AN INTERPOLATING FAMILY OF MEANS

RAJENDRA BHATIA* AND REN-CANG LI†

Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

ABSTRACT. This paper is concerned with a new family of binary symmetric means $M_p$ of two positive numbers $a$ and $b$:

$$
\frac{1}{M_p(a, b)} = c_p \int_0^\infty \frac{dx}{[(x^p + a^p)(x^p + b^p)]^{1/p}},
$$

where the constant $c_p$, depending on $p$, is chosen to have $M_p(a, a) = a$. Two distinctive members in the family are the well-known logarithmic mean ($p = 1$) and arithmetic-geometric mean ($p = 2$). Different expressions for $M_p$ are obtained to establish its other properties, including $M_2(a, b) \leq M_\infty(a, b)$ and the relation between $M_p$ and the power difference mean. Through investigating the induced operator norm of the integral operator with $M_1$ as its kernel, a generalization of the Hilbert inequality is obtained. Finally positive definiteness of certain matrices as implications of inequalities between two means is also investigated.

1. Introduction

Let $a$ and $b$ be positive numbers. The logarithmic mean $L(a, b)$ of $a$ and $b$ defined as

$$
L(a, b) := \frac{a - b}{\ln a - \ln b}
$$

has long been used in problems related to heat flow [16] and electrical conduction [17]. More recently it has been employed in differential geometry [2, 5]. The well-known arithmetic-geometric mean $AG(a, b)$ of Gauss is defined as follows: the sequences $\{a_n\}$ and $\{b_n\}$ defined inductively as

$$
a_0 = a, \quad b_0 = b,
$$

$$
a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},
$$

have a common limit, and

$$
AG(a, b) := \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.
$$

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This mean, introduced by Legendre and then by Gauss, is related to the evaluation of elliptic integrals, and several other problems in analysis [11, 8]. The expressions (1.1) and (1.2) do not carry any hint that these two means could belong to a common family. There are alternative descriptions for both. It can be seen that
\[
\frac{1}{L(a, b)} = \int_0^\infty \frac{dx}{(x + a)(x + b)},
\]
and an ingenious calculation, due to Gauss [12], is used to show
\[
\frac{1}{M(a, b)} = \frac{2}{\pi} \int_0^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}.
\]
This similarity between the expressions (1.3) and (1.4) is the motivation for us to introduce a family of means \(m_p(a, b)\), \(0 < p < 1\), defined by the relation
\[
\frac{1}{m_p(a, b)} := c_p \int_0^\infty \frac{dx}{(x^p + a^p)(x^p + b^p)^{1/p}}, \quad 0 < p < \infty,
\]
where the constant \(c_p\), depending on \(p\), will be chosen to have
\[
m_p(a, a) = a.
\]
Thus
\[
\frac{1}{c_p} = a \int_0^\infty \frac{dx}{(x^p + a^p)^{2/p}} = \int_0^\infty \frac{dy}{(y^p + 1)^{2/p}}.
\]
The means \(M_0\) and \(M_\infty\) are defined by taking limits:
\[
M_0(a, b) := \lim_{p \to 0^+} M_p(a, b), \quad M_\infty(a, b) := \lim_{p \to \infty} M_p(a, b).
\]

A binary symmetric mean \(M(a, b)\) of positive numbers \(a\) and \(b\) is a function that satisfies the following properties:
\begin{enumerate}
  \item \(\min\{a, b\} \leq M(a, b) \leq \max\{a, b\}\) (In particular, \(M(a, a) = a\));
  \item \(M(a, b) = M(b, a)\);
  \item \(M(\alpha a, \alpha b) = \alpha M(a, b)\) for all \(\alpha > 0\);
  \item \(M(a, b)\) is non-decreasing in \(a\) and \(b\).
\end{enumerate}

It is obvious from the definition that the mean \(M_p\) satisfies the properties (ii) – (iv). We will give different expressions for \(M_p\) from which other properties, including (i) above, become apparent. In particular, we will show that
\[
M_0(a, b) = \sqrt{ab}, \quad M_\infty(a, b) = \frac{2 \max\{a, b\}}{2 + \ln \frac{\max\{a, b\}}{\min\{a, b\}}} \leq \frac{a + b}{2}.
\]
We conjecture that for fixed \(a\) and \(b\), \(M_p(a, b)\) is an increasing function of \(p\). At this time, we can prove that
\[
M_2(a, b) \leq M_\infty(a, b).
\]
Inequalities already known then lead to the chain
\[
M_0(a, b) \leq M_1(a, b) \leq M_2(a, b) \leq M_\infty(a, b).
\]
The first of these inequalities is well-known and easy to prove; the second has been given different proofs in [9, 10, 19].
The rest of this paper is organized as follows. Section 2 investigates various properties of $M_p$ in detail, including different expressions for $M_p$ and the inequality (1.10). Section 3 gives a relation between $M_p$ and the power difference mean $K_p$. In section 4, we evaluate the norm of the integral operator induced on the space $L_2(\mathbb{R}_+)$ by the kernel $1/M_p(x, y)$. This gives an extension of the famous Hilbert inequality. In section 5, we discuss positive definiteness of certain matrices as implications of some relations between $M_p$ and $K_p$ for which another conjecture is also proposed.

2. Mean $M_p$

Expressions (1.8) for $M_0$ and (1.9) for $M_\infty$ will be proved after a detailed investigation on $M_p$ for $0 < p < \infty$ is completed. Then we will prove the inequality (1.10).

2.1. $0 < p < \infty$.

**Theorem 2.1.** $\min\{a, b\} \leq \sqrt{ab} \leq M_p(a, b) \leq \left(\frac{a^p + b^p}{2}\right)^{1/p} \leq \max\{a, b\}$.

