AN INTERPOLATING FAMILY OF MEANS

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ABSTRACT. This paper is concerned with a new family of binary symmetric means M_p of two positive numbers a and b:

$$\frac{1}{\mathrm{M}_p(a,b)} = c_p \int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}}, \quad 0<p<\infty,$$

where the constant c_p, depending on p, is chosen to have $\mathrm{M}_p(a,a) = a$. Two distinctive members in the family are the well-known logarithmic mean ($p=1$) and arithmetic-geometric mean ($p=2$). Different expressions for M_p are obtained to establish its other properties, including $\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b)$ and the relation between M_p and the power difference mean. Through investigating the induced operator norm of the integral operator with M_p^{-1} as its kernel, a generalization of the Hilbert inequality is obtained. Finally positive definiteness of certain matrices as implications of inequalities between two means is also investigated.

Dedicated to K. R. Parthasarathy on the occasion of his 75th birthday

1. Introduction

Let a and b be positive numbers. The *logarithmic mean* $\mathrm{L}(a,b)$ of a and b defined as

$$\mathrm{L}(a,b) := \frac{a-b}{\ln a - \ln b} \tag{1.1}$$

has long been used in problems related to heat flow [16] and electrical conduction [17]. More recently it has been employed in differential geometry [2, 5]. The well-known *arithmetic-geometric mean* $\mathrm{AG}(a,b)$ of Gauss is defined as follows: the sequences $\{a_n\}$ and $\{b_n\}$ defined inductively as

$$a_0 = a, \qquad b_0 = b,$$
$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

have a common limit, and

$$\mathrm{AG}(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 26E60; Secondary 15A45, 42A82, 47A30.
Key words and phrases. binary symmetric mean, logarithmic mean, arithmetic-geometric mean, power difference mean, integral operator, positive definite matrix.
* Supported in part by a J. C. Bose National Fellowship. He thanks the University of Texas at Arlington for a visit in March 2007 when this work was begun.
§ Supported in part by the National Science Foundation Grant DMS-0810506 and DMS-1115834.

This mean, introduced by Legendre and then by Gauss, is related to the evaluation of elliptic integrals, and several other problems in analysis [11, 8].

The expressions (1.1) and (1.2) do not carry any hint that these two means could belong to a common family. There are alternative descriptions for both. It can be seen that

$$\frac{1}{\mathrm{L}(a,b)} = \int_0^\infty \frac{dx}{(x+a)(x+b)}, \tag{1.3}$$

and an ingenious calculation, due to Gauss [12], is used to show

$$\frac{1}{\mathrm{AG}(a,b)} = \frac{2}{\pi}\int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}. \tag{1.4}$$

This similarity between the expressions (1.3) and (1.4) is the motivation for us to introduce a family of means $\mathrm{M}_p(a,b)$, $0 \le p \le \infty$, defined by the relation

$$\frac{1}{\mathrm{M}_p(a,b)} := c_p \int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}}, \quad 0 < p < \infty, \tag{1.5}$$

where the constant c_p, depending on p, will be chosen to have

$$\mathrm{M}_p(a,a) = a.$$

Thus

$$\frac{1}{c_p} = a\int_0^\infty \frac{dx}{(x^p+a^p)^{2/p}} = \int_0^\infty \frac{dy}{(y^p+1)^{2/p}}. \tag{1.6}$$

The means M_0 and M_∞ are defined by taking limits:

$$\mathrm{M}_0(a,b) := \lim_{p\to 0^+} \mathrm{M}_p(a,b), \quad \mathrm{M}_\infty(a,b) := \lim_{p\to\infty} \mathrm{M}_p(a,b). \tag{1.7}$$

A *binary symmetric mean* $M(a,b)$ of positive numbers a and b is a function that satisfies the following properties:

(i) $\min\{a,b\} \le M(a,b) \le \max\{a,b\}$ (In particular, $M(a,a) = a$);
(ii) $M(a,b) = M(b,a)$;
(iii) $M(\alpha a, \alpha b) = \alpha\, M(a,b)$ for all $\alpha > 0$;
(iv) $M(a,b)$ is non-decreasing in a and b.

It is obvious from the definition that the mean M_p satisfies the properties (ii) – (iv). We will give different expressions for M_p from which other properties, including (i) above, become apparent. In particular, we will show that

$$\mathrm{M}_0(a,b) = \sqrt{ab}, \tag{1.8}$$

$$\mathrm{M}_\infty(a,b) = \frac{2\max\{a,b\}}{2+\ln\frac{\max\{a,b\}}{\min\{a,b\}}} \le \frac{a+b}{2}. \tag{1.9}$$

We conjecture that *for fixed a and b, $\mathrm{M}_p(a,b)$ is an increasing function of p.* At this time, we can prove that

$$\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b). \tag{1.10}$$

Inequalities already known then lead to the chain

$$\mathrm{M}_0(a,b) \le \mathrm{M}_1(a,b) \le \mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b). \tag{1.11}$$

The first of these inequalities is well-known and easy to prove; the second has been given different proofs in [9, 10, 19].

The rest of this paper is organized as follows. Section 2 investigates various properties of M_p in detail, including different expressions for M_p and the inequality (1.10). Section 3 gives a relation between M_p and the *power difference mean* K_p. In section 4, we evaluate the norm of the integral operator induced on the space $L_2(\mathbb{R}_+)$ by the kernel $1/\mathrm{M}_p(x,y)$. This gives an extension of the famous Hilbert inequality. In section 5, we discuss positive definiteness of certain matrices as implications of some relations between M_p and K_p for which another conjecture is also proposed.

