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# Krylov type subspace methods for matrix polynomials

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## Abstract

We consider solving eigenvalue problems or model reduction problems for a quadratic matrix polynomial  $I\lambda^2 - A\lambda - B$  with large and sparse  $A$  and  $B$ . We propose new Arnoldi and Lanczos type processes which operate on the same space as  $A$  and  $B$  live and construct projections of  $A$  and  $B$  to produce a quadratic matrix polynomial with the coefficient matrices of much smaller size, which is used to approximate the original problem. We shall apply the new processes to solve eigenvalue problems and model reductions of a second order linear input–output system and discuss convergence properties. Our new processes are also extendable to cover a general matrix polynomial of any degree.

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### 1. Introduction

Krylov subspace techniques are widely used for solving linear systems of equations and eigenvalue problems involving large and sparse matrices [7,13]. They have found applications in many other large scale matrix problems such as model reductions of linear input–output systems [10,11]. The basic idea of the techniques is to extract information of an  $n \times n$  matrix  $A$  most relevant to the underlying computational problem through utilizing the so-called *Krylov* subspace

$$\mathcal{K}_k(A, v) = \text{span}\{v, Av, \dots, A^{k-1}v\},$$

or through utilizing two (row and column) Krylov subspaces  $\mathcal{K}_k(A, v)$  and  $\mathcal{K}_k(A^*, w)$  simultaneously, where  $v$  and  $w$  are vectors of dimension  $n$ , the asterisk denotes the conjugate transpose. This is realized by either the *Arnoldi* (or *Lanczos*) process when only  $\mathcal{K}_k(A, v)$  is employed or by the *nonsymmetric Lanczos process* if both  $\mathcal{K}_k(A, v)$  and  $\mathcal{K}_k(A^*, w)$  are used [1,17]. See also [7,13,20,24,25].

When  $\mathcal{K}_k(A, v)$  has dimension  $k$ , the Arnoldi process generates an orthonormal basis  $\{q_1, q_2, \dots, q_k\}$  for  $\mathcal{K}_k(A, v)$ , and an upper Hessenberg matrix  $H_k = Q_k^* A Q_k$ , which is the projection of  $A$  onto  $\mathcal{K}_k(A, v)$ , where  $Q_k = [q_1, q_2, \dots, q_k]$ . On the other hand, the Lanczos process generates a basis for  $\mathcal{K}_k(A, v)$  and a basis for  $\mathcal{K}_k(A^*, w)$  such that the two bases are biorthogonal. Simultaneously, a tridiagonal matrix  $T_k$  is obtained, which is the projection of  $A$  onto  $\mathcal{K}_k(A, v)$  along  $\mathcal{K}_k(A^*, w)$ . For Hermitian  $A$ , usually  $w$  is taken to be  $v$  and the process coincides with the Arnoldi process and is called the (symmetric) Lanczos process.

For modest  $k \ll n$ , some of the eigenvalues of  $H_k$  (and  $T_k$ ) are good approximations to some eigenvalues (usually extreme part) of  $A$ . This approximation of  $A$  by  $H_k$  (and  $T_k$ ) can be used in many other applications as well, such as the model reduction of linear input–output systems [5,10,11]. Over the years, many technical inventions, including a shift-and-invert strategy, look-ahead techniques, rational Krylov, (adaptive) block versions, implicit restart strategies, and so on, have been developed for the Lanczos/Arnoldi algorithm for better numerical efficiency and stability. But we shall not discuss them here, see for example [3,4,9,14,15,21,23,24,27,31] and references therein.

In this paper, we consider a related problem for a large and sparse  $n \times n$  *monic matrix polynomial*

$$A(\lambda) = I_n \lambda^m - \sum_{i=0}^{m-1} A_i \lambda^i, \tag{1.1}$$

where  $I_n$  is the  $n \times n$  identity matrix (later we may simply write  $I$  if its dimension is clear from the context). Eq. (1.1) is typically associated with the initial value problem for

$$x^{(m)}(t) - \sum_{i=0}^{m-1} A_i x^{(i)}(t) = g(t), \tag{1.2}$$

where  $A_i$  is an  $n \times n$  constant matrix and  $x(t)$  is a vector of dimension  $n$ , depending on  $t$ , and the initial values are usually given to  $x^{(i)}(t)$  at  $t = 0$  for  $0 \leq i \leq m - 1$ . A modal analysis of (1.2) is to find those scalars  $\mu$  and nonzero vectors  $x$  and/or  $y$  such that

$$A(\mu)x = 0, \quad y^*A(\mu) = 0.$$

$\mu$  is called an eigenvalue of (1.1) and  $x$  (and  $y$ ) a right (and left, resp.) eigenvector. On the other hand, in a linear input–output system with the state governed by (1.2), we are interested in approximations and computations of the transfer function

$$f(s) = c^*A(s)^{-1}b. \quad (1.3)$$

Specifically, we would like to find a lower dimensional linear input–output system whose input–output relation (i.e., the transfer function) gives a good approximation to those of the given system. This is often referred to as model reductions.

In most cases, a problem (e.g., eigenvalue problem) concerning the matrix polynomial can be reduced to one for the following  $nm \times nm$  matrix [12]:

$$A_{\text{LIN}} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{m-1} \end{pmatrix},$$

to which well-established methods can be applied. This is called *linearization*. For the eigenvalue problem or the model reduction problem, one can use the Arnoldi or the Lanczos algorithm on  $A_{\text{LIN}}$  to produce a Krylov subspace and then a projection, which is used to approximate  $A_{\text{LIN}}$ . Such a process will have to operate with vectors of dimension  $mn$ . Furthermore, the projection of  $A_{\text{LIN}}$  on a Krylov subspace is usually not a linearization of any matrix polynomial and thus the approximation as obtained loses its intrinsic physical connection to the original problem. In model reductions, for example, a consequence of this is that the reduced model that is obtained by applying the Arnoldi or the Lanczos process to the linearization problem  $A_{\text{LIN}}$  cannot be synthesized with a physical model of an  $m$ th order input–output system [2].

In this paper, we study extensions of the *standard Arnoldi process* and the *standard Lanczos process* for matrix polynomials without going through any linearization. We note that several other methods [19,26] have been developed that do not rely on the linearization processes (see also [4,28]). Here, we develop Krylov type projection methods that generate a basis  $\{q_1, q_2, \dots, q_k\}$  for a subspace as defined by  $A_i$  and its powers, and then apply projection simultaneously to each matrix  $A_i$  of the matrix polynomial to obtain  $H_k^{(i)} = Q_k^*A_iQ_k$  which is also in some condensed form, where  $Q_k = [q_1, q_2, \dots, q_k]$ . Then we approximate  $A(\lambda)$  by the lower dimensional matrix polynomial

$$H_k(\lambda) \equiv I_k \lambda^m - \sum_{i=0}^{m-1} H_k^{(i)} \lambda^i.$$

Compared with the linearization, this approach has advantages of preserving certain properties of the original system  $A(\lambda)$  such as

- If the field of values of  $A(\lambda)$  (i.e.,  $\mathcal{F} = \{\lambda : x^* A(\lambda)x = 0 \text{ for some } x \neq 0\}$ ) is on the left half complex plane, which guarantees stability of the system (1.2), then the field of values of  $H_k(\lambda)$  is also on the left half plane, preserving the stability.
- The property that  $A_j$  is symmetric, positive definite, etc. is preserved by  $H_k^{(i)}$ . In particular, if  $A(\lambda)$  is an overdamped vibrating system [8,12], so is  $H_k(\lambda)$ . So the property that all eigenvalues are real is preserved. The same is true if  $A(\lambda)$  comes from a weakly damped system, which has the eigenvalues near the imaginary axis.
- A gyroscopic system about a stable equilibrium [8] has  $A_0 > 0$  and  $A_1$  skew-Hermitian with  $m = 2$ . In this case, the eigenvalues are all imaginary. Again, the projection problem preserves this property.

