max}^{-1}n^{1/p'} \) if \( \alpha_{\max} > 1 \). (5.12) is slightly weaker than (5.9). But we need it for later use in the proof of Theorem 6.3.

6. **Condition numbers for** \( V \) **with** \( \alpha_j \in [\alpha, \beta] \) **and** \( -\alpha = \beta \). In this section, we apply Theorem 5.2 and (5.12) to the case \( -\alpha = \beta \).

**Theorem 6.1.**

\[
\kappa_p(V) \geq \begin{cases} 
S_{n-1,p'}(\alpha_{\max}, 0), & \text{if } \alpha_{\max} \leq n^{1/p'(n-1)} \\
\alpha_{\max}^{-1}S_{n-1,p'}(\alpha_{\max}, 0)/n^{1/p'}, & \text{if } \alpha_{\max} > n^{1/p'(n-1)} 
\end{cases}
\]

**Proof.** Apply Theorem 5.2 to the case \( -\alpha = \beta = \alpha_{\max} \) (and thus \( \omega = \alpha_{\max} \) and \( \tau = 0 \)) to get

\[
\kappa_p(V) \geq \max \left\{ S_{n-1,p'}(\alpha_{\max}, 0), \alpha_{\max}^{-1}S_{n-1,p'}(\alpha_{\max}, 0)/n^{1/p'} \right\}.
\]

We shall minimize the right-hand side of (6.2). By Theorem 2.1, the first quantity within \( \max \{ \cdots \} \) in (6.2) is decreasing in \( \alpha_{\max} \); while the second one is increasing in \( \alpha_{\max} \). Therefore the right-hand side of (6.2) achieves its minimum when the two equal, i.e.,

\[
n^{1/p'} = \alpha_{\max}^{-1} \Rightarrow \alpha_{\max} = n^{1/p'(n-1)}
\]

which yields (6.1). \( \square \)

**Theorem 6.2.**

\[
S_{n-1,p'}(n^{1/p'(n-1)}, 0) \leq \min_{\alpha} \kappa_p(V) \leq \min_{\alpha} \kappa_p(V_{\text{sym}}) \leq n^{1/p} \cdot \frac{3^{3/4}}{2} \cdot S_{n,1}(1, 0).
\]

**Proof.** The right-hand side of (6.2), as a function of \( \alpha_{\max} \), achieves its minimum at \( \alpha_{\max} = n^{1/p'(n-1)} \). That gives the first inequality. The second inequality is true because \( \{V_{\text{sym}}\} \) is a subset of all Vandermonde matrices. We now prove the third one. To this end, consider \( V = V_{\text{sym}} \) with Chebyshev nodes \( t_j \). Then \( \|V\|_1 \leq n \) and \( \|V\|_\infty = n \), and thus by (3.3)

\[
\|V\|_p \leq \|V\|_1^{1/p} \|V\|_\infty^{1/p'} \leq n.
\]

Apply Theorem 4.2 to the case \( -\alpha = \beta = 1 = \omega \) to get \( \|V^{-1}\|_\infty \leq \frac{3^{3/4}}{2} \cdot n^{-1}S_{n,1}(1, 0) \) and then to get

\[
\|V^{-1}\|_p \leq n^{1/p}\|V^{-1}\|_\infty \leq n^{1/p} \cdot \frac{3^{3/4}}{2} \cdot n^{-1}S_{n,1}(1, 0).
\]

So for this \( V_{\text{sym}} \), \( \kappa_p(V_{\text{sym}}) \leq n^{1/p} \cdot \frac{3^{3/4}}{2} \cdot S_{n,1}(1, 0) \), as needed. \( \square \)
We include \( \min \kappa_p(V_{\text{sym}}) \) in (6.3) mainly because Vandermonde matrices with symmetric nodes were heavily studied by Gautschi and his coauthor \([7, 8, 12]\). Moreover, assuming that the optimally conditioned \( V \) is unique, Gautschi \([8]\) showed that the optimally conditioned \( V \) must have symmetric nodes.

Remark 6.1. The leftmost inequality in (6.3) can be replaced by

\[
S_{n-1,1}(1,0)/n^{1/p'} \leq \min_{\alpha_j} \kappa_p(V),
\]

upon using (5.11) with \( \eta = 1 \) and \( \tau = 0 \). This is due to \([1]\) for \( p = 2 \), and turns out to be less sharp at least for \( p = \infty \). See Table 6.1. For any \( 1 \leq p \leq \infty \), the ratio of the upper bound in (6.3) over the lower bound here is \( n^{3/4} \).

The third inequality in (6.3) was proved by simply picking a special \( V \) which turns out to be good enough, as we shall see later, in yielding the correct asymptotic speed in our notation \( O_n \), but it does not produce the best possible factor \( n^d \) hidden in the notation. For \( p = \infty \), however, a tighter upper bound is possible by using the \( V \) with the translated Chebyshev nodes in \([-\beta_{\text{opt}}, \beta_{\text{opt}}]\), where \( \beta_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)} \) as in Theorem 4.4. Of course, one may use this \( V \) for all \( p \), but doing so will not only lead to a more complicated bound but also the resulted bound may not be much better due to more complicated estimation of \( \|V\|_p \). For this reason, we shall state a sharper version of Theorem 6.2 for \( p = \infty \) only, as a consequence of Theorem 6.2 for \( p = \infty \) and Theorem 4.4. It is sharper because of \( \beta_{\text{opt}} > 1 \) and Theorem 2.4.

Theorem 6.2'.

\[
S_{n-1,1}(n^{1/(n-1)},0) \leq \min_{\alpha_j} \kappa_{\infty}(V) \leq \min_{\alpha_j} \kappa_{\infty}(V_{\text{sym}}) \leq \frac{3^{3/4}}{2} \beta_{\text{opt}} S_{n,1}(\beta_{\text{opt}},0),
\]

where \( \beta_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)} \).

In what follows, we shall establish theorems that are of the same spirit as Theorems 6.2 and 6.2' but with \( \alpha_{\text{max}} \) subject to a constraint.