2. Mean M_p

Expressions (1.8) for M_0 and (1.9) for M_∞ will be proved after a detailed investigation on M_p for $0<p<\infty$ is completed. Then we will prove the inequality (1.10).

2.1. $0 < p < \infty$.

Theorem 2.1. $\min\{a,b\} \le \sqrt{ab} \le \mathrm{M}_p(a,b) \le \left(\frac{a^p+b^p}{2}\right)^{1/p} \le \max\{a,b\}$.

Proof. The first inequality is easy to see, and the last one is easy to see too by replacing both a and b in $\left(\frac{a^p+b^p}{2}\right)^{1/p}$ with $\max\{a,b\}$. We now prove the second and the third inequalities. Since $(x^p+a^p)(x^p+b^p) = (x^p)^2 + (a^p+b^p)x^p + a^pb^p$, we have

$$[x^p + (\sqrt{ab})^p]^2 \le (x^p+a^p)(x^p+b^p) \le \left[x^p + \frac{a^p+b^p}{2}\right]^2.$$

Therefore by (1.5),

$$\frac{1}{\mathrm{M}_p\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}, \left[\frac{a^p+b^p}{2}\right]^{1/p}\right)} \le \frac{1}{\mathrm{M}_p(a,b)} \le \frac{1}{\mathrm{M}_p(\sqrt{ab},\sqrt{ab})}$$

which, together with the condition $\mathrm{M}_p(z,z) = z$ for any $z>0$, lead to the desired inequalities. □

In the last integral in (1.6), substitute $t = (y^p+1)^{-1}$ to get

$$y^p = \frac{1}{t} - 1 = \frac{1-t}{t}, \tag{2.1}$$

$$py^{p-1}\,dy = -\frac{1}{t^2}\,dt, \tag{2.2}$$

$$dy = -\frac{1}{p}\left(\frac{t}{1-t}\right)^{\frac{p-1}{p}}\frac{1}{t^2}\,dt$$

$$= -\frac{1}{p}t^{-1-1/p}(1-t)^{1/p-1}\,dt, \tag{2.3}$$

$$\frac{1}{c_p} = \frac{1}{p}\int_0^1 t^{1/p-1}(1-t)^{1/p-1}\,dt \tag{2.4}$$

$$= \frac{B(\frac{1}{p},\frac{1}{p})}{p}, \tag{2.5}$$

where $B(\cdot,\cdot)$ is the Beta-function [1]. In the integral in (1.5), substitute $x^p + a^p = a^p t^{-1}$ to get

$$\begin{aligned}
x^p &= a^p\left(\frac{1}{t}-1\right) = a^p\frac{1-t}{t},\\
px^{p-1}\,dx &= -a^p\frac{1}{t^2}\,dt,\\
dx &= -\frac{a}{p}\left(\frac{t}{1-t}\right)^{\frac{p-1}{p}}\frac{1}{t^2}\,dt\\
&= -\frac{a}{p}t^{-1-1/p}(1-t)^{1/p-1}\,dt,\\
\frac{1}{\mathrm{M}_p(a,b)} &= c_p\frac{a}{p}\int_0^1\frac{t^{-1/p-1}(1-t)^{1/p-1}}{[(a^pt^{-1})(a^p\frac{1-t}{t}+b^p)]^{1/p}}\,dt\\
&= \frac{c_p}{p}\int_0^1\frac{t^{1/p-1}(1-t)^{1/p-1}}{[a^p(1-t)+b^pt]^{1/p}}\,dt. \qquad (2.6)
\end{aligned}$$

Combine (2.4) and (2.6) to get

$$\frac{1}{\mathrm{M}_p(a,b)} = \frac{\displaystyle\int_0^1\frac{t^{1/p-1}(1-t)^{1/p-1}}{[a^p(1-t)+b^pt]^{1/p}}\,dt}{\displaystyle\int_0^1 t^{1/p-1}(1-t)^{1/p-1}\,dt}. \qquad (2.7)$$

Theorem 2.2. *Given $a, b > 0$ and $0 < p < \infty$, we have*

$$\frac{1}{\mathrm{M}_p(a,b)} = (\max\{a,b\})^{-1}\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\left(\frac{1}{p}+i\right)^2}{\frac{2}{p}+i}\frac{1}{k!}\left[1-\left(\frac{\min\{a,b\}}{\max\{a,b\}}\right)^p\right]^k, \qquad (2.8)$$

where, by convention, $\prod_{i=0}^{-1}(\cdots)\equiv 1$ and $0! = 1$.

Proof. Both sides of (2.8) are equal to a if $a = b$. Assume without loss of generality that $a > b > 0$. Let $\alpha = 1-(b/a)^p$ and then $0 < \alpha < 1$. We have

$$\begin{aligned}
a^p(1-t)+b^pt &= a^p[1-t+(b/a)^pt] = a^p(1-\alpha t),\\
[a^p(1-t)+b^pt]^{-1/p} &= a^{-1}(1-\alpha t)^{-1/p}\\
&= a^{-1}\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^kt^k, \qquad (2.9)
\end{aligned}$$

The series in (2.9) converges for $\alpha < 1$ which justifies the term-by-term integration below. Equation (2.9), together with (2.7), yield

$$\frac{1}{\mathrm{M}_p(a,b)} = \frac{a^{-1}\displaystyle\int_0^1\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^kt^{k+1/p-1}(1-t)^{1/p-1}\,dt}{\displaystyle\int_0^1 t^{1/p-1}(1-t)^{1/p-1}}$$

$$= \frac{a^{-1} \sum_{k=0}^{\infty} \int_0^1 \frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!} \alpha^k t^{k+1/p-1}(1-t)^{1/p-1}\, dt}{\int_0^1 t^{1/p-1}(1-t)^{1/p-1}}$$

$$= a^{-1} \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!} \alpha^k \cdot \frac{B(k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})}. \tag{2.10}$$