An Arnoldi process of this type has already been developed recently for a monic quadratic polynomial [18], where a special case is also considered in which a linear combination of the two matrices is of low rank. The present paper, as a continuation of [18], will first investigate the subspaces associated with the Arnoldi type process, especially for the commutative case. This is done in Section 2. In Section 3, we develop a Lanczos type process for monic quadratic polynomials. Briefly analogous extensions to a general monic polynomial of degree  $m$  are mentioned. The uses of the Arnoldi/Lanczos type processes for model reductions and eigenvalue computations, and their convergent properties are given in Sections 4 and 5. We shall also present a numerical example to illustrate our new algorithms in Section 6. Throughout the paper, however, we will be focusing mostly on explaining the new ideas and leave subtle but important numerical considerations and numerical applications (e.g., more extensive numerical testing) to future studies. We also note that our focus on monic matrix polynomials does not lose any generality because non-monic cases can be handled implicitly as a monic one through some combination of shifting and a factorization of the leading matrix.

**Notation.** The  $j$ th column of an identity matrix (of size that will be clear from the context) is denoted as  $e_j$ .  $\mathbb{C}^{m \times n}$  is the set of all  $m$ -by- $n$  complex matrices,  $\mathbb{C}^n = \mathbb{C}^{n \times 1}$  (vectors), and  $\mathbb{C} = \mathbb{C}^1$  (scalars). We shall use MATLAB-like notation  $X_{(i:j,k:\ell)}$  to denote the submatrix of  $X$ , consisting of the intersections of rows  $i$  to  $j$  and columns  $k$  to  $\ell$ , and when  $i : j$  is replaced by  $:$ , it means all rows, similarly for columns. The generic notation  $\alpha$  is for a possible nonzero scalar, vector, and  $X$  for a possible nonzero matrix.  $\|x\|_2$  is the Euclidean norm of a vector  $x$  and  $\|X\|_2$  is the spectral norm of a matrix  $X$ . Given a real number  $\gamma$ ,  $\lfloor \gamma \rfloor$  is the largest integer that is not greater than  $\gamma$ .

## 2. Arnoldi type process for monic matrix polynomials

In this section, we shall first describe the Arnoldi type process for  $I\lambda^2 - A\lambda - B$  derived in [18], and then study the Krylov type subspace that it computes. In particular, we discuss the special case when  $A$  and  $B$  commute. We shall also show how this can be generalized to a general  $m$ th degree monic matrix polynomial.

### 2.1. Arnoldi type process for $I\lambda^2 - A\lambda - B$

The Arnoldi type algorithm for  $I\lambda^2 - A\lambda - B$  of [18] is based on the fact that given  $q_1 \in \mathbb{C}^n$  with  $\|q_1\|_2 = 1$ , there is a unitary matrix  $Q \in \mathbb{C}^{n \times n}$  with  $Qe_1 = q_1$  such that<sup>4</sup>

$$Q^*AQ = H_a \equiv (h_{a;ij}), \quad Q^*BQ = H_b \equiv (h_{b;ij}) \quad (2.1)$$

with

$$h_{a;ij} = 0 \text{ for } i \geq 2j + 1, \quad h_{b;ij} = 0 \text{ for } i \geq 2j + 2. \quad (2.2)$$

From this, the following algorithm is derived in [18].

#### Algorithm 2.1 (ARNOLDI TYPE PROCESS)

1. **Given**  $q_1$  **with**  $\|q_1\|_2 = 1$ ;
2.  $N = 1$ ;
3. **for**  $j = 1, 2, \dots, k$  **do**
4.   **if**  $j > N$  **then BREAK**;
5.    $\hat{q} = Aq_j$ ;
6.   **for**  $i = 1, 2, \dots, N$  **do**
7.      $h_{a;ij} = q_i^* \hat{q}$ ;  $\hat{q} = \hat{q} - q_i h_{a;ij}$ ;
8.   **end for**
9.    $h_{a;N+1,j} = \|\hat{q}\|_2$ ;
10.   **if**  $h_{a;N+1,j} > 0$  **then**
11.      $N = N + 1$ ,  $q_N = \hat{q} / h_{a;Nj}$ ;
12.   **end if**
13.    $\hat{q} = Bq_j$ ;
14.   **for**  $i = 1, 2, \dots, N$  **do**
15.      $h_{b;ij} = q_i^* \hat{q}$ ;  $\hat{q} = \hat{q} - q_i h_{b;ij}$ ;
16.   **end for**
17.    $h_{b;N+1,j} = \|\hat{q}\|_2$ ;
18.   **if**  $h_{b;N+1,j} > 0$  **then**

<sup>4</sup> The entries of  $H_a$  and  $H_b$  are unfortunately heavily subscripted – with a lower case letter before the separator “;” to indicate the association to  $A$  or  $B$ , and with two integers  $i$  and  $j$  as their row and column indexes. For better readability, sometimes we insert a comma between the two integers.

- 19.  $N = N + 1; q_N = \hat{q}/h_{a;Nj};$
- 20. **end if**
- 21. **end for**

In the algorithm,  $N$  tracks the number of vectors  $q_i$  already generated at any given point. Let  $\alpha_k$  and  $\beta_k$  be the values of  $N$  at the ends of Lines 12 and 20, respectively, for the trip  $j = k$ . It can be seen that  $\alpha_k \leq \beta_k \leq \alpha_k + 1$ . Upon completion of the above process, we have (see [18])

$$AQ_{(:,1:k)} = Q_{(:,1:\alpha_k)}H_{a(1:\alpha_k,1:k)}, \tag{2.3}$$

$$BQ_{(:,1:k)} = Q_{(:,1:\beta_k)}H_{b(1:\beta_k,1:k)}, \tag{2.4}$$

unless the  $j$ -loop is forced to **BREAK** out at Line 4, in which case,

$$AQ_{(:,1:N)} = Q_{(:,1:N)}H_{a(1:N,1:N)}, \tag{2.5}$$

$$BQ_{(:,1:N)} = Q_{(:,1:N)}H_{b(1:N,1:N)}. \tag{2.6}$$

While  $H_a$  and  $H_b$  are lower banded, their lower bandwidths increase quickly. In [18], the special case that a linear combination of  $A$  and  $B$  is of low rank is considered. In that case, the lower bandwidth of  $H_a$  and  $H_b$  is bounded by the rank of the combination plus 1. Later, we shall consider a special case when  $A$  and  $B$  commute. As we shall see, the lower bandwidth can also be significantly reduced in this case. We first need to describe the subspace  $\text{span}\{q_1, q_2, \dots, q_\ell\}$  in terms of  $q_1$ ,  $A$ , and  $B$ .