Theorem 6.3. Let \( \delta > 0 \) and set \( \delta' = \delta / \cos \frac{\pi}{2n} \). If \( \delta \leq 1 \), then

\[
S_{n-1,1,p}(\delta,0) \leq \min_{\alpha_{\text{max}} \leq \delta} \kappa_p(V) \leq \min_{\alpha_{\text{max}} = \delta} \kappa_p(V) \leq n^{1/p} \frac{(\sqrt{2} + 1)^{3/4}}{4} \delta' S_{n,1}(\delta',0);
\]

and if \( \delta > 1 \), then

\[
\delta^{n-1} S_{n-1,1,p}(\delta,0) \leq \min_{\alpha_{\text{max}} = \delta} \kappa_p(V) \leq n^{1/p} \frac{3^{3/4} (\cos \frac{\pi}{2n})^{n-1}}{2} (\delta')^n S_{n,1}(\delta',0),
\]

\[
S_{n-1,1,p}(n^{1/p}(n-1),0) \leq \min_{\alpha_{\text{max}} \leq \delta} \kappa_p(V) \leq n^{1/p} \frac{3^{3/4}}{2} S_{n,1}(1,0).
\]

Inequalities (6.6), (6.7), and (6.8) remain valid with \( V \) replaced by \( V_{\text{sym}} \).

Proof. 1) Observe that \( \{V : \alpha_{\text{max}} = \delta\} \subset \{V : \alpha_{\text{max}} \leq \delta\} \). So

\[
\min_{\alpha_{\text{max}} \leq \delta} \kappa_p(V) \leq \min_{\alpha_{\text{max}} = \delta} \kappa_p(V).
\]

This is the middle inequality in (6.6).
2) Apply (5.12) to the case \(-\alpha = \beta = \alpha_{\text{max}} \leq \delta \leq 1\) (and thus \(\omega = \alpha_{\text{max}}\) and \(\tau = 0\)) to obtain
\[
\kappa_p(V) \geq S_{n-1,p'}(\alpha_{\text{max}}, 0) \geq S_{n-1,p'}(\delta, 0)
\]
By Theorem 2.1. This gives the first inequality in (6.6).

3) Apply (5.12) to the case \(-\alpha = \beta = \delta = \alpha_{\text{max}}\) (and thus \(\omega = \delta\) and \(\tau = 0\)) to obtain the first inequality in (6.7).

4) Take \(-\alpha = \beta = \delta / \cos \frac{\pi}{2n} = \delta', \) and \(\alpha_j = x_j\) \((1 \leq j \leq n)\), the translated Chebyshev nodes as in (4.2). Then
\[
\tau = 0, \quad \alpha_{\text{max}} = \max \{\alpha_j\} = \beta \cos \frac{\pi}{2n} = \delta, \quad \delta \leq \omega = \beta = \delta'.
\]

Theorem 4.2 says that for the \(V\) with those nodes
\[
\frac{\|V\|_{\infty}}{S_{n,1}(\omega, 0)} \leq \omega \max \left\{1, \frac{1 + \omega}{1 + \omega^2}\right\} \frac{3^{3/4}}{2n} \leq \begin{cases} \\
\delta' (\sqrt{2} + 1) 3^{3/4} / 4n, & \text{if } \delta \leq 1, \\
\delta' 3^{3/4} / 2n, & \text{if } \delta \geq 1,
\end{cases}
\]
where we have used
\[
(6.10) \max_{\omega > 0} \left\{1, \frac{1 + \omega}{1 + \omega^2}\right\} = \frac{1 + \omega}{1 + \omega^2} \big|_{\omega = \sqrt{2} - 1} = \frac{\sqrt{2} + 1}{2}, \quad \max_{\omega \geq 1} \left\{1, \frac{1 + \omega}{1 + \omega^2}\right\} = 1.
\]
Now employ \(\|V\|_p \leq n\) if \(\delta \leq 1\) and \(\|V\|_p \leq n^{3n-1}\) if \(\delta \geq 1\), and \(\|V^{-1}\|_p \leq n^{1/p} \|V^{-1}\|_{\infty}\) to get the last inequalities in (6.6) and in (6.7).

5) A proof of (6.8) can be done in the same way as for Theorem 6.2.

6) Finally, when \(V\) is replaced by \(V_{\text{sym}}\), the first inequalities in (6.6), (6.7), and (6.8) still hold. The middle inequality in (6.6) remains valid, too, because (6.9) is true with \(V\) replaced by \(V_{\text{sym}}\). The last inequalities in (6.6), (6.7), and (6.8) hold because they all were proved by bounding some \(\kappa_p(V_{\text{sym}})\). \(\square\)

There are stronger versions of (6.7) and (6.8) for \(p = \infty\), too, just as we did for Theorem 6.2.

**Theorem 6.3'.** Let \(\delta > 1\) and set \(\delta' = \delta / \cos \frac{\pi}{2n}\). Then
\[
\begin{align*}
\frac{\delta^{n-1} S_{n-1,1}(\delta, 0)}{n} &\leq \min_{\alpha_{\text{max}} = \delta} \kappa_{\infty}(V) \leq \frac{3^{3/4}}{2} \max \{n, \Lambda_n(0)(\delta')^{n-1}\} \delta' S_{n,1}(\delta', 0), \\
(6.12) S_{n,1}(n^{1/(n-1)}, 0) &\leq \min_{\alpha_{\text{max}} \leq \delta} \kappa_{\infty}(V) \leq \frac{3^{3/4}}{2} \omega_1 S_{n,1}(\omega_1, 0),
\end{align*}
\]
where \(\omega_1 = \min \{\delta', (n/\Lambda_n(0))^{1/(n-1)}\}\). It can be seen that \(\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}\) for \(n\) sufficiently large. Inequalities (6.11), and (6.12) remain valid with \(V\) replaced by \(V_{\text{sym}}\).