Using the well-known properties of the Beta and Gamma functions [1]

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \Gamma(z) = (z-1)\Gamma(z-1),$$

we have

$$\begin{aligned} \frac{B(k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})} &= \frac{\Gamma(k+\frac{1}{p})\Gamma(\frac{1}{p})}{\Gamma(k+\frac{2}{p})} \frac{\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p})} \\ &= \frac{\Gamma(k+\frac{1}{p})}{\Gamma(\frac{1}{p})} \frac{\Gamma(\frac{2}{p})}{\Gamma(k+\frac{2}{p})} \\ &= \prod_{i=0}^{k-1} \frac{\frac{1}{p}+i}{\frac{2}{p}+i}. \end{aligned}$$

Substituting this into (2.10) gives (2.8). □

Theorem 2.3. *Given $a, b > 0$ and $0 < p < \infty$, we have*

$$\frac{1}{\mathrm{M}_p(a,b)} = \left(\frac{a^p+b^p}{2}\right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{k!} \left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] \left[\prod_{i=0}^{2k-1} \frac{\frac{1}{p}+i}{\frac{2}{p}+i}\right] \left(\frac{a^p-b^p}{a^p+b^p}\right)^{2k}. \tag{2.11}$$

Proof. We have

$$\begin{aligned} (x^p+a^p)(x^p+b^p) &= \left(x^p + \frac{a^p+b^p}{2}\right)^2 - \left(\frac{a^p-b^p}{2}\right)^2 \\ &= \left(x^p + \frac{a^p+b^p}{2}\right)^2 \left(1-r^2\right), \end{aligned}$$

where $r = \frac{a^p-b^p}{2} / \left(x^p + \frac{a^p+b^p}{2}\right)$. Therefore $|r| < 1$ and

$$\begin{aligned} [(x^p+a^p)(x^p+b^p)]^{-1/p} &= \left(x^p + \frac{a^p+b^p}{2}\right)^{-2/p} \left(1-r^2\right)^{-1/p}, \\ &= \left(x^p + \frac{a^p+b^p}{2}\right)^{-2/p} \sum_{k=0}^{\infty} \left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] \frac{r^{2k}}{k!}, \end{aligned}$$

$$\frac{1}{\mathrm{M}_p(a,b)} = c_p \int_0^{\infty} \frac{1}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}} \sum_{k=0}^{\infty} \left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] \frac{r^{2k}}{k!}\, dx$$

$$
\begin{aligned}
&= c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}} \\
&\quad + c_p \sum_{k=1}^\infty \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] \int_0^\infty \frac{r^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}}\, dx \\
&= \left(\frac{a^p+b^p}{2}\right)^{-1/p} \\
&\quad + \sum_{k=1}^\infty \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] c_p \int_0^\infty \frac{\left(\frac{a^p-b^p}{2}\right)^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}}\, dx.
\end{aligned}
\tag{2.12}
$$

Substitute $x = \left(\frac{a^p+b^p}{2}\right)^{1/p} y$ and $y^p + 1 = t^{-1}$ as in (2.1) – (2.3) to get

$$
\begin{aligned}
\int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \int_0^\infty \frac{dy}{(y^p+1)^{2k+2/p}} \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \int_0^1 \frac{1}{p} t^{2k+1/p-1}(1-t)^{1/p-1}\, dt \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \frac{B(2k+\frac{1}{p},\frac{1}{p})}{p}.
\end{aligned}
$$

This together with (2.5) lead to

$$
\begin{aligned}
c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \frac{B(2k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})} \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \prod_{i=0}^{2k-1} \frac{\frac{1}{p}+i}{\frac{2}{p}+i}.
\end{aligned}
\tag{2.13}
$$

Now (2.11) is a consequence of (2.12) and (2.13). ☐

2.2. $p = 0$.

Theorem 2.4. *Given* $a, b > 0$, *we have*

$$
\mathrm{M}_0(a,b) = \sqrt{ab}. \tag{2.14}
$$

Proof. It can be verified that $\lim_{p\to 0^+} \left(\frac{a^p+b^p}{2}\right)^{1/p} = \sqrt{ab}$. The equality $\mathrm{M}_0(a,b) = \sqrt{ab}$ is then a consequence of Theorem 2.1. ☐

2.3. $p = \infty$.

Theorem 2.5. *Given* $a, b > 0$, *we have*

$$
\mathrm{M}_\infty(a,b) = \frac{2\max\{a,b\}}{2 + \ln\frac{\max\{a,b\}}{\min\{a,b\}}}, \tag{2.15}
$$

and

$$
\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b) \le (a+b)/2. \tag{2.16}
$$

Proof. Both (2.15) and (2.16) are obvious if $a = b$. Assume that $a > b > 0$. Then

$$\lim_{p\to\infty}\int_0^\infty \frac{dy}{(y^p+1)^{2/p}} = \int_0^1 dy + \int_1^\infty \frac{dy}{y^2} = 2,$$

$$\begin{aligned}\lim_{p\to\infty}\int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}} &= \int_0^b \frac{dx}{ab} + \int_b^a \frac{dx}{xa} + \int_a^\infty \frac{dx}{x^2}\\ &= \frac{1}{a} + \frac{\ln a - \ln b}{a} + \frac{1}{a}\\ &= \frac{2 + (\ln a - \ln b)}{a}.\end{aligned}$$