### 2.2. Subspace $\text{span}\{q_1, q_2, \dots, q_\ell\}$

Suppose first that in line 10 and 18 of Algorithm 2.1, all

$$h_{a;N+1,j} > 0, \quad h_{b;N+1,j} > 0.$$

We notice that the process starts with  $q_1$ , and at  $j = 1$  it generates new directions in vectors  $Aq_1$  first and then  $Bq_1$ ; at  $j = 2$ , new directions in vectors  $A^2q_1$  first and then  $BAq_1$ ; at  $j = 3$  new directions in vectors  $ABq_1$  and then  $B^2q_1$ ; and so on. The following table displays new vectors that expand the same subspace as the vectors generated at step  $j$ .

	$j$						
	1	2	3	4	5	6	7
New	$Aq_1$	$A^2q_1$	$ABq_1$	$A^3q_1$	$ABAq_1$	$A^2Bq_1$	$AB^2q_1$
Vectors	$Bq_1$	$BAq_1$	$B^2q_1$	$BA^2q_1$	$B^2Aq_1$	$BABq_1$	$B^3q_1$





Krylov subspace. First we need to know the corresponding position for  $q_j$  in (2.10). To this end, let integers  $s$  and  $t$  be such that

$$j = t(t+1)/2 + s \quad \text{for some } 0 < s \leq t+1,$$

i.e.,  $q_j$  belongs to Group  $t$  in (2.10), and  $q_j \sim B^{s-1}A^{t-(s-1)}q_1$ . Thus

$$Aq_j \sim B^{s-1}A^{t+1-(s-1)}q_1 \sim q_{(t+1)(t+2)/2+s} = q_{j+t+1}, \quad (2.11)$$

$$Bq_j \sim B^sA^{t+1-s}q_1 \sim q_{(t+1)(t+2)/2+s+1} = q_{j+t+2}. \quad (2.12)$$

So the  $j$ th column of  $H_a$  has nonzero entries from position 1 to  $j+t+1$  and the  $j$ th column of  $H_b$  has nonzero entries from position 1 to  $j+t+2$ . Fig. 1 shows the structures of  $H_a$  and  $H_b$  for the commutative case.

**Theorem 2.2.** *Given  $q_1 \in \mathbb{C}^n$  with  $\|q_1\|_2 = 1$ , suppose that  $A$  and  $B$  commute and that the first  $n$  vectors in (2.10) are linearly independent. Then there is a unitary matrix  $Q \in \mathbb{C}^{n \times n}$  with  $Qe_1 = q_1$  such that*

$$Q^*AQ = H_a \equiv (h_{a;ij}), \quad Q^*BQ = H_b \equiv (h_{b;ij}) \quad (2.13)$$

satisfy

$$h_{a;ij} = 0 \text{ for } i \geq j+t+2, \quad h_{b;ij} = 0 \text{ for } i \geq j+t+3, \quad (2.14)$$

where  $t$  is the unique integer such that  $t(t+1)/2 + (t+1) \geq j > t(t+1)/2$ .

It is of interest to compare nonzero patterns for  $H_a$  and  $H_b$  here with those for  $H_a$  and  $H_b$  in (2.1) for general  $A$  and  $B$ , where there are about  $j$  nonzero entries below the diagonal entries in the  $j$ th columns. Theorem 2.2, however, says when  $A$  and  $B$  commute there are about  $\sqrt{2j}$  nonzero entries below the diagonal entries since  $t \approx \sqrt{2j}$  for large  $j$ .

It is of an independent interest to see how many trips to the  $j$ -loop in Algorithm 3.1 for commutative  $A$  and  $B$  must be made in order to produce an orthonormal basis for the entire space  $\mathbb{C}^n$ , assuming that the first  $n$  vectors in (2.10) are linearly independent. (Recall that in the generic case and noncommutative  $A$  and  $B$ ,  $(n-1)/2$  trips are enough.) To this end, by (2.11) and (2.12) we need to find the minimal  $j$  so that

$$\begin{aligned} j+t+1 = n \quad \text{or} \quad j+t+2 = n, \\ \text{subject to } j = t(t+1)/2 + s, \quad 1 \leq s \leq t+1, \end{aligned}$$

or equivalently  $t(t+1)/2 + 1 \leq j \leq t(t+1)/2 + t + 1$ . Write  $j+t+2 = \hat{n}$ , where  $\hat{n} = n$  or  $n+1$  as needed. Then

$$\begin{aligned} t(t+1)/2 + t + 3 \leq \hat{n} \leq t(t+1)/2 + 2t + 3 \\ \iff t^2 + 3t + 6 - 2\hat{n} \leq 0 \leq t^2 + 5t + 6 - 2\hat{n}, \end{aligned}$$

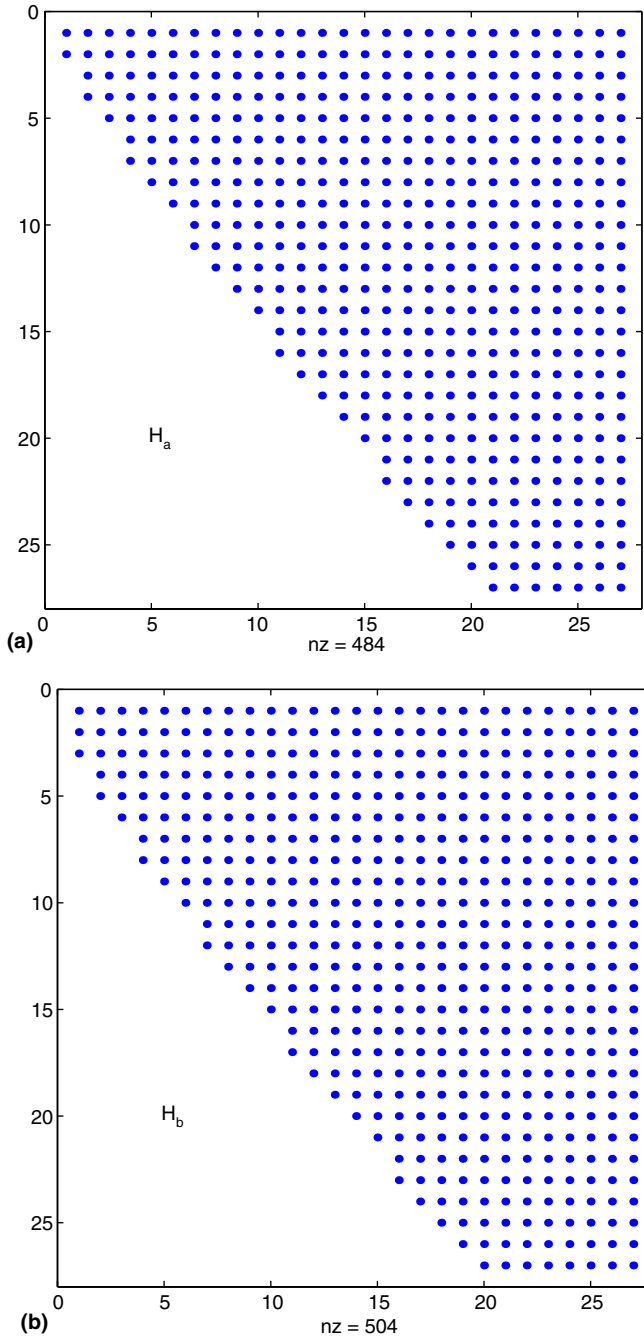


Fig. 1. The sparsity patterns of  $H_a$  and  $H_b$ ; the commutative case.