**Proof.** Only the second inequalities in (6.11) and (6.12) need proofs. For (6.11), it follows from the proof of Theorem 6.3, upon using \(\|V\|_{\infty} = \max \{n, \omega^{n-1} \Lambda_n(0)\}\)
which for large $n$ is proportional to $\sqrt{n} \delta^{n-1}$ instead of $\|V\|_\infty \leq n \delta^{n-1}$. The second inequality in (6.12) is obtained by approximately minimizing
\[
\frac{3^{1/4}}{2} \max \left\{ n, \Lambda_n(0)(\alpha'_{\text{max}})^{n-1} \right\} \alpha'_{\text{max}} \, S_{n,1}(\alpha'_{\text{max}}, 0),
\]
subject to $\alpha_{\text{max}} \leq \delta$, where $\alpha'_{\text{max}} = \alpha_{\text{max}} / \cos \frac{\pi}{2n}$. That $\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}$ for $n$ sufficiently large is due to $(n/\Lambda_n(0))^{1/(n-1)} \sim (2\pi/n)^{1/2(n-1)} \sim 1$. \[\square\]

We shall now investigate the tightness of the upper bounds and the lower bounds we have established so far, as well as the asymptotical speeds of $\kappa_p(V)$ minimized over certain set of Vandermonde matrices. We shall do so only for $p = \infty$; for any other $p$, $S_{n-1,p}$ in the lower bounds will have to be weakened by using (2.9) so as to apply the same lines of arguments here.

For $p = \infty$, $p' = 1$. Since
\[
n^{1/(n-1)} = 1 + \ln(n-1) - \frac{2 + \ln^2(n-1)}{2(n-1)^2} + \ldots,
\]
Theorem 6.2 says that $\kappa_\infty(V)$ is no smaller than
\[
S_{n-1,1}(n^{1/(n-1)}, 0) = \left( \frac{n-1}{\sqrt{2}} \right) (1 + \sqrt{2})^{n-1} \left[ 1 + O \left( \frac{\ln(n)}{n} \right) \right] + (-1)^{n-1} \left( \frac{n-1}{\sqrt{2}} \right) (1 + \sqrt{2})^{n-1} \left[ 1 - O \left( \frac{\ln(n)}{n} \right) \right]
\]
\[
\sim \frac{(1 + \sqrt{2})^{n-1}}{2(n-1)^{1/\sqrt{2}}}. \tag{6.13}
\]
On the other hand, $S_{n,1}(1, 0) = \frac{1}{2} (1 + \sqrt{2})^n + \frac{(-1)^n}{2} (1 + \sqrt{2})^{-n}$, and for $\beta_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)},$
\[
S_{n,1}(\beta_{\text{opt}}, 0) = \left( \frac{2}{\pi} \right) \frac{\sqrt{n}}{\sqrt{2}} \left( \frac{n}{\sqrt{2}} \right) (1 + \sqrt{2})^n \left[ 1 + O \left( \frac{\ln(n)}{n} \right) \right] + (-1)^n \left( \frac{2}{\pi} \right) \frac{n}{\sqrt{2}} \left( \frac{n}{\sqrt{2}} \right) (1 + \sqrt{2})^n \left[ 1 - O \left( \frac{\ln(n)}{n} \right) \right]
\]
\[
\sim \frac{2}{\pi} \frac{\sqrt{n}}{n^{1/2}} (1 + \sqrt{2})^n. \tag{6.14}
\]
Therefore
\[
\frac{S_{n,1}(1, 0)}{S_{n-1,1}(n^{1/(n-1)}, 0)} \sim (1 + \sqrt{2})^{n^{1/\sqrt{2}}}, \tag{6.15}
\]
\[
\frac{S_{n,1}(\beta_{\text{opt}}, 0)}{S_{n-1,1}(n^{1/(n-1)}, 0)} \sim \left( \frac{2}{\pi} \right) \frac{\sqrt{n}}{n^{1/2}} (1 + \sqrt{2})^{n^{1/2}}, \tag{6.16}
\]
\[
\min_{\alpha_j} \kappa_\infty(V_{\text{sym}}), \min_{\alpha_j} \kappa_\infty(V) = O_n \left( (1 + \sqrt{2})^n \right). \tag{6.17}
\]
We now turn to the bounds in Theorem 6.3. We claim
\[
\left( 1 + \sqrt{\delta^2 + 1} \right)^n \sim (1 + \sqrt{\delta^2 + 1}) \sim 1. \tag{6.18}
\]
Table 6.1

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Ratio ( \sim )</th>
<th>for bounds on</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6.3) ( \frac{(1+\sqrt{2})^{3/4}}{2} \times n^{1/2} )</td>
<td>( \min_{\alpha} )</td>
<td>( \min_{\alpha_{\text{max}}} ) for ( \delta \leq 1 )</td>
</tr>
<tr>
<td>(6.5) ( \frac{(1+\sqrt{2})^{3/4}}{2} \times (\frac{2}{\pi})^{\sqrt{2}/4} \times n^{\sqrt{2}/4} )</td>
<td>( \min_{\alpha_{\text{max}}} ) for ( \delta &gt; 1 )</td>
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<td>(6.7) ( \frac{\pi^{3/4}}{2} \times (1 + \sqrt{\delta^2 + 1}) \times n^{1/2} )</td>
<td>( \min_{\alpha_{\text{max}}} ) for ( \delta &gt; 1 )</td>
<td></td>
</tr>
<tr>
<td>(6.8) ( \frac{(1+\sqrt{2})^{3/4}}{2} \times \delta^{n} \times n^{1/2} )</td>
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</tr>
<tr>
<td>(6.12) ( \frac{(1+\sqrt{2})^{3/4}}{2} \times (\frac{2}{\pi})^{\sqrt{2}/4} \times n^{\sqrt{2}/4} )</td>
<td>( \min_{\alpha_{\text{max}}} ) for ( \delta &gt; 1 )</td>
<td></td>
</tr>
</tbody>
</table>