Therefore $c_\infty = 1/2$, and (2.15) holds by definition. The change of order of taking limits and the integrals above is justified by Lebesgue's Dominated Convergence Theorem [21, p.76] because

$$\frac{1}{(y^p+1)^{2/p}} \le \begin{cases} 1 & \text{for } 0 \le y \le 1,\\ y^{-2} & \text{for } 1 < y,\end{cases}$$

$$\frac{1}{[(x^p+a^p)(x^p+b^p)]^{1/p}} \le \begin{cases} (ab)^{-1} & \text{for } 0 \le x \le a,\\ x^{-2} & \text{for } a < x.\end{cases}$$

The second inequality in (2.16) is relatively easy to show. It goes as follows. Since

$$\mathrm{M}_1(a,b) = \frac{a-b}{\ln a - \ln b} \le \frac{a+b}{2},$$

we have successively

$$\begin{aligned} 2(a-b) &=\le (a+b)(\ln a - \ln b),\\ 2a &\le 2b + (a+b)(\ln a - \ln b),\\ 4a &= 2(a+b) + (a+b)(\ln a - \ln b),\\ \frac{2a}{2+\ln a - \ln b} &\le \frac{a+b}{2},\end{aligned}$$

as expected.

Let us focus on the first inequality in (2.16) now. As in the proof by John Todd for *Problem 19-17* in [10], let $r = (a-b)/(a+b)$. Then $0 < r < 1$ and $a = \frac{a+b}{2}(1+r)$ and $b = \frac{a+b}{2}(1-r)$. It suffices to show that $\mathrm{M}_2(1+r, 1-r) \le \mathrm{M}_\infty(1+r, 1-r)$. It was shown by Gauss [12] (see also [8, p.7]) that for $|r| < 1$

$$\frac{1}{\mathrm{M}_2(1+r,1-r)} = 1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots.$$

On the other hand, by (1.9),

$$\begin{aligned}\frac{1}{\mathrm{M}_\infty(1+r,1-r)} &= \frac{2+\ln\frac{1+r}{1-r}}{2(1+r)}\\ &= \frac{2 + 2r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right)}{2(1+r)}\end{aligned}$$

$$= \frac{1 + r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right)}{1+r}.$$

So for $\mathrm{M}_2(1+r, 1-r) \leq \mathrm{M}_\infty(1+r, 1-r)$ to hold, it suffices to have

$$(1+r)\left[1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \\ \geq 1 + r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right), \quad (2.17)$$

or, equivalently,

$$(1+r)\left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \\ \geq r\left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right). \quad (2.18)$$

Since $1 + r > 2r$, (2.18) holds if

$$2r\left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \\ \geq r\left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right) \quad (2.19)$$

which is guaranteed if the corresponding coefficients of r^{2n+1} from both sides satisfy

$$2\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 \geq \frac{1}{2n+1} \quad (2.20)$$

for $n \geq 1$. This is what we shall prove now. To this end, we shall use the following estimate for factorial $n!$ [18, 20]

$$\sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n+1)} < n! < \sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n)}. \quad (2.21)$$

We have

$$\begin{aligned} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &> \frac{\sqrt{2\pi}(2n)^{2n+1/2}e^{-2n+1/(24n+1)}}{2^{2n}[\sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n)}]^2} \\ &= \frac{e^{1/(24n+1)}}{\sqrt{\pi n}\, e^{1/(6n)}}, \end{aligned}$$

$$\begin{aligned} 2(2n+1)\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 &> 4n\left(\frac{e^{1/(24n+1)}}{\sqrt{\pi n}\, e^{1/(6n)}}\right)^2 \\ &= \frac{4}{\pi}\, e^{2/(24n+1)-1/(3n)} \\ &= \frac{4}{\pi}\, e^{-(18n+1)/[3n(24n+1)]} \end{aligned}$$

$$\geq 1.12 \quad \text{for } n \geq 2.$$

This proves that (2.19) holds for $n \geq 2$. It can be verified that (2.19) holds for $n = 1$ also. The proof is completed. □

Remark 2.6. One can use (2.21) to also show that

$$\left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 < \frac{1}{\pi n} < \frac{1}{2n+1}.$$

This is used by John Todd [10] to show $\mathrm{M}_1(1+r, 1-r) \leq \mathrm{M}_2(1+r, 1-r)$. ◇

3. Relation to the Power Difference Mean

The *power difference mean* $\mathrm{K}_p(a, b)$ is defined for any p and a, $b > 0$ as follows [14, 6].

$$\mathrm{K}_p(a, b) := \frac{p-1}{p} \frac{a^p - b^p}{a^{p-1} - b^{p-1}}, \tag{3.1}$$

where it is understood that

$$\mathrm{K}_p(a, a) := a, \quad \mathrm{K}_1(a, b) := \lim_{p \to 1} \mathrm{K}_p(a, b) = \mathrm{L}(a, b). \tag{3.2}$$

Alternatively, $\mathrm{K}_p(a, b)$ admits the following integral expression:

$$\frac{1}{\mathrm{K}_p(a, b)} = \int_0^1 \frac{dt}{[(1-t)a^p + tb^p]^{1/p}}. \tag{3.3}$$

By (1.1), (1.3), and (3.2), we have $\mathrm{M}_1(a, b) = \mathrm{L}(a, b) = \mathrm{K}_1(a, b)$. It makes us wonder what kind of relations are between $\mathrm{M}_p(a, b)$ and $\mathrm{K}_p(a, b)$ for $p \neq 1$. Theorem 3.2 below provides an answer. But first we establish an expansion formula for $\mathrm{K}_p(a, b)$.