**Proof.** The first inequality is easy to see, and the last one is easy to see too by replacing both $a$ and $b$ in $\left(\frac{a^p + b^p}{2}\right)^{1/p}$ with $\max\{a, b\}$. We now prove the second and the third inequalities. Since $(x^p + a^p)(x^p + b^p) = (x^p)^2 + (a^p + b^p)x^p + a^pb^p$, we have

$$[x^p + (\sqrt{ab})^p]^2 \leq (x^p + a^p)(x^p + b^p) \leq \left[x^p + \frac{a^p + b^p}{2}\right]^2.$$  

Therefore by (1.5),

$$\frac{1}{M_p\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}, \left[\frac{a^p + b^p}{2}\right]^{1/p}\right)} \leq \frac{1}{M_p(a, b)} \leq \frac{1}{M_p(\sqrt{ab}, \sqrt{ab})},$$

which, together with the condition $M_p(z, z) = z$ for any $z > 0$, leads to the desired inequalities. $\square$

In the last integral in (1.6), substitute $t = (y^p + 1)^{-1}$ to get

$$y^p = \frac{1}{t} - 1 = \frac{1 - t}{t},$$

$$py^{p-1}dy = -\frac{1}{t^2}dt,$$

$$dy = -\frac{1}{p} \left(\frac{t}{1-t}\right)^{\frac{p-1}{p}} \frac{1}{t^2}dt$$

$$= -\frac{1}{p}t^{-1-1/p}(1-t)^{1/p-1}dt,$$

$$\frac{1}{c_p} = \frac{1}{p} \int_0^1 t^{1/p-1}(1-t)^{1/p-1} dt$$

$$= B\left(\frac{1}{p}, \frac{1}{p}\right)$$

(2.5)
where $B(\cdot, \cdot)$ is the Beta-function [1]. In the integral in (1.5), substitute $x^p + a^p = a^p t^{-1}$ to get

$$x^p = a^p \left( \frac{1}{t} - 1 \right) = a^p \frac{1}{t} - t,$$

$$p x^{p-1} dx = -a^p \frac{1}{t^2} dt,$$

$$dx = -\frac{a}{p} \left( \frac{t}{1-t} \right)^{\frac{p-1}{p}} \frac{1}{t^2} dt$$

$$= -\frac{a}{p} t^{-1/p} (1-t)^{1/p-1} dt,$$

$$\frac{1}{M_p(a, b)} = c_p \frac{a}{p} \int_0^1 t^{-1/p} (1-t)^{1/p-1} \left[ (ap t^{-1})(ap t^{-1} + b) \right]^{1/p} dt$$

$$= c_p \frac{a}{p} \int_0^1 \left[ (1-t)^{1/p-1} + b/(1-t)^{1/p} \right] dt.$$ (2.6)

Combine (2.4) and (2.6) to get

$$\frac{1}{M_p(a, b)} = \frac{\int_0^1 t^{1/p-1} (1-t)^{1/p-1}}{\int_0^1 (1-t)^{1/p-1} dt}.$$ (2.7)

**Theorem 2.2.** Given $a, b > 0$ and $0 < p < \infty$, we have

$$\frac{1}{M_p(a, b)} = (\max\{a, b\})^{-1} \sum_{k=0}^{\infty} k! \left[ \frac{1}{p} + i \right] \frac{1}{k!} \left[ 1 - \left( \min\{a, b\} \right) \right] \left[ 1 - \left( \max\{a, b\} \right) \right] \frac{1}{k!}.$$

where, by convention, $\prod_{k=0}^{-1} (\cdots) = 1$ and $0! = 1$.

**Proof.** Both sides of (2.8) are equal to $a$ if $a = b$. Assume without loss of generality that $a > b > 0$. Let $\alpha = 1 - (b/a)^p$ and then $0 < \alpha < 1$. We have

$$a^p (1-t) + b^p t = a^p [1-t + (b/a)^p t] = a^p (1-\alpha t),$$

$$a^p (1-t) + b^p t^{-1/p} = \alpha^{-1} (1-\alpha t)^{-1/p}$$

$$= \alpha^{-1} \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} (1/p + i)}{k!} \alpha^k k.$$ (2.9)

The series in (2.9) converges for $\alpha < 1$ which justifies the term-by-term integration below. Equation (2.9), together with (2.7), yields

$$\frac{1}{M_p(a, b)} = a^{-1} \int_0^1 \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} (1/p + i)}{k!} \alpha^k k+1/p-1 (1-t)^{1/p-1} dt$$

$$\int_0^1 t^{1/p-1} (1-t)^{1/p-1}$$
where 

\[ \Pr \]

Substituting we have

\[ \text{Given Theorem 2.3. Using} \]

\[ \text{the well-known properties of the Beta and Gamma functions [1]} \]

\[ \text{we have} \]

\[ B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s + t)}, \quad \Gamma(z) = (z - 1)\Gamma(z - 1), \]

\[ \text{we have} \]

\[ \frac{B(k + \frac{1}{p}, \frac{1}{p})}{B(\frac{1}{p}, \frac{1}{p})} = \frac{\Gamma(k + \frac{1}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p})\Gamma(k + \frac{1}{p})} \]

\[ = \frac{\Gamma(k + \frac{1}{p})}{\Gamma(k + 1)} \]

\[ = \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \]

Substituting this into (2.10) gives (2.8). \[ \square \]

**Theorem 2.3.** Given \( a, b > 0 \) and \( 0 < p < \infty \), we have

\[ \frac{1}{M_p(a, b)} = \left( \frac{a^p + b^p}{2} \right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \left( \frac{2k-1}{p} + i \right) \left( \frac{a^p - b^p}{a^p + b^p} \right)^{2k} \]

\[ \text{(2.11)} \]

