



Problems: 10494-10500

WMC Problems Group; Dennis Spellman; David Joyner; William P. Wardlaw; Robert A. Russell; Klaus Huber; Ray Redheffer; David Day; Jeffrey C. Lagarias; Peter W. Shor

The American Mathematical Monthly, Volume 103, Issue 1 (Jan., 1996), 74-75.

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The American Mathematical Monthly
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PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

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The published solution is likely to be based on a solution that is complete and correct. Additional information, such as references to other appearances of the problem or its solution, is also welcome.

An asterisk () after the number of a problem, or part of a problem, indicates that no solution is currently available.*

PROBLEMS

10494. *Proposed by WMC Problems Group, Western Maryland College, Westminster, MD.*

For each positive integer n , evaluate the sum

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} / \binom{2n}{k}.$$

10495. *Proposed by Dennis Spellman, Philadelphia, PA, and David Joyner and William P. Wardlaw, United States Naval Academy, Annapolis, MD.*

Let R be a *principal ideal ring*; that is, R is a commutative ring with 1 in which every ideal is of the form Ra for some $a \in R$. Prove or give a counterexample: If $a, b \in R$ are multiples of one another, then they are *unit* multiples of one another (that is, there is an invertible element $u \in R$ such that $a = ub$).

10496. Proposed by Robert A. Russell, New York, NY.

Let C_m^n denote the number of cells in an n dimensional polyomino formed by adding m coats, as described below, to a monomino (one-celled polyomino). A coat consists of just enough cells to cover each previously exposed $n - 1$ dimensional cell face. Thus $C_0^n = 1$, $C_1^n = 2n + 1$, and $C_2^n = 2n^2 + 2n + 1$. Show that $C_m^n = C_n^m$.

10497. Proposed by Klaus Huber, Darmstadt, Germany.

The Gaussian integers are those complex numbers $x + iy$ for which x and y are integers. Given a complex number z , let $[z]$ denote the closest Gaussian integer to z , let z^* denote the complex conjugate of z , and let $N(z) = zz^*$. It is known that, if p is a rational prime with $p \equiv 1 \pmod{4}$, then $p = a^2 + b^2$ with integer a and b in an essentially unique way, and hence $p = \pi\pi^*$ with π a Gaussian integer in an essentially unique way. Reduction modulo π is defined by

$$\gamma \bmod \pi = \gamma - \left[\frac{\gamma \cdot \pi^*}{\pi \cdot \pi^*} \right] \cdot \pi.$$

A reduced set of residues $\{\alpha_i : i = 1 \dots p - 1\}$ modulo the Gaussian integer π can be defined by choosing g to be a primitive root modulo p and setting $\alpha_i = g^i \bmod \pi$. Show that

$$\sum_{i=1}^{p-1} N(\alpha_i) = \frac{p^2 - 1}{6}.$$

10498. Proposed by Ray Redheffer, University of California, Los Angeles, CA.

Consider the system of differential equations

$$\frac{dx}{dt} = -(x + a(t)y) \quad \frac{dy}{dt} = -(b(t)x + y) \quad (*)$$

where $a(t)$ and $b(t)$ are positive, continuous and bounded for $0 \leq t < \infty$.

If $(\sup a(t))(\sup b(t)) < 1$, it is easy to prove that all solutions of $(*)$ tend to 0 as $t \rightarrow \infty$. Does the same conclusion follow if one assumes only that $\sup (a(t)b(t)) < 1$?

10499. Proposed by David Day, University of Kentucky, Lexington, KY, and Ren-Cang Li, University of California, Berkeley, CA.

Let $M = T + \text{diag}(\alpha_i)$, where T is Hermitian Toeplitz and $\alpha_1, \dots, \alpha_n$ are real numbers with $\alpha_1 < \dots < \alpha_n$. Let $\lambda_1 \leq \dots \leq \lambda_n$ denote the eigenvalues of M . Show that

$$\min_{1 \leq i \leq n-1} (\lambda_{i+1} - \lambda_i) \geq \min_{1 \leq i \leq n-1} (\alpha_{i+1} - \alpha_i).$$

10500. Proposed by Jeffrey C. Lagarias and Peter W. Shor, AT&T Bell Laboratories, Murray Hill, NJ.

Consider the following three properties that a sequence $\{f(n) : n = 1, 2, \dots\}$ of real numbers may have.

(P1) The sequence $\{f(n) : n = 1, 2, \dots\}$ is bounded.

(P2) For each real $\lambda > 1$, the subsequence $\{f(\lfloor 2^{\lambda^n} \rfloor) : n = 1, 2, \dots\}$ is bounded.

(P3) For each real $\lambda > 1$, the subsequence $\{f(\lfloor \lambda^{2^n} \rfloor) : n = 1, 2, \dots\}$ is bounded.

Obviously (P1) \implies (P2) and (P1) \implies (P3). What other implications hold, if any?