which gives

$$\frac{-5 + \sqrt{8\hat{n} + 1}}{2} \leq t \leq \frac{-3 + \sqrt{8\hat{n} - 15}}{2}.$$

Thus

$$\hat{n} - \frac{1}{2} - \frac{\sqrt{8\hat{n} - 15}}{2} \leq j \leq \hat{n} + \frac{1}{2} - \frac{\sqrt{8\hat{n} + 1}}{2}.$$

For example when  $n = 1000$ , this requires  $j = 955$ . Asymptotically it needs  $n$  steps.

#### 2.4. General monic matrix polynomial

The Arnoldi type process developed for  $A$  and  $B$  can be extended to a general monic matrix polynomial (1.1) of degree  $m$ . Recall the theoretical backbone of the algorithm is the decomposition (2.1). So we shall just state a corresponding decomposition result for  $A(\lambda)$  but omit a detailed statement of an algorithm. The actual algorithm follows readily from this.

**Theorem 2.3.** *Let  $A_\ell \in \mathbb{C}^{n \times n}$ ,  $0 \leq \ell \leq m - 1$ . Given  $q_1 \in \mathbb{C}^n$  with  $\|q_1\|_2 = 1$ , there is a unitary matrix  $Q \in \mathbb{C}^{n \times n}$  with  $Qe_1 = q_1$  such that*

$$Q^* A_\ell Q = H_\ell \equiv (h_{\ell;ij}), \quad \text{for } 0 \leq \ell \leq m - 1 \quad (2.15)$$

satisfying

$$h_{\ell;ij} = 0, \quad \text{for } i \geq mj + m - \ell. \quad (2.16)$$

### 3. Lanczos type process for monic quadratic matrix polynomials

In this section, we develop a Lanczos type processes in parallel to the Arnoldi type process presented in the previous section. We shall present the derivation for monic quadratic matrix polynomial  $I\lambda^2 - A\lambda - B$ , and indicate a generalization to general  $m$ th degree monic matrix polynomials.

#### 3.1. Lanczos type process

Given  $v_1, w_1 \in \mathbb{C}^n$  with  $w_1^* v_1 = 1$ , a sequence of similarity transformations  $V_1, V_2, \dots, V_k$  ( $k \leq n$ ) can be constructed (unless a division by zero is encountered in the process) such that

$$V^{-1} A V = T_a \equiv (t_{a;ij}), \quad V^{-1} B V = T_b \equiv (t_{b;ij}) \quad (3.1)$$

with

$$\begin{aligned} t_{a;ij} &= 0, & \text{for } i \geq 2j + 1 \text{ or } j \geq 2i + 1, \\ t_{b;ij} &= 0, & \text{for } i \geq 2j + 2 \text{ or } j \geq 2i + 2. \end{aligned} \quad (3.2)$$

where  $V = V_1 V_2 \cdots V_k$  satisfies  $V e_1 = v_1$  and  $V^{-*} e_1 = w_1$ , see [16] for the construction. Eq. (3.1) or its partial reduction can also be realized by a Lanczos type process, which we present now.

Let  $W = V^{-*}$  and rewrite (3.1) to get

$$\begin{aligned} AV &= VT_a, & A^*W &= WT_a^*, & W^*V &= I_n, \\ BV &= VT_b, & B^*W &= WT_b^*. \end{aligned} \tag{3.3}$$

Notice that  $v_1$  and  $w_1$  are given and  $w_1^* v_1 = 1$ . We shall show how to progressively compute the first few columns of  $V = (v_1, v_2, \dots, v_n)$ ,  $W = (w_1, w_2, \dots, w_n)$ ,  $T_a$  and  $T_b$  from  $v_1$  and  $w_1$  until a breakdown occurs. We have

$$Av_1 = v_1 t_{a;11} + v_2 t_{a;21}, \tag{3.4}$$

$$A^*w_1 = w_1 \bar{t}_{a;11} + w_2 \bar{t}_{a;12}, \tag{3.5}$$

$$Bv_1 = v_1 t_{b;11} + v_2 t_{b;21} + v_3 t_{b;31}, \tag{3.6}$$

$$B^*w_1 = w_1 \bar{t}_{b;11} + w_2 \bar{t}_{b;12} + w_3 \bar{t}_{b;13}. \tag{3.7}$$

Eqs. (3.4) and (3.5), and biorthogonality between  $V$ 's columns and  $W$ 's columns yield  $t_{a;11} = w_1^* A v_1$ . Set

$$\hat{v}_2 = Av_1 - v_1 t_{a;11}, \quad \hat{w}_2 = A^*w_1 - w_1 \bar{t}_{a;11}.$$

We may take  $t_{a;21} = \sqrt{|\hat{w}_2^* \hat{v}_2|}$ . A breakdown occurs if  $t_{a;21} = 0$ ; otherwise we assign

$$t_{a;12} = \hat{w}_2^* \hat{v}_2 / t_{a;21}, \quad v_2 = \hat{v}_2 / t_{a;21}, \quad w_2 = \hat{w}_2 / \bar{t}_{a;12}.$$

Then,  $w_2^* v_2 = 1$ . Next we turn to (3.6) and (3.7). Analogously we obtain

$$t_{b;11} = w_1^* B v_1, \quad t_{b;21} = w_2^* B v_1, \quad t_{b;12} = w_1^* B v_2. \tag{3.8}$$

Set

$$\hat{v}_3 = Bv_1 - v_1 t_{b;11} - v_2 t_{b;21}, \quad \hat{w}_3 = B^*w_1 - w_1 \bar{t}_{b;11} - w_2 \bar{t}_{b;12}. \tag{3.9}$$

We may take  $t_{b;31} = \sqrt{|\hat{w}_3^* \hat{v}_3|}$ . A breakdown occurs if  $t_{b;31} = 0$ ; otherwise we assign

$$t_{b;13} = \hat{w}_3^* \hat{v}_3 / t_{b;31}, \quad v_3 = \hat{v}_3 / t_{b;31}, \quad w_3 = \hat{w}_3 / \bar{t}_{b;13}.$$