**Proof.** We have

\[ (x^p + a^p)(x^p + b^p) = \left( x^p + \frac{a^p + b^p}{2} \right)^2 - \left( \frac{a^p - b^p}{2} \right)^2 \]

\[ = \left( x^p + \frac{a^p + b^p}{2} \right)^2 (1 - r^2), \]

where \( r = \frac{a^p - b^p}{2} / \left( x^p + \frac{a^p + b^p}{2} \right) \). Therefore \( |r| < 1 \) and

\[ [(x^p + a^p)(x^p + b^p)]^{-1/p} = \left( x^p + \frac{a^p + b^p}{2} \right)^{-2/p} (1 - r^2)^{-1/p}, \]

\[ = \left( x^p + \frac{a^p + b^p}{2} \right)^{-2/p} \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2k}}{k!}, \]

\[ \frac{1}{M_p(a, b)} = c_p \int_0^\infty \frac{1}{(x^p + a^p + b^p)^{2/p}} \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2k}}{k!} \, dx \]
\[ c_p \int_0^\infty \frac{dx}{(x^p + \frac{a^p + b^p}{2})^{2/p}} \]
\[ + c_p \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] \int_0^\infty \frac{r^{2k}}{(x^p + \frac{a^p + b^p}{2})^{2/p}} dx \]
\[ = \left( \frac{a^p + b^p}{2} \right)^{-1/p} \]
\[ + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] c_p \int_0^\infty \frac{(a^p - b^p)^{2k}}{(x^p + \frac{a^p + b^p}{2})^{2k+2/p}} dx. \] 

(2.12)

Substitute \( x = \left( \frac{a^p + b^p}{2} \right)^{1/p} y \) and \( y^p + 1 = t^{-1} \) as in (2.1) – (2.3) to get
\[ \int_0^\infty \frac{dx}{(x^p + \frac{a^p + b^p}{2})^{2k+2/p}} = \left( \frac{a^p + b^p}{2} \right)^{-2k-1/p} \int_0^\infty \frac{dy}{(y^p + 1)^{2k+2/p}} \]
\[ = \left( \frac{a^p + b^p}{2} \right)^{-2k-1/p} \int_0^\frac{1}{y} t^{2k+1/p-1} (1 - t)^{1/p-1} dt \]
\[ = \left( \frac{a^p + b^p}{2} \right)^{-2k-1/p} \frac{B(2k + \frac{1}{p}, \frac{1}{p})}{p}. \]

This together with (2.5) leads to
\[ c_p \int_0^\infty \frac{dx}{(x^p + \frac{a^p + b^p}{2})^{2k+2/p}} = \left( \frac{a^p + b^p}{2} \right)^{-2k-1/p} \frac{B(2k + \frac{1}{p}, \frac{1}{p})}{p} \]
\[ \times B \left( \frac{1}{p}, \frac{1}{p} \right) \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right). \] 

(2.13)

Now (2.11) is a consequence of (2.12) and (2.13). \( \square \)

2.2. \( p = 0. \)

**Theorem 2.4.** Given \( a, b > 0, \) we have
\[ M_0(a, b) = \sqrt{ab}. \]

(2.14)

**Proof.** It can be verified that \( \lim_{p \to 0^+} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \sqrt{ab}. \) The equality \( M_0(a, b) = \sqrt{ab} \) is then a consequence of Theorem 2.1. \( \square \)

2.3. \( p = \infty. \)

**Theorem 2.5.** Given \( a, b > 0, \) we have
\[ M_\infty(a, b) = \frac{2 \max\{a, b\}}{2 + \ln \max\{a, b\}} \]
\[ \frac{\min\{a, b\}}{\min\{a, b\}}. \] 

(2.15)

and
\[ M_2(a, b) \leq M_\infty(a, b) \leq \frac{(a + b)}{2}. \] 

(2.16)
Proof. Both (2.15) and (2.16) are obvious if \(a = b\). Assume that \(a > b > 0\). Then

\[
\lim_{p \to \infty} \int_0^\infty \frac{dy}{(y^p + 1)^{2/p}} = \int_0^1 dy + \int_1^\infty \frac{dy}{y^2} = 2,
\]

\[
\lim_{p \to \infty} \int_0^\infty \frac{dx}{[(x^p + a^p)(x^p + b^p)]^{1/p}} = \int_0^b \frac{dx}{ab} + \int_b^a \frac{dx}{xa} + \int_a^\infty \frac{dx}{x^2} = \frac{1}{a} + \frac{\ln a - \ln b}{a} + \frac{1}{a} = \frac{2 + (\ln a - \ln b)}{a}.
\]

Therefore \(c_\infty = 1/2\), and (2.15) holds by definition. The change of order of taking limits and the integrals above is justified by Lebesgue’s Dominated Convergence Theorem [21, p.76] because

\[
\frac{1}{(y^p + 1)^{2/p}} \leq \begin{cases} 
1 & \text{for } 0 \leq y \leq 1, \\
y^{-2} & \text{for } 1 < y,
\end{cases}
\]

\[
\frac{1}{[(x^p + a^p)(x^p + b^p)]^{1/p}} \leq \begin{cases} 
(ab)^{-1} & \text{for } 0 \leq x \leq a, \\
x^{-2} & \text{for } a < x.
\end{cases}
\]

The second inequality in (2.16) is relatively easy to show. It goes as follows. Since

\[
M_1(a, b) = \frac{a - b}{\ln a - \ln b} \leq \frac{a + b}{2},
\]

we have successively

\[
2(a - b) \leq (a + b)(\ln a - \ln b),
\]

\[
2a \leq 2b + (a + b)(\ln a - \ln b),
\]

\[
4a = 2(a + b) + (a + b)(\ln a - \ln b),
\]

\[
\frac{2a}{2 + \ln a - \ln b} \leq \frac{a + b}{2},
\]

as expected.