This is because

\[
1 \leq \frac{1 + \sqrt{\delta^2 + 1}}{1 + \sqrt{\delta^2 + 1}} \leq \left( \cos \frac{\pi}{2n} \right)^{-n} \Rightarrow 1 \leq \left( \frac{1 + \sqrt{\delta^2 + 1}}{1 + \sqrt{\delta^2 + 1}} \right)^n \leq \left( \cos \frac{\pi}{2n} \right)^{-n} \sim 1
\]

since \( -n \ln \cos \frac{\pi}{2n} \sim \frac{\pi^2}{2n} \rightarrow 0 \Rightarrow \left( \cos \frac{\pi}{2n} \right)^{-n} \sim 1 \). With (6.18), we have

\[
\frac{\delta' S_{n,1}(\delta', 0)}{S_{n-1,1}(\delta, 0)} = \frac{\delta' S_{n,1}(\delta', 0)}{\delta S_{n,1}(\delta, 0)} \times \frac{\delta S_{n,1}(\delta, 0)}{S_{n-1,1}(\delta, 0)}
\]

\[
\sim \left( \cos \frac{\pi}{2n} \right)^{-n} \times \left( \frac{1 + \sqrt{\delta^2 + 1}}{1 + \sqrt{\delta^2 + 1}} \right)^n \times (1 + \sqrt{\delta^2 + 1})
\]

\[
\sim 1 + \sqrt{\delta^2 + 1},
\]

\[
\frac{\delta^n S_{n,1}(\delta', 0)}{\delta'^n S_{n-1,1}(\delta, 0)} \sim \left( \frac{1 + \sqrt{\delta^2 + 1}}{1 + \sqrt{\delta^2 + 1}} \right)^n \times (1 + \sqrt{\delta^2 + 1})
\]

\[
\sim 1 + \sqrt{\delta^2 + 1},
\]

For the bounds in Theorem 6.3’, we notice

\[
\max\{n, \Lambda_n(0)(\delta')^{n-1}\} = \Lambda_n(0)(\delta')^{n-1} \sim \sqrt{\frac{\pi}{2}} \sqrt{n} \delta^{n-1}, \quad \omega_1 = (n/\Lambda_n(0))^{1/(n-1)} \sim 1
\]

for any given \( \delta > 1 \) and \( n \) sufficiently large. Using the analysis above, we arrive at Table 6.1 for the asymptotic behaviors for the ratios of the upper bounds over the lower bounds in the corresponding inequalities. Given that \( S_{n,1}(\delta, 0) \) goes to \( +\infty \) exponentially, our upper bounds and the lower bounds in Theorems 6.2, 6.2’, 6.3, and 6.3’ are very tight. These bounds, together with Theorem 2.1, lead to the qualitative behavior of \( \min_{\alpha_j} \kappa_p(V) \) as \( \sigma_{\text{max}} \) varies, depicted in Figure 1.1. From how we got the
upper bounds by these inequalities, we conclude that

\[
\text{For a fixed } \alpha_{\text{max}}, \text{ a nearly optimally conditioned } V \text{ is the one with the translated Chebyshev nodes on the symmetric interval that is slightly larger than } [-\alpha_{\text{max}}, \alpha_{\text{max}}] \text{ (so that } \pm \alpha_{\text{max}} \text{ are part of the nodes).}
\]

In addition to Table 6.1, the analysis above yields the asymptotical speeds of \( \min \kappa_\infty(V) \) for various cases, summarized in the following theorem.

**Theorem 6.4.** We have

\[
\begin{align*}
\text{(6.22)} & \quad \min_{\alpha_j} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{2})^n \right), \\
\text{(6.23)} & \quad \min_{\alpha_{\text{max}} \leq \delta} \kappa_\infty(V), \quad \min_{\alpha_{\text{max}} = \delta} \kappa_\infty(V) = \mathcal{O} \left( (\delta^{-1} + \sqrt{1 + \delta^{-2}})^n \right) \text{ for } \delta \leq 1, \\
\text{(6.24)} & \quad \min_{\alpha_{\text{max}} = \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{1 + \delta^2})^n \right) \text{ for } \delta > 1, \\
\text{(6.25)} & \quad \min_{\alpha_{\text{max}} \leq \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{2})^n \right) \text{ for } \delta > 1.
\end{align*}
\]

Equations (6.22) – (6.25) remain valid with \( V \) replaced by \( V_{\text{sym}} \).

This is a very informative theorem, for example,

\[
\begin{align*}
\text{(6.26)} & \quad \min_{\alpha_{\text{max}} \leq 1/2} \kappa_\infty(V), \quad \min_{\alpha_{\text{max}} = 1/2} \kappa_\infty(V) = \mathcal{O} \left( (2 + \sqrt{5})^n \right), \\
\text{(6.27)} & \quad \min_{\alpha_{\text{max}} = 2} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{5})^n \right).
\end{align*}
\]

Except (6.23), all other equations in Theorem 6.4 are in terms of \( \mathcal{O}_n(\cdots) \). It is quite natural to wonder whether this is really necessary. It was argued and conjectured in [18] that all \( \mathcal{O}_n(\cdots) \) in Theorem 6.4 could be replaced by \( \mathcal{O}(\cdots) \).

**7. Condition numbers for \( V \) with \( \alpha_i \in [\alpha, \beta] \) and \( 0 \leq \alpha < \beta \).** Assume throughout this section

\[
0 \leq \alpha < \beta.
\]

Notice that the case \( \alpha < \beta \leq 0 \) can be turned into this case by reversing the signs of all \( \alpha_j \) while leaving norms \( \|V\|_p \) and \( \|V^{-1}\|_p \) unchanged. So results in what follows apply to the case \( \alpha < \beta \leq 0 \) as well after minor modifications.

Let \( 0 \leq \alpha \leq \alpha_j \leq \max_j \alpha_j = \alpha_{\text{max}} \leq \beta \). Then \( \tau \leq -1 \), and thus by Theorem 2.1 \( S_{n-1,p'}(\omega, \tau) \) is decreasing in \( \omega \) and increasing in \( |\tau| \). Given \( \alpha_j \) and thus the constraints \( 0 \leq \alpha \leq \alpha_j \leq \beta \) on \( \alpha \) and \( \beta \), it follows from (4.3) that \( \omega \) achieves its smallest value and \( |\tau| \) its biggest values at the same time when

\[
\text{(7.1)} \quad \alpha = \min_j \alpha_j, \quad \beta = \alpha_{\text{max}} = \max_j \alpha_j
\]

at which the right-hand side of (5.9) is maximized over all possible feasible \( \alpha \) and \( \beta \) subject to \( 0 \leq \alpha \leq \alpha_j \leq \beta \). For this reason assignments (7.1) will be kept throughout this section, unless otherwise explicitly stated.
Theorem 7.1. Suppose \( \alpha_j \geq 0 \), and \( \alpha \) and \( \beta \) defined by (7.1). Then

\[
\kappa_p(V) \geq \max \left\{ S_{n-1,p}(\omega, \tau), \alpha_{\max}^{-1} S_{n-1,p}(\omega, \tau) / n^{1/p'} \right\},
\]

\[
\geq \max \left\{ S_{n-1,p}(\alpha_{\max}/2, 1), \alpha_{\max}^{-1} S_{n-1,p}(\alpha_{\max}/2, 1) / n^{1/p'} \right\},
\]

\[
= \begin{cases} 
S_{n-1,p}(\alpha_{\max}/2, 1), & \text{if } \alpha_{\max} \leq n^{1/[p'(n-1)]} \\
\alpha_{\max}^{-1} S_{n-1,p}(\alpha_{\max}/2, 1) / n^{1/p'}, & \text{if } \alpha_{\max} > n^{1/[p'(n-1)]}.
\end{cases}
\]