Then,  $w_3^* v_3 = 1$ . In general, we have for  $j \geq 2$ :

$$Av_j = \sum_{i=\lfloor (j+1)/2 \rfloor}^{2j-1} v_i t_{a;ij} + v_{2j} t_{a;2jj}, \tag{3.10}$$

$$A^*w_j = \sum_{i=\lfloor (j+1)/2 \rfloor}^{2j-1} w_i \bar{t}_{a;ji} + w_{2j} \bar{t}_{a;j2j}, \tag{3.11}$$

$$Bv_j = \sum_{i=\lfloor j/2 \rfloor}^{2j} v_i t_{b;ij} + v_{2j+1} t_{b;2j+1j}, \tag{3.12}$$

$$B^*w_j = \sum_{i=\lfloor j/2 \rfloor}^{2j} w_i \bar{t}_{b;ji} + w_{2j+1} \bar{t}_{b;j2j+1}. \quad (3.13)$$

with similar expressions for  $B$  and  $B^*$ . Eqs. (3.10) and (3.11), and biorthogonality between  $V$ 's columns and  $W$ 's columns yield

$$t_{a;ij} = w_i^* A v_j, \quad t_{a;ji} = w_j^* A v_i, \quad \text{for } \lfloor (j+1)/2 \rfloor \leq i \leq 2j-1. \quad (3.14)$$

Set

$$\hat{v}_{2j} = A v_j - \sum_{i=\lfloor (j+1)/2 \rfloor}^{2j-1} v_i t_{a;ij}, \quad \hat{w}_{2j} = A^* w_j - \sum_{i=\lfloor (j+1)/2 \rfloor}^{2j-1} w_i \bar{t}_{a;ji}. \quad (3.15)$$

We may take  $t_{a;2jj} = \sqrt{|\hat{w}_{2j}^* \hat{v}_{2j}|}$ . A breakdown occurs if  $t_{a;2jj} = 0$ ; otherwise we assign

$$t_{a;j2j} = \hat{w}_{2j}^* \hat{v}_{2j} / t_{a;2jj}, \quad v_{2j} = \hat{v}_{2j} / t_{a;2jj}, \quad w_{2j} = \hat{w}_{2j} / \bar{t}_{a;j2j}.$$

Then,  $w_{2j}^* v_{2j} = 1$ . Similarly we get  $v_{2j+1}$  and  $w_{2j+1}$  from  $B v_j$  and  $B^* w_j$ , respectively. Analogously we obtain

$$t_{b;ij} = w_i^* B v_j, \quad t_{b;ji} = w_j^* B v_i, \quad \text{for } \lfloor j/2 \rfloor \leq i \leq 2j. \quad (3.16)$$

Set

$$\hat{v}_{2j+1} = B v_j - \sum_{i=\lfloor j/2 \rfloor}^{2j} v_i t_{b;ij}, \quad \hat{w}_{2j+1} = B^* w_j - \sum_{i=\lfloor j/2 \rfloor}^{2j} w_i \bar{t}_{b;ji}. \quad (3.17)$$

We may take  $t_{b;2j+1j} = \sqrt{|\hat{w}_{2j+1}^* \hat{v}_{2j+1}|}$ . A breakdown occurs if  $t_{b;2j+1j} = 0$ ; otherwise we assign

$$t_{b;j2j+1} = \hat{w}_{2j+1}^* \hat{v}_{2j+1} / t_{b;2j+1j}, \quad v_{2j+1} = \hat{v}_{2j+1} / t_{b;2j+1j}, \\ w_{2j+1} = \hat{w}_{2j+1} / \bar{t}_{b;j2j+1}.$$

Then,  $w_{2j+1}^* v_{2j+1} = 1$ .

If  $A$  and  $B$  are Hermitian and  $v_1 = w_1$ , then  $T_a$  and  $T_b$  are symmetric and  $v_i = w_i$ , and the above computation is equivalent to the Arnoldi type process for the symmetric problem.

The following figures in (3.18) show what the computed part of  $T_a$  and  $T_b$  look like for  $j$ , as in (3.10)–(3.13), running from 1 to  $k = 5$ , where the entries marked by *unfilled circles* have not been computed yet, but they can be readily computed afterwards by

$$t_{a;ij} = w_i^* A v_j, \quad t_{b;ij} = w_i^* B v_j, \quad \text{for } k+1 \leq i, j \leq 2k+1.$$



```

17.  $\hat{v} = Bv_j; \hat{w} = B^*w_j;$ 
18. if  $j > \beta_{\ell_b}$  then  $\ell_b = \ell_b + 1;$ 
19. for  $i = \ell_b, \dots, N$  do
20.    $t_{b;ij} = w_i^* \hat{v}; \hat{v} = \hat{v} - v_i t_{b;ij};$ 
21.    $t_{b;ji} = \hat{w}^* v_i; \hat{w} = \hat{w} - w_i t_{b;ji};$ 
22. end for
23.  $t_{b;N+1,j} = \sqrt{|\hat{w}^* \hat{v}|};$ 
24. if  $t_{b;N+1,j} = 0$  then
25.   if  $\hat{v} \neq 0$  or  $\hat{w} \neq 0$  then BREAK;
26. else
27.    $N = N + 1, t_{b;jN} = \hat{w}^* \hat{v} / t_{b;Nj}; v_N = \hat{v} / t_{b;Nj}; w_N = \hat{w} / \bar{t}_{b;jN}, \beta_j = N;$ 
28. end if
29. end for

```

In Algorithm 3.1,  $N$  tracks the number of vectors  $v_i$  already constructed at any given point, which is also the number of vectors  $w_i$  already constructed at that point;  $\alpha_j$  is the value of  $N$  at the end of Line 15, i.e., the row number of the last nonzero entry in the  $j$ th column of  $T_a$  and  $\beta_j$  is the value of  $N$  at the end of Line 27, i.e., the row number of the last nonzero entry in the  $j$ th column of  $T_b$ ;  $\ell_a$  (and  $\ell_b$ ) tracks the row number of the first nonzero entry in the  $j$ th column of  $T_a$  (and  $T_b$  resp.). Then  $\ell_a$  is the smallest integer such that  $\alpha_{\ell_a} \geq j$ , and  $\ell_b$  is the smallest integer such that  $\beta_{\ell_b} \geq j$ .

If the execution of Algorithm 3.1 is completed by the **BREAK** statement at Line 4, a common (right) invariant subspace and a common left invariant subspace for  $A$  and  $B$  have been computed, and

$$AV_{(:,1:N)} = V_{(:,1:N)}T_{a(1:N,1:N)}, \quad A^*W_{(:,1:N)} = W_{(:,1:N)}[T_{a(1:N,1:N)}]^*, \quad (3.19)$$

$$BV_{(:,1:N)} = V_{(:,1:N)}T_{b(1:N,1:N)}, \quad B^*W_{(:,1:N)} = W_{(:,1:N)}[T_{b(1:N,1:N)}]^*, \quad (3.20)$$

If none of the **BREAK** statements is executed, we have

$$AV_{(:,1:k)} = V_{(:,1:\alpha_k)}T_{a(1:\alpha_k,1:k)}, \quad A^*W_{(:,1:k)} = W_{(:,1:\alpha_k)}[T_{a(1:k,1:\alpha_k)}]^*, \quad (3.21)$$

$$BV_{(:,1:k)} = V_{(:,1:\beta_k)}T_{b(1:\beta_k,1:k)}, \quad B^*W_{(:,1:k)} = W_{(:,1:\beta_k)}[T_{b(1:k,1:\beta_k)}]^*, \quad (3.22)$$

In this case, parts of projections  $T_a$  and  $T_b$  are not computed, but they can be readily obtained afterwards by

$$t_{a;ij} = q_i^* Aq_j, \quad t_{b;ij} = q_i^* Bq_j, \quad \text{for } k + 1 \leq i \leq N \text{ and } k + 1 \leq j \leq N.$$

It can be seen that the nonzero patterns, i.e., the positions of nonzero entries, in  $T_a$  and  $T_b$  here, are contained in those as described in (3.18).