Let us focus on the first inequality in (2.16) now. As in the proof by John Todd for Problem 19-17 in [10], let \(r = (a - b)/(a + b)\). Then \(0 < r < 1\) and \(a = \frac{a+b}{2}(1+r)\) and \(b = \frac{a+b}{2}(1-r)\). It suffices to show that \(M_2(1 + r, 1 - r) \leq M_\infty(1 + r, 1 - r)\).

It was shown by Gauss [12] (see also [8, p.7]) that for \(|r| < 1\)

\[
\frac{1}{M_2(1 + r, 1 - r)} = 1 + \frac{1}{4} r^2 + \frac{9}{64} r^4 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)}\right)^2 r^{2n} + \cdots.
\]

On the other hand, by (1.9),

\[
\frac{1}{M_\infty(1 + r, 1 - r)} = \frac{2 + \ln \frac{1+r}{1-r}}{2(1+r)} = \frac{2 + \ln \frac{1+r}{1-r}}{2(1+r)}
\]

\[
= \frac{2 + 2r \left(1 + \frac{1}{3} r^2 + \frac{1}{5} r^4 + \cdots + \frac{1}{2n+1} r^{2n} + \cdots\right)}{2(1+r)}.
\]
So for $M_2(1 + r, 1 - r) \leq M_\infty(1 + r, 1 - r)$ to hold, it suffices to have

\[
(1 + r) \left[ 1 + \frac{1}{4} r^2 + \frac{9}{64} r^4 + \cdots + \left( \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \right)^2 r^{2n} + \cdots \right]
\geq 1 + r \left( 1 + \frac{1}{3} r^2 + \frac{1}{5} r^4 + \cdots + \frac{1}{2n + 1} r^{2n} + \cdots \right),
\tag{2.17}
\]

or, equivalently,

(1 + r) \left[ \frac{1}{4} r^2 + \frac{9}{64} r^4 + \cdots + \left( \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \right)^2 r^{2n} + \cdots \right]
\geq r \left( \frac{1}{3} r^2 + \frac{1}{5} r^4 + \cdots + \frac{1}{2n + 1} r^{2n} + \cdots \right).
\tag{2.18}

Since $1 + r > 2r$, (2.18) holds if

\[
2r \left[ \frac{1}{4} r^2 + \frac{9}{64} r^4 + \cdots + \left( \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \right)^2 r^{2n} + \cdots \right]
\geq r \left( \frac{1}{3} r^2 + \frac{1}{5} r^4 + \cdots + \frac{1}{2n + 1} r^{2n} + \cdots \right),
\tag{2.19}
\]

which is guaranteed if the corresponding coefficients of $r^{2n+1}$ from both sides satisfy

\[
2 \left( \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \right)^2 \geq \frac{1}{2n + 1}
\tag{2.20}
\]

for $n \geq 1$. This is what we shall prove now. To this end, we shall use the following estimate for factorial $n!$ \cite{[18, 20]}

\[
\sqrt{2\pi n^{n+1/2} e^{-n+1/(12n+1)}} < n! < \sqrt{2\pi n^{n+1/2} e^{-n+1/(12n+1)}}.
\tag{2.21}
\]

We have

\[
\frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} = \frac{(2n)!}{2^{2n}(n!)^2} > \frac{\sqrt{2\pi (2n)^{2n+1/2} e^{-2n+1/(24n+1)}}}{2^{2n} \left[ \sqrt{2\pi n^{n+1/2} e^{-n+1/(12n+1)}} \right]^2}
\]

\[
= \frac{e^{1/(24n+1)}}{\sqrt{\pi n} e^{1/(6n)}},
\]

\[
2(2n + 1) \left( \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \right)^2 > 4n \left( \frac{e^{1/(24n+1)}}{\sqrt{\pi n} e^{1/(6n)}} \right)^2
\]

\[
= \frac{4}{\pi} e^{2/(24n+1)-1/(3n)}
\]

\[
= \frac{4}{\pi} e^{-(18n+1)/[3n(24n+1)]}.
\]
This proves that (2.19) holds for \( n \geq 2 \). It can be verified that (2.19) holds for \( n = 1 \) also. The proof is completed. \( \Box \)

Remark 2.6. One can use (2.21) to also show that
\[
\left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 < \frac{1}{\pi n} < \frac{1}{2n+1}.
\]
This is used by John Todd [10] to show \( M_1(1 + r, 1 - r) \leq M_2(1 + r, 1 - r) \).

3. Relation to the Power Difference Mean

The power difference mean \( K_p(a, b) \) is defined for any \( p \) and \( a, b > 0 \) as follows [14, 6].
\[
K_p(a, b) := \frac{p - 1}{p} \frac{a^p - b^p}{a^{p-1} - b^{p-1}}, \tag{3.1}
\]
where it is understood that
\[
K_p(a, a) := a, \quad K_1(a, b) := \lim_{p \to 1} K_p(a, b) = L(a, b). \tag{3.2}
\]
Alternatively, \( K_p(a, b) \) admits the following integral expression:
\[
\frac{1}{K_p(a, b)} = \int_0^1 \frac{dt}{(1 - t)a^p + (1 + t)b^p}^{1/p}. \tag{3.3}
\]
By (1.1), (1.3), and (3.2), we have \( M_1(a, b) = L(a, b) = K_1(a, b) \). It makes us wonder what kind of relations are between \( M_p(a, b) \) and \( K_p(a, b) \) for \( p \neq 1 \). Theorem 3.2 below provides an answer. But first we establish an expansion formula for \( K_p(a, b) \).