Lemma 3.1. *Given $a, b > 0$, we have*

$$\frac{1}{\mathrm{K}_p(a, b)} = \left(\frac{a^p + b^p}{2}\right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left[\prod_{i=0}^{2k-1} \left(\frac{1}{p} + i\right)\right] \left(\frac{a^p - b^p}{a^p + b^p}\right)^{2k}. \tag{3.4}$$

Proof. K_p as defined by (3.1) has a removable singularity at $p = 1$. In this case, equation (3.4) can be verified either by using $\mathrm{K}_1(a, b) = \mathrm{L}(a, b)$ or by taking the limit as p goes to 1. In what follows, we shall assume $p \neq 1$. It suffices to show (3.4) for $a = 1$ and $0 < b \neq 1$. It follows from (3.1) that

$$\begin{aligned} \frac{1}{\mathrm{K}_p(a, b)} \left(\frac{a^p + b^p}{2}\right)^{1/p} &= \frac{p}{p-1} \frac{a^{p-1} - b^{p-1}}{a^p - b^p} \left(\frac{a^p + b^p}{2}\right)^{1/p} \\ &= \frac{p}{p-1} \frac{1 - b^{p-1}}{1 - b^p} \left(\frac{1 + b^p}{2}\right)^{1/p}. \end{aligned} \tag{3.5}$$

Let $r = (1 - b^p)/(1 + b^p)$. Then $|r| < 1$, and

$$b^p = \frac{1-r}{1+r}, \; \frac{1+b^p}{2} = \frac{1}{1+r}, \; 1 - b^p = \frac{2r}{1+r}, \; b^{p-1} = (b^p)^{(p-1)/p} = \left(\frac{1-r}{1+r}\right)^{1-1/p}.$$

Therefore by (3.5)

$$\frac{1}{\mathrm{K}_p(a,b)}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \frac{p}{p-1}\,\frac{1-\left(\frac{1-r}{1+r}\right)^{1-1/p}}{\frac{2r}{1+r}}(1+r)^{-1/p}$$
$$= \frac{p}{p-1}\,\frac{(1+r)^{1-1/p}-(1-r)^{1-1/p}}{2r}. \qquad (3.6)$$

Use the binomial series expansion to get

$$(1+r)^{1-1/p} = \sum_{k=0}^{\infty}\left[\prod_{i=0}^{k-1}\left(1-\frac{1}{p}-i\right)\right]\frac{r^k}{k!}$$
$$= 1+\sum_{k=1}^{\infty}(-1)^{k-1}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{r^k}{k!},$$
$$(1-r)^{1-1/p} = 1+\sum_{k=1}^{\infty}(-1)^{k-1}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{(-r)^k}{k!}$$
$$= 1-\sum_{k=1}^{\infty}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{r^k}{k!},$$
$$(1+r)^{1-1/p}-(1-r)^{1-1/p} = 2\left(1-\frac{1}{p}\right)\sum_{\ell=0}^{\infty}\left[\prod_{i=0}^{2\ell-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2\ell+1}}{(2\ell+1)!},$$

and from (3.6)

$$\frac{1}{\mathrm{K}_p(a,b)}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \sum_{\ell=0}^{\infty}\left[\prod_{i=0}^{2\ell-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2\ell}}{(2\ell+1)!},$$

as was to be shown. □

Theorem 3.2. *Given $a, b > 0$ and $a \neq b$, we have*

1. $\mathrm{M}_p(a,b) > \mathrm{K}_p(a,b)$ *for* $0 \le p < 1$,
2. $\mathrm{M}_1(a,b) = \mathrm{K}_1(a,b)$,
3. $\mathrm{M}_p(a,b) < \mathrm{K}_p(a,b)$ *for* $p > 1$.

Proof. We compare the right hand side of (2.11) and that of (3.4). For the purpose here, we may ignore the factor $\left(\frac{a^p+b^p}{2}\right)^{-1/p}$ in both expressions and compare the two series. Let

$$\alpha_k = \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\left[\prod_{i=0}^{2k-1}\frac{\frac{1}{p}+i}{\frac{2}{p}+i}\right], \quad \beta_k = \frac{1}{(2k+1)!}\left[\prod_{i=0}^{2k-1}\left(\frac{1}{p}+i\right)\right]$$

which are the coefficients of $\left(\frac{a^p-b^p}{a^p+b^p}\right)^{2k}$ in the two series, respectively. Since $\alpha_0 = 1 = \beta_0$, it suffices to show that for $k \ge 1$

$$\alpha_k < \beta_k \text{ for } 0 < p < 1;\ \alpha_k = \beta_k \text{ for } p = 1;\ \text{and } \alpha_k > \beta_k \text{ for } p > 1.$$

Comparing α_k and β_k, after canceling the common factor $\prod_{i=0}^{2k-1}\left(\frac{1}{p}+i\right)$ in α_k and β_k, is equivalent to comparing the two quantities

$$\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{\prod_{i=0}^{2k-1}\left(\frac{2}{p}+i\right)}=\frac{p^k\prod_{i=0}^{k-1}(1+pi)}{\prod_{i=0}^{2k-1}(2+pi)}=\frac{p^k}{2^k\prod_{i=1}^{k}[2+p(2i-1)]}$$

and $k!/(2k+1)!$. We have

$$\begin{aligned}\frac{p^k}{2^k\prod_{i=1}^{k}[2+p(2i-1)]}&-\frac{k!}{(2k+1)!}\\&=\frac{p^k\,(2k+1)!-k!\,2^k(2+p)(2+3p)\cdots[2+(2k-1)p]}{2^k\,(2k+1)!\,\prod_{i=1}^{k}[2+p(2i-1)]}\end{aligned}$$

whose numerator denoted by $g(p)$ is a polynomial of degree k in p with the leading coefficient (of p^k)

$$\begin{aligned}(2k+1)!-k!\,2^k\,(2k-1)!!&=(2k+1)!-k!\,2^k\,\frac{(2k+1)!}{(2k)!!\,(2k+1)}\\&=(2k+1)!\left(1-\frac{1}{2k+1}\right)>0,\end{aligned}$$

and the rest of the coefficients (of p^i for $i<k$) are all negative, and

$$g(1)=(2k+1)!-k!\,2^k\,(2k+1)!!=0.$$

Therefore $g(p)<g(1)p^k=0$ for $0<p<1$, and $g(p)>g(1)p^k=0$ for $p>1$. This completes the proof. □