The following theorem shows that indeed two sets  $\{w_i\}$  and  $\{v_i\}$  by Algorithm 3.1 enjoy the desired biorthogonality property.

**Theorem 3.1.** *Suppose Algorithm 3.1 runs to its completion without breakdowns to produce  $\{w_i\}_{i=1}^N$  and  $\{v_i\}_{i=1}^N$ , where  $N = \beta_k$ . Then*

$$w_i^* v_j = 0 \quad \text{if } i \neq j \text{ and } w_i^* v_i = 1.$$

**Proof.**  $w_i^* v_i = 1$  is clear from their definition. We shall only need to show  $w_i^* v_j = 0$  if  $i \neq j$ . Let  $\ell_{aj}$  and  $\ell_{bj}$  be the  $\ell_a$  and  $\ell_b$  used at Lines 7 and 19, respectively, at the  $j$ th trip of the  $j$ -loop,  $\hat{v}_i$  and  $\hat{w}_i$  be the ones finally used to give  $v_i$  and  $w_i$  by scalar scaling in Algorithm 3.1.

We prove the claim by induction on  $k$ . The case for  $k = 1$  is easy to establish. Suppose the claim in the theorem holds for  $k = j - 1$ . We now show it also holds for  $k = j$ , i.e.,  $w_s^* \hat{v}_{\alpha_j} = 0 = \hat{w}_{\alpha_j}^* v_s$  for  $s < \alpha_j$  and  $w_s^* \hat{v}_{\beta_j} = 0 = \hat{w}_{\beta_j}^* v_s$  for  $s < \beta_j$ .  $w_s^* \hat{v}_{\alpha_j} = 0 = \hat{w}_{\alpha_j}^* v_s$  for  $\ell_{aj} \leq s \leq \alpha_j - 1$  follows from the definition. For the case of  $s < \ell_{aj}$ , we have

$$\begin{aligned} w_s^* \hat{v}_{\alpha_j} &= w_s^* \left( Av_j - \sum_{i=\ell_{aj}}^{\alpha_j-1} v_i t_{a;ij} \right) \\ &= w_s^* Av_j = \left[ \hat{w}_{\alpha_s} + \sum_{i=\ell_{as}}^{\alpha_s-1} w_i \bar{t}_{a;si} \right]^* v_j \\ &= 0, \\ \hat{w}_{\alpha_j}^* v_s &= \left[ A^* w_j - \sum_{i=\ell_{aj}}^{\alpha_j-1} w_i \bar{t}_{a;ji} \right]^* v_s \\ &= w_j^* Av_s = w_j^* \left( \hat{v}_{\alpha_s} + \sum_{i=\ell_{as}}^{\alpha_s-1} v_i t_{a;is} \right) \\ &= 0, \end{aligned}$$

by induction hypothesis, since  $s < \ell_{aj}$  implies  $\alpha_s < j$  by Line 6. Similarly we have  $\hat{w}_{\alpha_j}^* v_s = 0$ .

Analogously we can show  $w_s^* \hat{v}_{\beta_j} = 0 = \hat{w}_{\beta_j}^* v_s$  for  $s < \beta_j$ .  $\square$

### 3.2. General monic matrix polynomial

As in Section 2.4, the Lanczos type process developed can also be extended to a general monic matrix polynomial (1.1) of degree  $m$ . Again, we just state the corresponding decomposition results and omit a detailed statement of an algorithm.

**Theorem 3.2.** *Let  $A_\ell \in \mathbb{C}^{n \times n}$ ,  $0 \leq \ell \leq m - 1$ . Given  $v_1, w_1 \in \mathbb{C}^n$  such that  $w_1^* v_1 = 1$ , (unless there is a breakdown) there is a matrix  $V \in \mathbb{C}^{n \times n}$  with  $V e_1 = v_1$  and  $V^{-*} e_1 = w_1$  such that*

$$V^{-1} A_\ell V = T_a \equiv (t_{\ell;ij}), \quad \text{for } 0 \leq \ell \leq m - 1 \quad (3.23)$$

satisfying

$$t_{\ell;ij} = 0, \quad \text{for } i \geq mj + m - \ell \text{ or } j \geq mi + m - \ell. \quad (3.24)$$

## 4. Application to model reduction

In a second order single-input and single-output linear system, a quadratic matrix polynomial is involved in its transfer function

$$f(s) = c^*(I - As - Bs^2)^{-1}b$$

where  $b$  and  $c$  are  $n$ -dimensional vectors,  $A$  and  $B$  are  $n \times n$ , either sparse or in some kinds of factored forms. In model reductions, it is desirable that the given system is approximated by another second order system of lower dimension that is called a reduced system. The approximation is usually in terms of the transfer functions and is often done by requiring that the transfer function of the reduced system  $g(s)$  and the original transfer function  $f(s)$  to have the same moments up to certain degree (i.e., terms associated with  $s^0, s^1, s^2, \dots$  of their Taylor expansions at  $s = 0$ ). In the case of first order systems, Feldman and Freund [10] show that the Lanczos algorithm is a powerful method that can be used to achieve this. Here we shall show that the Arnoldi/Lanczos type algorithms derived in the previous sections can be used in the same way for second order systems.

- For the Arnoldi type process with  $q_1 = b/\|b\|_2$ , let Algorithm 2.1 produce  $Q_{(:,1:N)}$ ,  $H_{a(1:N,1:N)}$ , and  $H_{b(1:N,1:N)}$ . Define

$$g_{\text{arnld}}(s) = \tilde{c}_N^*(I - H_{a(1:N,1:N)}s - H_{b(1:N,1:N)}s^2)^{-1}\|b\|_2 e_1, \quad (4.1)$$

where  $\tilde{c}_N = Q_{(:,1:N)}^* c$ .

- For the Lanczos type process with  $v_1 = b/\|b\|_2$  and  $w_1 = c/b^*c$ , suppose that Algorithm 3.1 runs to its completion without breakdowns and produces  $V_{(:,1:N)}$ ,  $W_{(:,1:N)}$ ,  $T_{a(1:N,1:N)}$ , and  $T_{b(1:N,1:N)}$ . Define

$$g_{\text{lanz}}(s) = (c^*b)e_1^*(I - T_{a(1:N,1:N)}s - T_{b(1:N,1:N)}s^2)^{-1}e_1. \quad (4.2)$$

We shall next find out how accurate  $g_{\text{arnd}}(s)$  and  $g_{\text{lancz}}(s)$  are as approximations to  $f(s)$  by determining the numbers of matching leading terms in their Taylor expansions at  $s = 0$ . We start by presenting a lemma.