Lemma 3.1. Given \( a, b > 0 \), we have
\[
\frac{1}{K_p(a, b)} = \left( \frac{a^p + b^p}{2} \right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right) \right) \left( \frac{a^p - b^p}{a^p + b^p} \right)^{2k}. \tag{3.4}
\]

Proof. \( K_p \) as defined by (3.1) has a removable singularity at \( p = 1 \). In this case, equation (3.4) can be verified either by using \( K_1(a, b) = L(a, b) \) or by taking the limit as \( p \) goes to 1. In what follows, we shall assume \( p \neq 1 \). It suffices to show (3.4) for \( a = 1 \) and \( 0 < b \neq 1 \). It follows from (3.1) that
\[
\frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \frac{p}{p - 1} \frac{a^{p-1} - b^{p-1}}{a^p - b^p} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \frac{p - 1}{p - 1} \frac{1 - b^{p-1}}{1 - b^p} \left( \frac{1 + b^p}{2} \right)^{1/p}.
\]
Let \( r = (1 - b^p)/(1 + b^p) \). Then \( |r| < 1 \), and
\[
b^p = \frac{1 - r}{1 + r}, \quad \frac{1 + b^p}{2} = \frac{1}{1 + r}, \quad 1 - b^p = \frac{2r}{1 + r}, \quad b^{p-1} = \left( b^p \right)^{(p-1)/p} = \left( \frac{1 - r}{1 + r} \right)^{(1-1/p)}.
\]
Therefore by (3.5)
\[
\frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \frac{p}{p-1} \frac{1 - \left( \frac{1-r}{1+r} \right)^{1-1/p}}{(1+r)^{-1/p}} = \frac{p}{p-1} \frac{(1+r)^{1-1/p} - (1-r)^{1-1/p}}{2r}.
\]
(3.6)

Use the binomial series expansion to get
\[
(1 + r)^{1-1/p} = \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left( 1 - \frac{1}{p} - i \right) \right] \frac{r^k}{k!},
\]
\[
= 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \left( 1 - \frac{1}{p} \right) \left[ \prod_{i=0}^{k-2} \left( \frac{1}{p} + i \right) \right] \frac{r^k}{k!},
\]
\[
(1 - r)^{1-1/p} = 1 - \sum_{k=1}^{\infty} \left( 1 - \frac{1}{p} \right) \left[ \prod_{i=0}^{k-2} \left( \frac{1}{p} + i \right) \right] \frac{(-r)^k}{k!},
\]
\[
(1 + r)^{1-1/p} - (1 - r)^{1-1/p} = 2 \left( 1 - \frac{1}{p} \right) \sum_{\ell=0}^{\infty} \left[ \prod_{i=0}^{2\ell-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2\ell+1}}{(2\ell + 1)!},
\]
and from (3.6)
\[
\frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \sum_{\ell=0}^{\infty} \left[ \prod_{i=0}^{2\ell-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2\ell}}{(2\ell + 1)!},
\]
as was to be shown. 

**Theorem 3.2.** Given \(a, b > 0\) and \(a \neq b\), we have
\[
(1) \ \mathcal{M}_p(a, b) > K_p(a, b) \quad \text{for} \quad 0 \leq p < 1,
\]
\[
(2) \ \mathcal{M}_1(a, b) = K_1(a, b),
\]
\[
(3) \ \mathcal{M}_p(a, b) < K_p(a, b) \quad \text{for} \quad p > 1.
\]

**Proof.** We compare the right hand side of (2.11) and that of (3.4). For the purpose here, we may ignore the factor \(\left( \frac{a^p + b^p}{2} \right)^{-1/p}\) in both expressions and compare the two series. Let
\[
\alpha_k = \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] \left[ \prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right) \right], \quad \beta_k = \frac{1}{(2k+1)!} \left[ \prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right) \right]
\]
which are the coefficients of \(\left( \frac{a^p + b^p}{2} \right)^{2k}\) in the two series, respectively. Since \(\alpha_0 = 1 = \beta_0\), it suffices to show that for \(k \geq 1\),
\[
\alpha_k < \beta_k \quad \text{for} \quad 0 < p < 1; \quad \alpha_k = \beta_k \quad \text{for} \quad p = 1; \quad \text{and} \quad \alpha_k > \beta_k \quad \text{for} \quad p > 1.
\]
Comparing $\alpha_k$ and $\beta_k$, after canceling the common factor $\prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right)$ in $\alpha_k$ and $\beta_k$, is equivalent to comparing the two quantities
\[
\frac{\prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right)}{\prod_{i=0}^{2k-1} \left( \frac{2}{p} + i \right)} = \frac{\prod_{i=0}^{k-1} (1 + pi)}{\prod_{i=0}^{2k-1} (2 + pi)} = \frac{p^k}{2^k \prod_{i=1}^{2k} [2 + p(2i - 1)]}
\]
and $k!/(2k + 1)!$. We have
\[
\frac{p^k}{2^k \prod_{i=1}^{k} [2 + p(2i - 1)]} = \frac{k!}{(2k + 1)!} = \frac{p^k (2k + 1)! - k! 2^k (2 + p)(2 + 3p) \cdots [2 + (2k - 1)p]}{2^k (2k + 1)! \prod_{i=1}^{k} [2 + p(2i - 1)]}
\]
whose numerator denoted by $g(p)$ is a polynomial of degree $k$ in $p$ with the leading coefficient (of $p^k$)
\[
(2k + 1)! - k! 2^k (2k - 1)!! = (2k + 1)! - k! 2^k \frac{(2k + 1)!}{(2k)!! (2k + 1)}
\]
\[
= (2k + 1)! \left( 1 - \frac{1}{2k + 1} \right) > 0,
\]
and the rest of the coefficients (of $p^i$ for $i < k$) are all negative, and
\[
g(1) = (2k + 1)! - k! 2^k (2k + 1)!! = 0.
\]
Therefore $g(p) < g(1)p^k = 0$ for $0 < p < 1$, and $g(p) > g(1)p^k = 0$ for $p > 1$. This completes the proof. \hfill \square