Proof. (7.2) is (5.9) with (7.1). (7.3) holds because of \( \omega \leq \alpha_{\max}/2 \) and \( \tau \leq -1 \), and the monotonicity of \( S_{n-1,p}(\omega, \tau) \) in its two arguments by Theorem 2.1. Also by Theorem 2.1, the first quantity within \( \max \{ \ldots \} \) in (7.3) is decreasing; while the second one is increasing. Therefore the right-hand side of (7.3) achieves its minimum when the two equal, i.e.,

\[ n^{1/p'} = \alpha_{\max}^{-1} \Rightarrow \alpha_{\max} = n^{1/[p'(n-1)]} \]

which yields (7.4). \( \square \)

Theorem 7.2.

\[
S_{n-1,p'} \left( n^{1/[p'(n-1)]} / 2, 1 \right) \leq \min_{\alpha_j \geq 0} \kappa_p(V) \leq n^{1/p} \frac{\sqrt{2}}{4} S_{n,1}(1/2, 1).
\]

Proof. The first inequality is true because the right-hand side of (7.3) achieves its minimum at \( \alpha_{\max} = n^{1/[p'(n-1)]} \). We now prove the second one. To this end, take \( 0 = \alpha < \beta = 1 \), and consider \( V \) with translated Chebyshev nodes \( x_j \). Then \( \|V\|_1 \leq n \) and \( \|V\|_\infty = n \), and thus by (3.3) \( \|V\|_p \leq \|V\|_1^{1/p} \|V\|_\infty^{1/p'} \leq n \). Apply Theorem 4.3 to the case \( 0 = \alpha < \beta = 1 \) (and thus \( \omega = 1/2 \) and \( \tau = -1 \)) to get \( \|V^{-1}\|_\infty \leq (2\sqrt{2}n)^{-1} S_{n,1}(1/2, 1) \) and then to get

\[ \|V^{-1}\|_p \leq n^{1/p} \|V^{-1}\|_\infty \leq n^{1/p} (2\sqrt{2}n)^{-1} S_{n,1}(1/2, 1). \]

So for this \( V \), \( \kappa_p(V) \leq n^{1/p} (2\sqrt{2})^{-1} S_{n,1}(1/2, 1) \), as needed. \( \square \)

Remark 7.1. The leftmost inequality in (7.5) can be replaced by

\[
S_{n-1,1}(1/2, 1) / n^{1/p'} \leq \min_{\alpha_j \geq 0} \kappa_p(V),
\]

upon using (5.11) with \( \eta = 1/2 \) and \( \tau = -1 \). This is due to [1] for \( p = 2 \), and turns out to be less sharp at least for \( p = \infty \). See Table 7.1. For any \( 1 \leq p \leq \infty \), the ratio of the upper bound in (7.5) over the lower bound here is \( n \frac{2\sqrt{2}}{2} \).

The second inequality in (7.5) was proved by simply picking a special \( V \) with the translated Chebyshev nodes in \([0, 1]\). For the same reason that led to Theorem 6.2’, for \( p = \infty \) a tighter upper bound is possible by using the \( V \) with the translated Chebyshev nodes in \([0, \beta_{opt}^+]\), where \( \beta_{opt}^+ = 2(n/\Lambda_n(1))^{1/(n-1)} \) as in Theorem 4.4. This gives the following theorem.

Theorem 7.2’.

\[
S_{n-1,1} \left( n^{1/(n-1)} / 2, 1 \right) \leq \min_{\alpha_j \geq 0} \kappa_\infty(V) \leq \frac{\beta_{opt}^+}{2 \sqrt{1 + \beta_{opt}^+}} S_{n,1}(\beta_{opt}^+/2, 1),
\]
where $\beta_{\text{opt}}^+ = 2(n/\Lambda_n(1))^{1/(n-1)}$.

**Theorem 7.3.** Given $0 \leq \gamma < \delta$, let

$$\omega_0 = \frac{\delta - \gamma}{2}, \quad \omega' = \frac{2}{1 + c} \omega_0, \quad \tau_0 = -\frac{\delta + \gamma}{\delta - \gamma}, \quad \tau' = \tau_0 \frac{1 + c}{2} \left(1 - \frac{1}{\tau_0} \frac{1 - c}{1 + c}\right),$$

where $c = \cos \frac{\pi}{2n}$. If $\delta < 1$, then

$$\begin{equation}
(7.8) \quad S_{n-1,p'}(\omega_0, \tau_0) \leq \min_{V \in \mathcal{V}_{[\gamma, \epsilon]}} \kappa_p(V) \leq n^{1/p} \Xi S_{n,1}(\omega', \tau') \leq n^{1/p} \Xi S_{n,1}(\omega_0, \tau_0),
\end{equation}$$

where

$$\begin{equation}
(7.9) \quad \Xi = \frac{\delta - \gamma}{\sqrt{1 + c} \sqrt{1 + \gamma} \sqrt{1 + c + 2\delta - \gamma + \gamma c}} = \frac{\delta - \gamma}{2\sqrt{1 + \gamma} \sqrt{1 + \delta}} + \frac{(\delta - \gamma)(\delta + \gamma + 2)}{16\sqrt{1 + \gamma}(1 + \delta)^{3/2}} \frac{\pi^2}{4n^2} + O \left(\frac{1}{n^2}\right);
\end{equation}$$

and if $\delta > 1$, then

$$\begin{equation}
(7.10) \quad \frac{\delta^{n-1} S_{n-1,p'}(\omega_0, \tau_0)}{n^{1/p'}} \leq \min_{V \in \mathcal{V}_{[\gamma, \epsilon]}} \kappa_p(V) \leq n^{1/p} \Xi^{n-1} S_{n,1}(\omega', \tau') \leq n^{1/p} \Xi^{n-1} S_{n,1}(\omega_0, \tau_0).
\end{equation}$$