**Lemma 4.1.** *Let  $X = (x_{ij})$  and  $Y = (y_{ij})$  be two  $n \times n$  matrices satisfying*

$$x_{ij} = 0, \quad \text{for } i > k_1j + \ell_1 \quad \text{and} \quad y_{ij} = 0, \quad \text{for } i > k_2j + \ell_2,$$

*and let  $Z = XY = (z_{ij})$ . Assume that no zero entries in  $Z$  are caused by exact arithmetic cancellations. Then  $z_{ij} = 0$  if and only if  $i > k_1k_2j + (k_1\ell_2 + \ell_1)$ .*

**Proof.** We have

$$\begin{aligned} z_{ij} &= \sum_{m=1}^n x_{im}y_{mj} = \sum_{\{m:i \leq k_1m + \ell_1\}} x_{im}y_{mj} + \sum_{\{m:i > k_1m + \ell_1\}} x_{im}y_{mj} \\ &= \sum_{\{m:i \leq k_1m + \ell_1\}} x_{im}y_{mj} + 0 \end{aligned} \tag{4.3}$$

since  $x_{im} = 0$  for  $i > k_1m + \ell_1$ . Now for  $i > (k_1k_2)j + (k_1\ell_2 + \ell_1)$  and  $i \leq k_1m + \ell_1$ , we have

$$k_1m + \ell_1 > (k_1k_2)j + (k_1\ell_2 + \ell_1) \Rightarrow m > k_2j + \ell_2;$$

so  $y_{mj} = 0$ . Therefore  $z_{ij} = 0$  for  $i > (k_1k_2)j + (k_1\ell_2 + \ell_1)$  by (4.3). It can be seen that in the generic case if  $i \leq (k_1k_2)j + (k_1\ell_2 + \ell_1)$ , then at least one summand in  $\sum_{\{m:i \leq k_1m + \ell_1\}} x_{im}y_{mj}$  is not zero.  $\square$

#### 4.1. Arnoldi type process

**Theorem 4.1.** *Let  $g_{\text{arnd}}(s)$  be as defined in (4.1) as a result of Algorithm 2.1. Then*

$$f(s) = g_{\text{arnd}}(s) + \mathcal{O}(s^{\lfloor \log_2 N \rfloor + 1}).$$

*If, in addition,  $c = b$ , then  $f(s) = g_{\text{arnd}}(s) + \mathcal{O}(s^{\lfloor \log_2 N \rfloor + 2})$ .*

This result is a bit of disappointing in terms of the order of approximation, compared unfavorably to the standard Arnoldi method for the linear model reduction which can achieve order  $\mathcal{O}(s^k)$  of approximation [29]. But Theorem 4.1 appears to be best possible in general unless there are additional structures in  $A$  and  $B$  such as commutativity or the low rank case as in [18]. If the Arnoldi Type Process is performed to its completion, we will have  $n \times n$  matrices  $Q$ ,  $H_a \equiv (h_{a;ij})$ , and  $H_b \equiv (h_{b;ij})$ , satisfying (2.1) and (2.2). We note that the sparsity pattern of (2.2) defines an envelope

enclosing nonzeros of  $H_a$  and  $H_b$ , while the matrices  $H_a$  and  $H_b$  as generated by the algorithm may be more condensed. From the reduction, we have

$$f(s)/\|b\|_2 = \tilde{c}^*(I - H_a s - H_b s^2)^{-1} e_1,$$

where  $\tilde{c} = Q^*c$ . The Taylor expansion of  $f(s)/\|b\|_2$  at  $s = 0$  is

$$f(s)/\|b\|_2 = \tilde{c}^* \left( \sum_{\ell=0}^{\infty} H_\ell s^\ell \right) e_1 = \sum_{\ell=0}^{\infty} (\tilde{c}^* H_\ell e_1) s^\ell,$$

where, as a result of

$$(I - H_a s - H_b s^2) \left( \sum_{\ell=0}^{\infty} H_\ell s^\ell \right) = I_n = \left( \sum_{\ell=0}^{\infty} H_\ell s^\ell \right) (I - H_a s - H_b s^2),$$

$H_\ell$  can be recursively defined as

$$H_0 = I_n, \tag{4.4}$$

$$H_1 = H_a, \tag{4.5}$$

$$H_\ell = H_a H_{\ell-1} + H_b H_{\ell-2}, \quad \text{for } \ell \geq 2, \tag{4.6}$$

$$= H_{\ell-1} H_a + H_{\ell-2} H_b, \quad \text{for } \ell \geq 2. \tag{4.7}$$

On the other hand, it can be derived similarly that

$$g_{\text{rand}}(s)/\|b\|_2 = \sum_{\ell=0}^{\infty} (\tilde{c}_N^* \tilde{H}_\ell e_1) s^\ell,$$

where  $\tilde{H}_\ell$  is recursively defined as

$$\tilde{H}_0 = I_N, \tag{4.8}$$

$$\tilde{H}_1 = H_{a(1:N,1:N)}, \tag{4.9}$$

$$\tilde{H}_\ell = H_{a(1:N,1:N)} \tilde{H}_{\ell-1} + H_{b(1:N,1:N)} \tilde{H}_{\ell-2}, \quad \text{for } \ell \geq 2, \tag{4.10}$$

$$= \tilde{H}_{\ell-1} H_{a(1:N,1:N)} + \tilde{H}_{\ell-2} H_{b(1:N,1:N)}, \quad \text{for } \ell \geq 2. \tag{4.11}$$

**Lemma 4.2.**  $H_\ell$  and  $\tilde{H}_\ell$  have the property that their  $(i, j)$ th entries are zero if  $i > 2^\ell j$ .

**Proof.** We prove this by induction for  $H_\ell$ ; the proof is similar for  $\tilde{H}_\ell$ . Clearly the result holds for  $H_0 = I_n$  and  $H_1 = H_a$ . Suppose the result holds for all matrices  $H_0, H_1, \dots, H_\ell$ , then the  $(i, j)$ th entry of  $H_a H_\ell$  is zero for  $i > 2^{\ell+1} j$  by Lemma 4.1 and by (2.2). It can be checked that the  $(i, j)$ th entry of  $H_b H_{\ell-1}$  is zero for  $i > 2^\ell j + 1$ . But for  $\ell \geq 1$  and  $j \geq 1$ ,  $2^{\ell+1} j \geq 2^\ell j + 2$ . Therefore, the result now follows from the recurrence relation.  $\square$

**Lemma 4.3.** *Suppose  $N \geq 3$ , and let  $m = \lfloor \log_2 N \rfloor$ , the largest possible integer such that  $2^m \leq N$ . Then*

- (1)  $H_\ell e_j = \begin{matrix} N \\ n - N \end{matrix} \begin{pmatrix} \tilde{H}_\ell e_j \\ 0 \end{pmatrix}$  for  $\ell = 0, 1, \dots, m$  and  $j = 1, 2, \dots, 2^{m-\ell}$ .
- (2)  $H_{m+1} e_1 = \begin{matrix} N \\ n - N \end{matrix} \begin{pmatrix} \tilde{H}_{m+1} e_1 \\ \mathbf{x} \end{pmatrix}$ .