4. Integral Operators Induced by $M_p$

Let $\phi^{[0]}(x) \geq 0$ be any function on $\mathbb{R}_+ = \{x : x > 0\}$. This introduces a function on $\mathbb{R}_+ \times \mathbb{R}_+$:
\[
\phi^{[1]}(x, y) = \int_0^\infty \phi^{[0]}(tx) \phi^{[0]}(ty) \, dt,
\]
and, in turn, another function
\[
\phi^{[2]}(x, y) = \int_0^\infty \phi^{[1]}(x, t) \phi^{[1]}(y, t) \, dt.
\]
For $0 < p < \infty$, let
\[
\phi^{[0]}_p(x) = e^{-x^p}.
\]
Then for $x, y > 0$
\[
\phi^{[1]}_p(x, y) = \int_0^\infty e^{-t^p(x^p + y^p)} \, dt \quad \text{(substitute } s = t^p(x^p + y^p))
\]
\[
= \int_0^\infty e^{-s} \frac{1}{p} s^{1/p-1} \, ds
\]
\[
= \frac{\Gamma(\frac{1}{p})}{p} \frac{1}{(x^p + y^p)^{1/p}}.
\]
\[ \phi_p^{[2]}(x, y) = \frac{\Gamma(\frac{1}{p})^2}{p^2} \int_0^{\infty} \frac{dt}{[(x^p + t^p)(y^p + t^p)]^{1/p}} \quad \text{(use (2.5))} \]
\[ = \frac{\Gamma(\frac{1}{p})^2}{p^2} B\left(\frac{1}{p}, \frac{1}{p}\right) \frac{1}{\cal M_p(x, y)}. \quad (4.5) \]

**Remark 4.1.** Instead of (4.3), we could have started with
\[ \phi_p^{[0]}(x) = \alpha_p e^{-x^p}, \quad \alpha_p = \frac{\Gamma(\frac{1}{p})^{1/4} \Gamma^{3/4}(\frac{1}{p})}{\Gamma(\frac{1}{p})}. \quad (4.3') \]
Then we will get
\[ \phi_p^{[2]}(x) = \frac{1}{M_p(x, y)}. \quad (4.5') \]
This provides another way of looking at the family of means \( M_p(x, y). \)

Note for \( p = 1, (4.5) \) is
\[ \phi_1^{[2]}(x, y) = \frac{1}{M_1(x, y)} = \frac{1}{L(x, y)}, \quad (4.6) \]
and for \( p = 2, \) it is
\[ \phi_2^{[2]}(x, y) = \frac{\pi^2}{8} \frac{1}{M_2(x, y)} = \frac{\pi^2}{8} \frac{1}{AG(x, y)}. \quad (4.7) \]
We obtain the values of the norms of the integral operators with kernel \( 1/M_p(x, y). \) These results are extensions of the famous Hilbert inequality. We use a familiar technique from Hardy, Littlewood, and Pólya [13].

**Theorem 4.2** ([13, Theorem 319, page 229]). Let \( \phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) be homogeneous of order \(-1, \) i.e., \( \phi(\lambda x, \lambda y) \equiv \lambda^{-1} \phi(x, y) \) for \( \lambda > 0, \) and that
\[ \int_0^{\infty} \frac{\phi(x, 1)}{\sqrt{x}} \, dx = \int_0^{\infty} \frac{\phi(1, y)}{\sqrt{y}} \, dy =: \kappa \quad (4.8) \]
Then the induced operator on \( L_2(\mathbb{R}_+) \)
\[ \Phi f(x) := \int_0^{\infty} \phi(x, y) f(y) \, dy \]
has norm \( \|\Phi\|_{L_2} \leq \kappa. \) If \( \phi(1, y) \) is uniformly bounded in \( y \in \mathbb{R}_+, \) then\(^1 \|\phi\|_{L_2} = \kappa. \)

For the kernel (4.4), \( \phi_p^{[1]}(1, y) \) is uniformly bounded in \( y \) and satisfies (4.8). Let \( \Phi_p^{[1]} \) be the integral operator with \( \phi_p^{[1]}(x, y) \) in (4.4) as its kernel. Apply Theorem 4.2 to get
\[ \|\Phi_p^{[1]}\|_{L_2} = \kappa_p := \int_0^{\infty} \frac{\phi_p^{[1]}(1, x)}{\sqrt{x}} \, dx \]
\[ = \frac{\Gamma(\frac{1}{p})}{p} \int_0^{\infty} \frac{dx}{(1 + x^p)^{1/p} x^{1/2}} \]
\(^1\)This is not explicitly asserted in [13], but can be inferred from the discussion there. See, e.g., [15, page 149].
\[\frac{\Gamma\left(\frac{1}{p}\right)}{p} \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{p}\right) = \frac{1}{p^2} (\Gamma\left(\frac{1}{p}\right))^2. \quad (4.9)\]

Since the operator \(\Phi_p^{[2]}\) induced by the kernel (4.5) is the square of \(\Phi_p^{[1]}\) induced by (4.4) and also \(\Phi_p^{[1]}\) is self-adjoint because \(\phi_p^{[1]}(x, y) = \phi_p^{[1]}(y, x)\), we have \(\|\Phi_p^{[2]}\|_{L_2} = \|\Phi_p^{[1]}\|_{L_2}^2\).