**Proof.** Pick any $0 \leq \alpha \leq \gamma < \delta \leq \beta$, we still have (5.12). For $\alpha = \gamma$ and $\beta = \delta$, in particular, we deduce the first inequalities in (7.8) and (7.10). To prove the second inequalities, we take $\alpha = \gamma$, and let $\beta$ be determined. Set $\alpha_j = x_j$, and then pick $\beta$ such that $x_1 = \delta$, i.e.,

$$\delta = x_1 = \omega(t_1 - \tau) = \frac{\beta - \alpha}{2} \cos \frac{\pi}{2n} + \frac{\beta + \alpha}{2} = b \frac{1 + c}{2} + a \frac{1 - c}{2},$$

which gives

$$b = \delta + (\delta - \gamma) \frac{1 - c}{1 + c}, \quad \omega = \omega_0 \frac{2}{1 + c} = \omega', \quad \tau = \tau_0 \frac{1 + c}{2} \left(1 - \frac{1}{\tau_0} \frac{1 - c}{1 + c}\right) = \tau'.$$

Apply Theorem 4.3 to get $\|V^{-1}\|_{\infty} \leq \Xi S_{n,1}(\omega, \tau)$. Now $\|V\|_p \leq n$ if $\delta \leq 1$ and $\|V\|_p \leq n^{\delta n-1}$ if $\delta \geq 1$ and $\|V^{-1}\|_p \leq n^{1/p} \|V^{-1}\|_{\infty}$. Thus if $\delta \leq 1$,

$$\kappa_p(V) \leq n^{1+1/p} \|V^{-1}\|_{\infty} \leq n^{1/p} \Xi S_{n,1}(\omega, \tau);$$

and if $\delta \geq 1$,

$$\kappa_p(V) \leq n^{1+1/p} \delta^{n-1} \|V^{-1}\|_{\infty} \leq n^{1/p} \delta^{n-1} \Xi S_{n,1}(\omega, \tau).$$

This completes the proofs for the second inequalities in (7.8) and (7.10). The third inequalities hold because $\omega' > \omega_0$ and $|\tau'| \leq |\tau_0|$. $\square$

The theorem below is essentially Theorem 7.3 for $\gamma = 0$, except (7.13) which can be proved in the same as for (7.5).
Theorem 7.4. Let \( \delta > 0 \), and let \( \delta' = [2/(1 + c)]\delta \geq \delta \), where \( c = \cos \frac{\pi}{2k} \). If \( \delta < 1 \), then

\[
(7.11) \quad S_{n-1,p'}(\delta/2, 1) \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_p(V) \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_p(V) \leq \frac{n^{1/p} \delta'}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1);
\]

and if \( \delta > 1 \), then

\[
(7.12) \quad \frac{\delta^{-1} S_{n-1,p'}(\delta/2, 1)}{n^{1/p'}} \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_p(V) \leq \left( \frac{1 + c}{2} \right) \frac{n^{1/p} (\delta')^n}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1),
\]

\[
(7.13) \quad S_{n-1,p'} \left( \frac{n^{1/p'(n-1)}}{2}, 1 \right) \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_p(V) \leq n^{1/p} \frac{\sqrt{2}}{4} S_{n,1}(1/2, 1).
\]

Theorem 7.4'. Let \( \delta > 1 \), and let \( \delta' = [2/(1 + c)]\delta \geq \delta \), where \( c = \cos \frac{\pi}{2k} \). Then

\[
(7.14) \quad \frac{\delta^{-1} S_{n-1,1}(\delta/2, 1)}{n} \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_\infty(V) \leq \frac{\max\{n, 2^{-(n-1)}\Lambda_n(1)(\delta')^{n-1}\}}{n} \times \frac{\delta'}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1),
\]

\[
(7.15) \quad S_{n-1,1} \left( \frac{n^{1/(n-1)}}{2}, 1 \right) \leq \min_{V \in \mathcal{V}[0, \delta]} \kappa_\infty(V) \leq \frac{\delta_1}{2\sqrt{1 + \delta_1}} S_{n,1}(\delta_1/2, 1),
\]

where \( \delta_1 = \min\{\delta', 2(n/\Lambda_n(1))^{1/(n-1)}\} \). It can be seen that \( \delta_1 = 2(n/\Lambda_n(1))^{1/(n-1)} \) for \( n \) sufficiently large.

Proof. Only the second inequalities in (7.14) and (7.15) need proofs. For (7.14), it follows from the proof of Theorem 7.3 for \( \gamma = 0 \), where instead of \( ||V||_\infty \leq n\delta^{-n-1} \) we use \( ||V||_\infty \leq \max\{n, \omega^{n-1}\Lambda_n(1)\} \), which is proportional to \( \sqrt{n}\delta^{-n-1} \). The second inequality in (7.15) is obtained by approximately minimizing

\[
\frac{\max\{n, 2^{-(n-1)}\Lambda_n(1)(\alpha'_{\max})^{n-1}\}}{n} \frac{\alpha'_\max}{2\sqrt{1 + \alpha'_\max}} S_{n,1}(\alpha'_\max/2, 1)
\]

subject to \( \alpha_{\max} \leq \delta \), where \( \alpha'_{\max} = [2/(1 + c)]\alpha_{\max} \). That \( \delta_1 = 2(n/\Lambda_n(1))^{1/(n-1)} \) for \( n \) sufficiently large is due to \( (n/\Lambda_n(1))^{1/(n-1)} \sim 2^{-1} (n\pi)^{1/[2(n-1)]} \sim 2^{-1} \). \( \square \)