4. Integral Operators Induced by M_p

Let $\phi^{[0]}(x)\geq 0$ be any function on $\mathbb{R}_+=\{x : x>0\}$. This introduces a function on $\mathbb{R}_+\times\mathbb{R}_+$:

$$\phi^{[1]}(x,y)=\int_0^\infty \phi^{[0]}(tx)\phi^{[0]}(ty)\,dt, \tag{4.1}$$

and, in turn, another function

$$\phi^{[2]}(x,y)=\int_0^\infty \phi^{[1]}(x,t)\phi^{[1]}(y,t)\,dt. \tag{4.2}$$

For $0<p<\infty$, let

$$\phi_p^{[0]}(x)=e^{-x^p}. \tag{4.3}$$

Then for $x,\,y>0$

$$\begin{aligned}\phi_p^{[1]}(x,y)&=\int_0^\infty e^{-t^p(x^p+y^p)}\,dt \qquad\qquad (\text{substitute } s=t^p(x^p+y^p))\\&=\int_0^\infty e^{-s}\,\frac{1}{p}\,\frac{s^{1/p-1}}{(x^p+y^p)^{1/p}}\,ds\\&=\frac{\Gamma(\frac{1}{p})}{p}\,\frac{1}{(x^p+y^p)^{1/p}},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\phi_p^{[2]}(x,y) &= \frac{\Gamma(\frac{1}{p})^2}{p^2}\int_0^\infty \frac{dt}{[(x^p+t^p)(y^p+t^p)]^{1/p}} \qquad \text{(use (2.5))}\\ &= \frac{\Gamma(\frac{1}{p})^2}{p^2}\,\frac{B(\frac{1}{p},\frac{1}{p})}{p}\,\frac{1}{\mathrm{M}_p(x,y)}. \end{aligned} \tag{4.5}$$

Remark 4.1. Instead of (4.3), we could have started with

$$\phi_p^{[0]}(x) = \alpha_p\, e^{-x^p}, \quad \alpha_p = \frac{\Gamma(\frac{2}{p})^{1/4}\, p^{3/4}}{\Gamma(\frac{1}{p})}. \tag{4.3$'$}$$

Then we will get

$$\phi_p^{[2]}(x) = \frac{1}{\mathrm{M}_p(x,y)}. \tag{4.5$'$}$$

This provides another way of looking at the family of means $\mathrm{M}_p(x,y)$. ◇

Note for $p=1$, (4.5) is

$$\phi_1^{[2]}(x,y) = \frac{1}{\mathrm{M}_1(x,y)} = \frac{1}{\mathrm{L}(x,y)}, \tag{4.6}$$

and for $p=2$, it is

$$\phi_2^{[2]}(x,y) = \frac{\pi^2}{8}\,\frac{1}{\mathrm{M}_2(x,y)} = \frac{\pi^2}{8}\,\frac{1}{\mathrm{AG}(x,y)}. \tag{4.7}$$

We obtain the values of the norms of the integral operators with kernel $1/\mathrm{M}_p(x,y)$. These results are extensions of the famous Hilbert inequality. We use a familiar technique from Hardy, Littlewood, and Pólya [13].

Theorem 4.2 ([13, Theorem 319 on p.229]). *Let $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be homogeneous of order -1, i.e., $\phi(\lambda x, \lambda y) \equiv \lambda^{-1}\phi(x,y)$ for $\lambda > 0$, and that*

$$\int_0^\infty \frac{\phi(x,1)}{\sqrt{x}}\,dx = \int_0^\infty \frac{\phi(1,y)}{\sqrt{y}}\,dy =: \kappa \tag{4.8}$$

Then the induced operator on $L_2(\mathbb{R}_+)$

$$\Phi f(x) := \int_0^\infty \phi(x,y) f(y)\,dy$$

has norm $\|\Phi\|_{L_2} \le \kappa$. If $\phi(1,y)$ is uniformly bounded in $y \in \mathbb{R}_+$, then[1] *$\|\Phi\|_{L_2} = \kappa$.*

For the kernel (4.4), $\phi_p^{[1]}(1,y)$ is uniformly bounded in y and satisfies (4.8). Let $\Phi_p^{[1]}$ be the integral operator with $\phi_p^{[1]}(x,y)$ in (4.4) as its kernel. Apply Theorem 4.2 to get

$$\begin{aligned}\|\Phi_p^{[1]}\|_{L_2} = \kappa_p &:= \int_0^\infty \frac{\phi_p^{[1]}(1,x)}{\sqrt{x}}\,dx\\ &= \frac{\Gamma(\frac{1}{p})}{p}\int_0^\infty \frac{dx}{(1+x^p)^{1/p}\,x^{1/2}}\\ &= \frac{\Gamma(\frac{1}{p})}{p}\,\frac{1}{p}\,B(\tfrac{1}{2p},\tfrac{1}{2p})\end{aligned}$$

[1] This is not explicitly asserted in [13], but can be inferred from the discussion there. See, e.g., [15, p.149].

$$= \frac{1}{p^2} [\Gamma(\tfrac{1}{2p})]^2. \tag{4.9}$$

Since the operator $\Phi_p^{[2]}$ induced by the kernel (4.5) is the square of $\Phi_p^{[1]}$ induced by (4.4) and also $\Phi_p^{[1]}$ is self-adjoint because $\phi_p^{[1]}(x,y) = \phi_p^{[1]}(y,x)$, we have $\|\Phi_p^{[2]}\|_{L_2} = \|\Phi_p^{[1]}\|_{L_2}^2$.