**Theorem 4.3.** Let \(0 < p < \infty\) and let
\[\mathcal{M}_p f(x) := \int_0^x \frac{1}{M_p(x, y)} f(y) \, dy.\]
Then \(\mathcal{M}_p\) is a bounded linear operator on \(L_2(\mathbb{R}^+)\) with
\[\|\mathcal{M}_p\|_{L_2} = \frac{\Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{1}{2}\right)^4}{p \Gamma\left(\frac{1}{2}\right)^2}. \quad (4.10)\]

**Proof.** Note by (4.5)
\[\Phi_p^{[2]} = \frac{\Gamma\left(\frac{1}{p}\right)^2}{p^2} \frac{1}{p} \mathcal{M}_p.\]
By the consideration above,
\[\|\mathcal{M}_p\|_{L_2} = \kappa_p^2 \frac{p^2 \Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{1}{2}\right)^4}{p \Gamma\left(\frac{1}{2}\right)^2} = \frac{\Gamma\left(\frac{1}{p}\right)^4}{\Gamma\left(\frac{1}{2}\right)^4} \frac{\Gamma\left(\frac{1}{2}\right)^4}{p \Gamma\left(\frac{1}{2}\right)^2} = \frac{\Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{1}{2}\right)^4}{p \Gamma\left(\frac{1}{2}\right)^2}.\]
as expected. \(\square\)

Special case \(p = 1\) gives
\[\|\mathcal{M}_1\|_{L_2} = \pi^2. \quad (4.11)\]
This is noted on [13, page 257] (the last statement of §355). Special case \(p = 2\) gives
\[\|\mathcal{M}_2\|_{L_2} = \frac{\Gamma(1/4)^2}{2\pi^2} = 8.753758 \cdots \quad (4.12)\]
which happens to be \(2\pi/\lambda(\sqrt{2}, 1)^2\).

**Remark 4.4.** Recall the famous Hilbert inequality that says the norm of the operator induced on \(L_2(\mathbb{R}^+)\) by the kernel \(1/(x + y)\) is \(\pi\). This is \(\kappa_1\) in (4.9).

More generally, consider the space \(L_r(\mathbb{R}^+)\), where \(r > 1\). [13, Theorem 319, page 229] says that if \(\phi\) and \(\Phi\) are as in Theorem 4.2 and
\[\kappa(r) := \int_0^\infty \frac{\phi(1, x)}{x^{1/r}} \, dx = \int_0^\infty \frac{\phi(1, y)}{y^{1/r}} \, dy < \infty, \quad (4.13)\]
then \(\Phi\) is a bounded operator on \(L_r(\mathbb{R}^+)\) with norm \(\|\Phi\|_{L_r} \leq \kappa(r)\).

In our case, for the kernel (4.4), (4.13) gives
\[\kappa_p(r) := \int_0^\infty \frac{dx}{(1 + x^p)^{1/p} x^{1/r}} \text{ (substitute } t = (1 + x^p)^{-1})\]
\[= \frac{1}{p} B\left(\frac{1-1/p}{p}, \frac{1/r}{p}\right)\]
\[
\frac{1}{p} \frac{\Gamma(\frac{1}{p} - \frac{1}{r'})}{\Gamma(\frac{1}{r'})} = 1
\]
where \(1/r + 1/r' = 1\). So we have
\[\|M_p\|_{L_r} \leq \kappa_p(r)^2 \frac{p^3 \Gamma(\frac{2}{r})}{\Gamma(\frac{1}{r})^4}.\]

Special case \(p = 1\)
\[\kappa_1(r) = \frac{\Gamma(\frac{2}{r})}{\Gamma(\frac{1}{r})} = \pi \csc(\pi/r)\]
is given in [13, pages 226 and 255].

5. Positive Definiteness of Certain Matrices

An interesting connection between binary means of positive real numbers and positive definite matrices has been developed in the last few years. See [5, Chapters 4 and 5], [7], [14], and references therein.

Let \(M\) and \(\tilde{M}\) be two binary means. We say that \(M \preceq \tilde{M}\) if for every \(n\) and for every choice of positive real numbers \(\lambda_1, \lambda_2, \ldots, \lambda_n\), the \(n \times n\) matrix
\[
\begin{pmatrix}
M(\lambda_i, \lambda_j) \\
M(\lambda_i, \lambda_j)
\end{pmatrix}
\]
is positive semidefinite. For many interesting means, it has been found that the inequality \(M \preceq \tilde{M}\) implies the stronger relation \(M \ll \tilde{M}\).

We explore this for the two families \(K_p\) and \(M_p\). First we observe that for every \(x \geq 0\), the matrix
\[
\frac{1}{(x^p + \lambda_1^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}
\]
is positive semidefinite, since it is congruent to the flat matrix \(E\) (the matrix with all its entries equal to one). It follows from (1.5) that for \(0 < p < \infty\), the \(n \times n\) matrices with \((i, j)\) entries
\[
\frac{1}{M_p(\lambda_i, \lambda_j)} = c_p \int_0^\infty \frac{dx}{(x^p + \lambda_i^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}
\]
are positive semidefinite. By a limiting argument, we see that the matrices
\[
\left(\frac{1}{M_{\tilde{p}}(\lambda_i, \lambda_j)}\right)_{n \times n} = \left(\frac{1}{\sqrt{\lambda_i \lambda_j}}\right)_{n \times n}
\]
and
\[
\left(\frac{1}{M_{\tilde{p}}(\lambda_i, \lambda_j)}\right)_{n \times n} = \left(\frac{2 + \ln \max\{\lambda_i, \lambda_j\}}{2 \max\{\lambda_i, \lambda_j\}}\right)_{n \times n}
\]
are also positive semidefinite.

In [14], Hiai and Kosaki have proved that for \(p \leq 1/2\), the matrices
\[
(K_p(\lambda_i, \lambda_j))_{n \times n}
\]
are positive semidefinite. Hence, the matrix
\[
\begin{pmatrix}
K_p(\lambda_i, \lambda_j) \\
M_p(\lambda_i, \lambda_j)
\end{pmatrix}_{n \times n}
\]
is positive semidefinite for \( p \leq 1/2 \), being the Schur product of two such matrices.