We shall now investigate the tightness of the upper bounds and the lower bounds in this section. Again, we shall do so only for \( p = \infty \) (and thus \( p' = 1 \)). Since

\[
1 + 2n^{1/(n-1)} = 3 - 2 \ln(n - 1) + \frac{2 + \ln^2(n - 1)}{(n - 1)^2} + \ldots,
\]
Theorem 7.2 says that $\kappa_{\infty}(V)$ is no smaller than
\[
T_{n-1}(1+2n^{-1/(n-1)}) = \left(\frac{n-1}{2}\right)^{-1/\sqrt{n}}(3+2\sqrt{2})^{n-1} \left[1 + O\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\
+ \left(\frac{n-1}{2}\right)^{1/\sqrt{n}}(3+2\sqrt{2})^{-n} \left[1 - O\left(\frac{\ln^2(n-1)}{n-1}\right)\right] \\
\sim \frac{(3+2\sqrt{2})^{n-1}}{2(n-1)^{1/\sqrt{n}}}.
\] (7.16)

On the other hand, $S_{n,1}(1/2,1) = \frac{1}{2}(3+2\sqrt{2})^n + \frac{1}{2}(3+2\sqrt{2})^{-n}$, and for $\beta_{\text{opt}}^+ = 2(n/\Lambda_n(1))^{1/(n-1)}$,
\[
S_{n,1}(\beta_{\text{opt}}^+/2,1) = \frac{(n\pi)^{-\sqrt{n}/4}}{2}(3+2\sqrt{2})^n \left[1 + O\left(\frac{\ln(n-1)}{n-1}\right)\right] \\
+ \frac{(n\pi)^{\sqrt{n}/4}}{2}(3+2\sqrt{2})^{-n} \left[1 - O\left(\frac{\ln(n-1)}{n-1}\right)\right] \\
\sim \frac{(3+2\sqrt{2})^n}{2(n\pi)^{\sqrt{n}/4}}.
\] (7.17)

Therefore
\[
\frac{S_{n,1}(1/2,1)}{S_{n-1,1}(n^{1/(n-1)}/2,1)} \sim (3+2\sqrt{2})^{n^{1/\sqrt{n}}}, \\
\frac{S_{n,1}(\beta_{\text{opt}}^+/2,1)}{S_{n-1,1}(n^{1/(n-1)}/2,1)} \sim (3+2\sqrt{2})^{(n\pi)^{\sqrt{n}/4}},
\] (7.18)

Let us turn to Theorem 7.3. First, we claim
\[
\phi \triangleq \left(\frac{\frac{1}{\omega_0} + |\tau'| + \sqrt{\left(\frac{1}{\omega_0} + |\tau'| \right)^2 - 1}}{\frac{1}{\omega_0} + |\tau_0| + \sqrt{\left(\frac{1}{\omega_0} + |\tau_0| \right)^2 - 1}}\right)^n \sim 1.
\] (7.19)

To this end, we notice that $\zeta \triangleq \left(1 - \frac{\frac{1}{\omega_0} + \frac{\xi}{2\zeta}}{1 + \frac{\xi}{2\zeta}}\right) \geq 1$, and $\frac{\xi}{2\zeta} \xi \leq 1$, and
\[
\frac{1}{\omega_0} + |\tau'| + \sqrt{\left(\frac{1}{\omega_0} + |\tau'| \right)^2 - 1} = \frac{1+\epsilon}{2} \left(\frac{1}{\omega_0} + |\tau_0| + \sqrt{\left(\frac{1}{\omega_0} + |\tau_0| \right)^2 - 1}\right) \\
\geq \frac{1+\epsilon}{2} \left(\frac{1}{\omega_0} + |\tau_0| + \sqrt{\left(\frac{1}{\omega_0} + |\tau_0| \right)^2 - 1}\right).
\]

Therefore $1 \geq \phi \geq \left(\frac{1+\epsilon}{2}\right)^n \sim 1$ because $n \ln \left(\frac{1+\epsilon}{2}\right) \sim -\frac{\pi^2}{16n} \to 0$. This proves (7.19).

With (7.19), we have
\[
\frac{S_{n,1}(\omega', \tau')}{{S_{n-1,1}(\omega_0, \tau_0)} \sim \frac{S_{n,1}(\omega_0, \tau_0)}{S_{n-1,1}(\omega_0, \tau_0)} \sim \left(\frac{\frac{1}{\omega_0} + |\tau_0| + \sqrt{\left(\frac{1}{\omega_0} + |\tau_0| \right)^2 - 1}}{\frac{1}{\omega_0} + |\tau_0| + \sqrt{\left(\frac{1}{\omega_0} + |\tau_0| \right)^2 - 1}}\right)^n.
\] (7.20)
For the bounds in Theorem 7.4', we notice
\[
\max(n, 2^{-(n-1)} \Lambda_n(1)(\delta')^{n-1}) = 2^{-(n-1)} \Lambda_n(1)(\delta')^{n-1} \sim \sqrt{n} \delta^{n-1},
\]
for any given \( \delta > 1 \) and \( n \) sufficiently large. Using the analysis above, we arrive at Table 7.1 for the asymptotic behaviors for the ratios of the upper bounds over the lower bounds in the corresponding inequalities. The conclusions are that these bounds are very tight. These bounds, together with Theorem 2.1, lead to the qualitative behavior of \( \min_{\alpha_j \geq 0} \kappa_{\infty}(V) \) as \( \alpha_{\max} \) varies, depicted in Figure 1.1. Also,
\[
(7.21) \quad \text{For a fixed } \alpha_{\max}, \text{ a nearly optimally conditioned } V \text{ is the one with the translated Chebyshev nodes on an interval slightly larger than } [0, \alpha_{\max}] \text{ (so that } \alpha_{\max} \text{ is a node)}.
\]

In addition to Table 7.1, the analysis above yields the asymptotical speeds of \( \min \kappa_{\infty}(V) \) for various cases, summarized in the following theorem.

