PROBLEMS AND SOLUTIONS

Edited by:
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Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions and relevant references. Three copies of all items needed to evaluate the problem should be sent.

Solutions of published problems should arrive at the MONTHLY PROBLEMS address given on the inside front cover before June 30, 1996. If possible, solutions should be typed with double spacing. Two copies suffice. Several solutions may be mailed together, but they should be on separate sheets of paper. The problem number and the solver's name and mailing address should appear on each solution. A mailing label should be included if an acknowledgment is desired.

The published solution is likely to be based on a solution that is complete and correct. Additional information, such as references to other appearances of the problem or its solution, is also welcome.

An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available.

PROBLEMS

10494. *Proposed by WMC Problems Group, Western Maryland College, Westminster, MD.*

For each positive integer n, evaluate the sum

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}\Big/\binom{2n}{k}.$$

10495. *Proposed by Dennis Spellman, Philadelphia, PA, and David Joyner and William P. Wardlaw, United States Naval Academy, Annapolis, MD.*

Let R be a *principal ideal ring*; that is, R is a commutative ring with 1 in which every ideal is of the form Ra for some $a \in R$. Prove or give a counterexample: If $a, b \in R$ are multiples of one another, then they are *unit* multiples of one another (that is, there is an invertible element $u \in R$ such that $a = ub$).

10496. *Proposed by Robert A. Russell, New York, NY.*

Let C_m^n denote the number of cells in an n dimensional polyomino formed by adding m *coats*, as described below, to a *monomino* (one-celled polyomino). A coat consists of just enough cells to cover each previously exposed $n-1$ dimensional cell face. Thus $C_0^n = 1$, $C_1^n = 2n+1$, and $C_2^n = 2n^2+2n+1$. Show that $C_m^n = C_n^m$.

10497. *Proposed by Klaus Huber, Darmstadt, Germany.*

The *Gaussian integers* are those complex numbers $x+iy$ for which x and y are integers. Given a complex number z, let $[z]$ denote the closest Gaussian integer to z, let z^* denote the complex conjugate of z, and let $N(z) = zz^*$. It is known that, if p is a rational prime with $p \equiv 1 \pmod 4$, then $p = a^2 + b^2$ with integer a and b in an essentially unique way, and hence $p = \pi\pi^*$ with π a Gaussian integer in an essentially unique way. Reduction modulo π is defined by

$$\gamma \bmod \pi = \gamma - \left[\frac{\gamma \cdot \pi^*}{\pi \cdot \pi^*}\right] \cdot \pi.$$

A *reduced set of residues* $\{\alpha_i : i = 1 \dots p-1\}$ modulo the Gaussian integer π can be defined by choosing g to be a primitive root modulo p and setting $\alpha_i = g^i \bmod \pi$. Show that

$$\sum_{i=1}^{p-1} N(\alpha_i) = \frac{p^2-1}{6}.$$

10498. *Proposed by Ray Redheffer, University of California, Los Angeles, CA.*

Consider the system of differential equations

$$\frac{dx}{dt} = -\big(x + a(t)y\big) \qquad \frac{dy}{dt} = -\big(b(t)x + y\big) \tag{$*$}$$

where $a(t)$ and $b(t)$ are positive, continuous and bounded for $0 \le t < \infty$.

If $\big(\sup a(t)\big)\big(\sup b(t)\big) < 1$, it is easy to prove that all solutions of $(*)$ tend to 0 as $t \to \infty$. Does the same conclusion follow if one assumes only that $\sup\big(a(t)b(t)\big) < 1$?

10499. *Proposed by David Day, University of Kentucky, Lexington, KY, and Ren-Cang Li, University of California, Berkeley, CA.*

Let $M = T + \operatorname{diag}(\alpha_i)$, where T is Hermitian Toeplitz and $\alpha_1, \dots, \alpha_n$ are real numbers with $\alpha_1 < \cdots < \alpha_n$. Let $\lambda_1 \le \cdots \le \lambda_n$ denote the eigenvalues of M. Show that

$$\min_{1 \le i \le n-1} (\lambda_{i+1} - \lambda_i) \ge \min_{1 \le i \le n-1} (\alpha_{i+1} - \alpha_i).$$

10500. *Proposed by Jeffrey C. Lagarias and Peter W. Shor, AT&T Bell Laboratories, Murray Hill, NJ.*

Consider the following three properties that a sequence $\{f(n) : n = 1, 2, \dots\}$ of real numbers may have.

(P1) The sequence $\{f(n) : n = 1, 2, \dots\}$ is bounded.

(P2) For each real $\lambda > 1$, the subsequence $\{f(\lfloor 2^{\lambda^n} \rfloor) : n = 1, 2, \dots\}$ is bounded.

(P3) For each real $\lambda > 1$, the subsequence $\{f(\lfloor \lambda^{2^n} \rfloor) : n = 1, 2, \dots\}$ is bounded.

Obviously (P1) $\Longrightarrow$ (P2) and (P1) $\Longrightarrow$ (P3). What other implications hold, if any?