The mean \( K_{\infty}(a, b) \) is equal to \( \max\{a, b\} \). Hence we have
\[
M_{\infty}(\lambda_i, \lambda_j) = \frac{2}{2 + \ln \frac{\max\{\lambda_i, \lambda_j\}}{\min\{\lambda_i, \lambda_j\}}} = \frac{2}{2 + |\ln \lambda_i - \ln \lambda_j|} = \frac{1}{1 + |\ln \lambda_i^{1/2} - \ln \lambda_j^{1/2}|}.
\]
The matrix with this as its \((i, j)\) entry is positive semidefinite, in fact, infinitely divisible [5, p.153].

We have proved that \( K_p \ll M_p \) for \( 0 \leq p \leq 1/2 \), and that \( M_{\infty} \ll K_{\infty} \). We conjecture that
\[
K_p \ll M_p \quad \text{for} \quad 1/2 < p < 1, \quad \text{and} \quad M_p \ll K_p \quad \text{for} \quad 1 < p < \infty. \tag{5.2}
\]

**Remark 5.1.** The positive semidefiniteness of the matrices (5.1) can be expressed in another way: the matrix
\[
\begin{pmatrix}
1 + \frac{1}{2} |\ln \lambda_i - \ln \lambda_j| \\
\max\{\lambda_i, \lambda_j\}
\end{pmatrix}_{n \times n}
\]
is always positive semidefinite. It is interesting to note that the matrix
\[
\begin{pmatrix}
1 + |\ln \lambda_i - \ln \lambda_j| \\
\max\{\lambda_i, \lambda_j\}
\end{pmatrix}_{n \times n}
\]
is not necessarily positive semidefinite, as can be seen from the \( 2 \times 2 \) example in which \( \lambda_1 = 1 \) and \( \lambda_2 = e^2 \). In fact more can be said. Let \( r \) be any real nonnegative number. Then the matrix
\[
W = \begin{pmatrix}
1 + r|\ln \lambda_i - \ln \lambda_j| \\
\max\{\lambda_i, \lambda_j\}
\end{pmatrix}_{n \times n}
\]
is positive semidefinite for \( 0 \leq r \leq 1/2 \) and not necessarily positive semidefinite for \( r > 1/2 \). This can be seen as follows. For \( 0 \leq r \leq 1/2 \), we have
\[
w_{ij} = \frac{1 + |\ln \lambda_i^r - \ln \lambda_j^r|}{\max\{\lambda_i, \lambda_j\}} = \frac{1 + |\ln \lambda_i^r - \ln \lambda_j^r|}{\max\{\lambda_i^{2r}, \lambda_j^{2r}\}} \cdot \frac{1}{(\max\{\lambda_i, \lambda_j\})^{1-2r}} =: w_{ij} v_{ij}.
\]
The matrix \( U = \{u_{ij}\}_{n \times n} \) is positive semidefinite (by the case \( r = 1/2 \) already proved), and the matrix \( V = \{v_{ij}\}_{n \times n} \) is positive semidefinite since the matrix \( \{1/\max\{\lambda_i, \lambda_j\}\}_{n \times n} \) is infinitely divisible [3, 5]. So \( W = \{w_{ij}\}_{n \times n} \) being the Schur product of \( U \) and \( V \) is positive semidefinite. Now consider the case \( r > 1/2 \). Let \( \tilde{r} \) be any number such that \( r > \tilde{r} > 1/2 \), and \( \alpha \) be the unique positive root of \( x = 2\tilde{r}\ln(1 + x) \) (such a root exists because at \( x = 0 \), the derivative of \( x \) is 1 and the derivative of \( 2\tilde{r}\ln(1 + x) \) is \( 2\tilde{r} > 1 \)). With \( \lambda_1 = 1 \) and \( \lambda_2 = e^{\alpha/r} \), the \( 2 \times 2 \) matrix \( W \) is
\[
W = \begin{pmatrix}
1 & (1 + \alpha)e^{-\alpha/r} \\
(1 + \alpha)e^{-\alpha/r} & e^{-\alpha/r}
\end{pmatrix}
\]
whose determinant
\[
\det W = e^{-\alpha/r} - (1 + \alpha)^2 e^{-2\alpha/r} = e^{-\alpha/r} \left[ e^{\alpha/r} - (1 + \alpha)^2 \right] < 0
\]
since \( \alpha/r < \alpha/r = 2 \ln(1 + \alpha) = \ln(1 + \alpha)^2 \).

**Remark 5.2.** Examples of means for which \( M \leq \tilde{M} \) but the stronger relation \( M \ll \tilde{M} \) is not true were given in [4], and in [14]. To that list, we add another. We have seen that \( M_{\infty}(a, b) \leq \Lambda(a, b) := (a + b)/2 \), where \( \Lambda \) stands for the arithmetic mean. But the relation \( M_{\infty} \ll \Lambda \) is not true. For example, with \( \lambda_1 = 17/100 \), \( \lambda_2 = 18/100 \), and \( \lambda_3 = 72/100 \), the 3 \times 3 matrix with its \((i, j)\) entry being \( M_{\infty}(\lambda_i, \lambda_j)/\Lambda(\lambda_i, \lambda_j) \) has a negative eigenvalue \(-0.00011509756859\) computed by MATLAB.

**References**

20. Weisstein, Eric W.: *Stirling’s approximation*.

\[2\text{http://mathworld.wolfram.com/stirlingsapproximation.html.}\]
AN INTERPOLATING FAMILY OF MEANS

Rajendra Bhatia: Indian Statistical Institute, New Delhi 110 016, India.
E-mail address: rbb@isid.ac.in

Ren-Cang Li: Department of Mathematics, University of Texas at Arlington, P.O.
Box 19408, Arlington, TX 76019-0408, USA.
E-mail address: rcli@uta.edu
URL: http://www.uta.edu/faculty/rcli/