**Theorem 7.5.** For \( 0 \leq \gamma < \delta \),
\[
\begin{align*}
(7.22) \quad & \min_{V \in V_{\gamma, \delta}} \kappa_{\infty}(V) = \mathcal{O}\left( \left[ \frac{1}{\omega_0} + |\tau_0| + \left( \frac{1}{\omega_0} + |\tau_0| \right)^2 - 1 \right]^n \right) \text{ for } \delta \leq 1, \\
(7.23) \quad & \min_{V \in V_{\gamma, \delta}} \kappa_{\infty}(V) = \mathcal{O}\left( \delta^n \left[ \frac{1}{\omega_0} + |\tau_0| + \left( \frac{1}{\omega_0} + |\tau_0| \right)^2 - 1 \right]^n \right) \text{ for } \delta > 1;
\end{align*}
\]
\( \min_{V \in [0, \delta]} \kappa_\infty(V), \min_{V \in [0, \delta]} \kappa_\infty(V) = \mathcal{O}\left(\left[\delta^{-1/2} + 1 + \sqrt{1 + \delta^{-1}}\right]^{2n}\right) \) for \( \delta \leq 1 \),

\( \min_{\alpha_{\text{max}} = \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{1 + \delta})^{2n} \right) \) for \( \delta > 1 \),

\( \min_{\alpha_{\text{max}} = \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (3 + 2\sqrt{2})^n \right) \) for \( \delta > 1 \);

And finally,

\( \min_{\alpha_{\text{max}} = \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (3 + 2\sqrt{2})^n \right) \).

With constants and modest factors \( n^d \) hidden, this theorem is perhaps more informative than the previous ones, for example,

\[
\min_{\alpha_{\text{max}} = 1/2} \kappa_\infty(V) = \mathcal{O}_n \left( (11 + 2\sqrt{30})^n \right),
\]

\[
\min_{\alpha_{\text{max}} = 1/2} \kappa_\infty(V) = \mathcal{O}_n \left( (10 + 4\sqrt{6})^n \right),
\]

\[
\min_{\alpha_{\text{max}} = 1/2} \kappa_\infty(V) = \mathcal{O}_n \left( (5 + 2\sqrt{6})^n \right),
\]

\[
\min_{\alpha_{\text{max}} = 2} \kappa_\infty(V) = \mathcal{O}_n \left( (4 + 2\sqrt{3})^n \right).
\]

8. Possible extensions to Vandermonde-like matrices and complex Vandermonde matrices. This section outlines two possible extensions to Vandermonde-like matrices and complex Vandermonde Matrices.

A Vandermonde-like matrix \( V \) is defined by

\[
\tilde{V} \overset{\text{def}}{=} \begin{pmatrix}
    p_0(\alpha_1) & p_0(\alpha_2) & \cdots & p_0(\alpha_n) \\
    p_1(\alpha_1) & p_1(\alpha_2) & \cdots & p_1(\alpha_n) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_{n-1}(\alpha_1^{n-1}) & p_{n-1}(\alpha_2^{n-1}) & \cdots & p_{n-1}(\alpha_n^{n-1})
\end{pmatrix},
\]

where \( p_i \) is a polynomial of degree \( i \). For practical considerations, often these \( p_i \) form a family of orthogonal polynomials and thus satisfy a three-term recurrence relation, often \( \alpha_i \) are the zeros of \( p_i \) in the family. As to the conditioning of \( V \) with orthogonal polynomials, Gautschi [10] derived an explicit formula for its condition number in Frobenius norm in terms of the associated Christoffel numbers. Reichel and Opfer [19] developed a progressive way to control the condition number for \( \tilde{V} \) by Chebyshev polynomials. However what we are going to outline is a technique to obtain a lower bound on \( \kappa_\infty(V) \) in the most general setting, i.e., \( p_i \) is a polynomial of degree \( i \), and all \( \alpha_j \in [\alpha, \beta] \).
Asymptotically Optimal Condition Number of a Real Vandermonde Matrix

Expand \( T_n(x; \omega, \tau) \) as a linear combination of \( p_0(x), p_1(x), \ldots, p_n(x) \):

\[
T_n(x; \omega, \tau) = \tilde{a}_0 p_0(x) + \tilde{a}_1 p_1(x) + \cdots + \tilde{a}_n p_n(x),
\]

and define, similarly to \( S_{n,p}(\omega, \tau) \),

\[
\tilde{S}_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} |\tilde{a}_j|^p \right)^{1/p}.
\]

Then it can be shown that, using the lines of arguments from Lemma 5.1,

\[
\|V^{-1}\|_p \geq \tilde{S}_{n-1,p'}(\omega, \tau)/n^{1/p'}.
\] (8.2)

Often getting a lower bound on \( \|V\|_p \) is rather straightforward, and with that a lower bound on \( \kappa_p(V) \) is readily established.

Now we consider a complex Vandermonde matrix. We still use the symbol \( V \) but with complex \( \alpha_j \). Suppose that all \( \alpha_j \) lie in the ellipse \( E \) with foci \( \alpha, \beta \in \mathbb{R} \), and semimajor axis \( R \) and semiminor axis \( r \), as shown in Figure 8.1. It can be shown that \cite{20, Page 203}

\[
\max_{z \in E} |T_n(z; \omega, \tau)| = T_n(z; \omega, \tau)|_{z=(\beta+\alpha)/2+R} = T_n(R/\omega).
\]

Therefore, resembling Lemma 5.1, we have

\[
\|V^{-1}\|_p \geq \frac{\tilde{S}_{n-1,p'}(\omega, \tau)}{n^{1/p'} T_{n-1}(R/\omega)}.
\] (8.3)

9. Concluding remarks. We have obtained a series of lower and upper bounds on the condition number \( \kappa_p(V) \) of a real Vandermonde matrix \( V \). These bounds are
proved to be asymptotically optimal, except possibly the one in Theorem 5.2 in the case when interval \( \alpha, \beta \) is not one of the three kinds: 1) symmetrical \((-\alpha = \beta)\); 2) nonnegative \((\alpha \geq 0)\); 3) non-positive \((\beta \leq 0)\).

Our results led us to deduce the qualitative behaviors of optimally conditioned Vandermonde matrices as the largest absolute value \( \alpha_{\text{max}} \) of all nodes varies, as shown in Figure 1.1 at the beginning of this paper. Our proofs yielded nearly optimally conditioned Vandermonde matrices in various circumstances.

Similar bounds, though unclear about their asymptotically optimality, can be proved, too, for confluent Vandermonde matrices and rectangular Vandermonde matrices. Details are in [18].

REFERENCES