Theorem 4.3. *Let $0 < p < \infty$ and let*

$$\mathscr{M}_p f(x) := \int_0^\infty \frac{1}{\mathrm{M}_p(x,y)} f(y)\, dy.$$

Then $\mathscr{M}_p$ is a bounded linear operator on $L_2(\mathbb{R}_+)$ with

$$\|\mathscr{M}_p\|_{L_2} = \frac{\Gamma(\frac{2}{p})\,\Gamma(\frac{1}{2p})^4}{p\,\Gamma(\frac{1}{p})^2}. \tag{4.10}$$

Proof. Note by (4.5)

$$\Phi_p^{[2]} = \frac{\Gamma(\frac{1}{p})^2}{p^2} \frac{B(\frac{1}{p},\frac{1}{p})}{p} \mathscr{M}_p.$$

By the consideration above,

$$\|\mathscr{M}_p\|_{L_2} = \kappa_p^2 \frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{1}{2p})^4}{p^4} \frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{2}{p})\,\Gamma(\frac{1}{2p})^4}{p\,\Gamma(\frac{1}{p})^2},$$

as expected. □

Special case $p = 1$ gives

$$\|\mathscr{M}_1\|_{L_2} = \pi^2. \tag{4.11}$$

This is noted on [13, p.257] (the last statement of §355). Special case $p = 2$ gives

$$\|\mathscr{M}_2\|_{L_2} = \frac{\Gamma(1/4)^2}{2\pi^2} = 8.753758\cdots \tag{4.12}$$

which happens to be $2\pi/\mathrm{AG}(\sqrt{2},1)^2$.

Remark 4.4. Recall the famous Hilbert inequality that says the norm of the operator induced on $L_2(\mathbb{R}_+)$ by the kernel $1/(x+y)$ is π. This is κ_1 in (4.9). ◇

More generally, consider the space $L_r(\mathbb{R}_+)$, where $r > 1$. [13, Theorem 319 on p.229] says that if ϕ and Φ are as in Theorem 4.2 and

$$\kappa(r) := \int_0^\infty \frac{\phi(1,x)}{x^{1/r}}\, dx = \int_0^\infty \frac{\phi(1,y)}{y^{1/r'}}\, dx < \infty, \tag{4.13}$$

then Φ is a bounded operator on $L_r(\mathbb{R}_+)$ with norm $\|\Phi\|_{L_r} \le \kappa(r)$.

In our case, for the kernel (4.4), (4.13) gives

$$\begin{aligned}\kappa_p(r) &:= \int_0^\infty \frac{dx}{(1+x^p)^{1/p}\,x^{1/r}} \quad \text{(substitute } t = (1+x^p)^{-1}) \\ &= \frac{1}{p} B(\tfrac{1-1/r}{p}, \tfrac{1/r}{p}) \\ &= \frac{1}{p} \frac{\Gamma(\frac{1-1/r}{p})\,\Gamma(\frac{1/r}{p})}{\Gamma(\frac{1}{p})}\end{aligned}$$

$$= \frac{1}{p} \frac{\Gamma(\frac{1}{r'p})\,\Gamma(\frac{1}{rp})}{\Gamma(\frac{1}{p})}, \tag{4.14}$$

where $1/r + 1/r' = 1$. So we have

$$\|\mathscr{M}_p\|_{L_r} \le \kappa_p(r)^2 \frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4}. \tag{4.15}$$

Special case $p = 1$

$$\kappa_1(r) = \Gamma(\tfrac{1}{r'})\,\Gamma(\tfrac{1}{r}) = \pi \csc(\pi/r)$$

is given in [13, p.226 and p.255].

5. Positive Definiteness of Certain Matrices

An interesting connection between binary means of positive real numbers and positive definite matrices has been developed in the last few years. See [5, Chapters 4 and 5], [7], [14], and references therein.

Let M and $\widetilde{M}$ be two binary means. We say that $M \ll \widetilde{M}$ if for every n and for every choice of positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, the $n \times n$ matrix

$$\left(\frac{M(\lambda_i, \lambda_j)}{\widetilde{M}(\lambda_i, \lambda_j)}\right)_{n\times n}$$

is positive semidefinite. For many interesting means, it has been found that the inequality $M \le \widetilde{M}$ implies the stronger relation $M \ll \widetilde{M}$.

We explore this for the two families K_p and M_p. First we observe that for every $x \ge 0$, the matrix

$$\left(\frac{1}{(x^p + \lambda_i^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}\right)_{n\times n}$$

is positive semidefinite, since it is congruent to the *flat* matrix E (the matrix with all its entries equal to one). It follows from (1.5) that for $0 < p < \infty$, the $n \times n$ matrices with (i, j) entries

$$\frac{1}{\mathrm{M}_p(\lambda_i, \lambda_j)} = c_p \int_0^\infty \frac{dx}{(x^p + \lambda_i^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}$$

are positive semidefinite. By a limiting argument, we see that the matrices $\left(\frac{1}{\mathrm{M}_0(\lambda_i,\lambda_j)}\right)_{n\times n} = \left(\frac{1}{\sqrt{\lambda_i\lambda_j}}\right)_{n\times n}$ and

$$\left(\frac{1}{\mathrm{M}_\infty(\lambda_i, \lambda_j)}\right)_{n\times n} = \left(\frac{2 + \ln \frac{\max\{\lambda_i,\lambda_j\}}{\min\{\lambda_i,\lambda_j\}}}{2\max\{\lambda_i, \lambda_j\}}\right)_{n\times n} \tag{5.1}$$

are also positive semidefinite.

In [14], Hiai and Kosaki have proved that for $p \le 1/2$, the matrices

$$\big(\mathrm{K}_p(\lambda_i, \lambda_j)\big)_{n\times n}$$

are positive semidefinite. Hence, *the matrix*

$$\left(\frac{\mathrm{K}_p(\lambda_i, \lambda_j)}{\mathrm{M}_p(\lambda_i, \lambda_j)}\right)_{n\times n}$$

is positive semidefinite for $p \le 1/2$, being the Schur product of two such matrices.

The mean $\mathrm{K}_\infty(a,b)$ is equal to $\max\{a,b\}$. Hence we have

$$\frac{\mathrm{M}_\infty(\lambda_i,\lambda_j)}{\mathrm{K}_\infty(\lambda_i,\lambda_j)}=\frac{2}{2+\ln\frac{\max\{\lambda_i,\lambda_j\}}{\min\{\lambda_i,\lambda_j\}}}=\frac{2}{2+|\ln\lambda_i-\ln\lambda_j|}=\frac{1}{1+|\ln\lambda_i^{1/2}-\ln\lambda_j^{1/2}|}.$$

The matrix with this as its (i,j) entry is positive semidefinite, in fact, infinitely divisible [5, p.153].

We have proved that $\mathrm{K}_p \ll \mathrm{M}_p$ *for* $0\le p\le 1/2$, *and that* $\mathrm{M}_\infty \ll \mathrm{K}_\infty$. *We conjecture that*

$$\mathrm{K}_p \ll \mathrm{M}_p \quad \textit{for } 1/2<p<1, \textit{ and } \quad \mathrm{M}_p \ll \mathrm{K}_p \quad \textit{for } 1<p<\infty. \tag{5.2}$$

Remark 5.1. The positive semidefiniteness of the matrices (5.1) can be expressed in another way: the matrix

$$\left(\frac{1+\frac{1}{2}|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n} \tag{5.3}$$

is always positive semidefinite. It is interesting to note that the matrix

$$\left(\frac{1+|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n}$$

is not necessarily positive semidefinite, as can be seen from the 2×2 example in which $\lambda_1=1$ and $\lambda_2=e^2$. In fact more can be said. Let r be any real nonnegative number. Then *the matrix*

$$W=\left(\frac{1+r|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n}$$

is positive semidefinite for $0\le r\le 1/2$ *and not necessarily positive semidefinite for* $r>1/2$. This can be seen as follows. For $0\le r\le 1/2$, we have

$$w_{ij}=\frac{1+|\ln\lambda_i^r-\ln\lambda_j^r|}{\max\{\lambda_i,\lambda_j\}}=\frac{1+|\ln\lambda_i^r-\ln\lambda_j^r|}{\max\{\lambda_i^{2r},\lambda_j^{2r}\}}\cdot\frac{1}{[\max\{\lambda_i,\lambda_j\}]^{1-2r}}=:u_{ij}v_{ij}.$$

The matrix $U=\big(u_{ij}\big)_{n\times n}$ is positive semidefinite (by the case $r=1/2$ already proved), and the matrix $V=\big(v_{ij}\big)_{n\times n}$ is positive semidefinite since the matrix $\big(1/\max\{\lambda_i,\lambda_j\}\big)_{n\times n}$ is infinitely divisible [3, 5]. So $W=(w_{ij})_{n\times n}$ being the Schur product of U and V is positive semidefinite. Now consider the case $r>1/2$. Let $\widetilde{r}$ be any number such that $r>\widetilde{r}>1/2$, and α be the unique positive root of $x=2\,\widetilde{r}\ln(1+x)$ (such a root exists because at $x=0$, the derivative of x is 1 and the derivative of $2\,\widetilde{r}\ln(1+x)$ is $2\,\widetilde{r}>1$). With $\lambda_1=1$ and $\lambda_2=e^{\alpha/r}$, the 2×2 matrix W is

$$W=\begin{pmatrix}1 & (1+\alpha)e^{-\alpha/r}\\ (1+\alpha)e^{-\alpha/r} & e^{-\alpha/r}\end{pmatrix}$$

whose determinant

$$\det W=e^{-\alpha/r}-(1+\alpha)^2e^{-2\alpha/r}=e^{-2\alpha/r}\left[e^{\alpha/r}-(1+\alpha)^2\right]<0$$

since $\alpha/r<\alpha/\widetilde{r}=2\ln(1+\alpha)=\ln(1+\alpha)^2$. ◇

Remark 5.2. Examples of means for which $M \le \widetilde{M}$ but the stronger relation $M \ll \widetilde{M}$ is not true were given in [4], and in [14]. To that list, we add another. We have seen that $\mathrm{M}_\infty(a,b) \le \mathrm{A}(a,b) := (a+b)/2$, where A stands for the arithmetic mean. But the relation $\mathrm{M}_\infty \ll \mathrm{A}$ is not true. For example, with $\lambda_1 = 17/100$, $\lambda_2 = 18/100$, and $\lambda_3 = 72/100$, the 3×3 matrix with its (i,j) entry being $\mathrm{M}_\infty(\lambda_i,\lambda_j)/\mathrm{A}(\lambda_i,\lambda_j)$ has a negative eigenvalue -0.00011509756859 computed by MATLAB. $\diamond$

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