Here the integers to the left of the partitioned matrices are the row sizes of the corresponding blocks.

**Proof.** That the last  $n - N$  entries of  $H_\ell e_j$  for  $0 \leq \ell \leq m$  and  $1 \leq j \leq 2^{m-\ell}$  are all zeros is due to Lemma 4.2 since  $N + 1 > 2^m = 2^\ell 2^{m-\ell} \geq 2^\ell j$ . We shall prove Claim 1 by induction on  $\ell$ . It holds true for  $\ell = 0, 1$ . Suppose  $m \geq \ell \geq 2$  and that the claim holds for  $0, 1, \dots, \ell - 1$ . Then

$$\begin{aligned} H_\ell e_j &= H_a H_{\ell-1} e_j + H_b H_{\ell-2} e_j \\ &= H_a \begin{pmatrix} \tilde{H}_{\ell-1} e_j \\ 0 \end{pmatrix} + H_b \begin{pmatrix} \tilde{H}_{\ell-2} e_j \\ 0 \end{pmatrix} \\ &= H_a \begin{pmatrix} \tilde{H}_{\ell-1} & 0 \\ 0 & 0 \end{pmatrix} e_j + H_b \begin{pmatrix} \tilde{H}_{\ell-2} & 0 \\ 0 & 0 \end{pmatrix} e_j \\ &= \begin{pmatrix} H_a(1:N, 1:N) \tilde{H}_{\ell-1} e_j \\ \mathbf{x} \end{pmatrix} + \begin{pmatrix} H_b(1:N, 1:N) \tilde{H}_{\ell-2} e_j \\ \mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{H}_\ell e_j \\ \mathbf{x} \end{pmatrix}. \end{aligned}$$

The lower block marked by  $\mathbf{x}$  is 0 for  $\ell \leq m$  and  $j \leq 2^{m-\ell}$  as we commented; Claim 1 is proved. With Claim 1 proved, setting  $\ell = m + 1$  and  $j = 1$  in the above equations leads to Claim 2.  $\square$

**Proof of Theorem 4.1.**  $\ell \leq m = \lfloor \log_2 N \rfloor$  implies  $2^\ell \leq N$ ; and thus by Lemma 4.3

$$\tilde{c}^* H_\ell e_1 = \begin{pmatrix} \tilde{c}_N \\ \mathbf{x} \end{pmatrix}^* \begin{pmatrix} \tilde{H}_\ell e_1 \\ 0 \end{pmatrix} = \tilde{c}_N^* \tilde{H}_\ell e_1,$$

as expected. Now if  $c = b$ ,  $\tilde{c} = \tilde{c}_N = \|b\|_2 e_1$ , then also by Lemma 4.3

$$e_1^* H_{m+1} e_1 = \begin{pmatrix} e_1 \\ 0 \end{pmatrix}^* \begin{pmatrix} \tilde{H}_{m+1} e_1 \\ \mathbf{x} \end{pmatrix} = e_1^* \tilde{H}_{m+1} e_1,$$

as was to be shown.  $\square$

#### 4.2. Lanczos type process

**Theorem 4.2.** *Suppose Algorithm 3.1 runs to its completion without breakdowns with  $Ve_1 = \alpha b$  and  $We_1 = \beta c$ , and as a result let  $g_{\text{lancz}}(s)$  be as defined in (4.2). Assume also the process can be continued until all  $n$  columns of  $V$  (and  $W$ ) are obtained. Then*

$$f(s) = g_{\text{lancz}}(s) + \mathcal{O}(s^{2\lfloor \log_2 N \rfloor + 1}).$$

Theorem 4.2 is a consequence of Claim 5 of Lemma 4.5.

The assumption that the Lanczos type process can be continued until all  $n$  columns of  $V$  (and  $W$ ) are obtained is for our theoretical analysis only. In practice, the process will stop much earlier than that for large sparse matrices. With the assumption, we have  $V, W, T_a \equiv (t_{a;ij})$ , and  $T_b \equiv (t_{b;ij})$ , satisfying (3.1) and (3.2). Then,

$$f(s)/(c^*b) = e_1^*(I - T_a s - T_b s^2)^{-1} e_1$$

whose Taylor expansion at  $s = 0$  is

$$f(s)/(c^*b) = e_1^* \left( \sum_{\ell=0}^{\infty} T_{\ell} s^{\ell} \right) e_1 = \sum_{\ell=0}^{\infty} (e_1^* T_{\ell} e_1) s^{\ell},$$

where  $T_{\ell}$  can be recursively defined the same as (4.4)–(4.7) with  $H$  replaced by  $T$ . On the other hand, for  $g_{\text{lancz}}(s)$  we have

$$g_{\text{lancz}}(s)/(c^*b) = \sum_{\ell=0}^{\infty} (e_1^* \tilde{T}_{\ell} e_1) s^{\ell},$$

where  $\tilde{T}_{\ell}$  can also be recursively defined the same as (4.8)–(4.11) with  $H$  replaced by  $T$ .

**Lemma 4.4.**  $T_{\ell}$  and  $\tilde{T}_{\ell}$  have the property that their  $(i, j)$ th entries are zero if  $i > 2^{\ell} j$  or  $2^{\ell} i < j$ .

**Proof.** Analogous to the proof of Lemma 4.2.  $\square$

**Lemma 4.5.** *Suppose  $N \geq 3$ , and let  $m = \lfloor \log_2 N \rfloor$ , the largest possible integer such that  $2^m \leq N$ . Then*

- (1)  $T_{\ell} e_j = \begin{matrix} N \\ n - N \\ 0 \end{matrix} \binom{\tilde{T}_{\ell} e_j}{0}$  for  $\ell = 0, 1, \dots, m$  and  $j = 1, 2, \dots, 2^{m-\ell}$ .
- (2)  $T_{m+1} e_1 = \begin{matrix} N \\ n - N \\ x \end{matrix} \binom{\tilde{T}_{m+1} e_1}{x}$ .
- (3)  $e_i^* T_{\ell} e_1 = e_i^* \tilde{T}_{\ell} e_1$  for  $\ell = 0, 1, \dots, 2m$  and  $i = 1, 2, \dots, \min\{2^m, 2^{2m-\ell}\}$ .

**Proof.** Claims 1 and 2 follow from Lemma 4.3 since the nonzero patterns of  $T_a$  and  $T_b$  are contained in those of  $H_a$  and  $H_b$ . On the other hand, since the nonzero patterns of  $T_a^T$  and  $T_b^T$  are also contained in those of  $H_a$  and  $H_b$ , the conclusion of Lemma 4.3 applies to  $T_\ell^T$ , which implies  $e_i^* T_\ell = (e_i^* \tilde{T}_\ell, 0)$  for  $\ell = 0, 1, \dots, m$  and  $i = 1, 2, \dots, 2^{m-\ell}$ . and  $e_1^* T_{m+1} = (e_1^* \tilde{T}_{m+1}, 0)$ .

We now prove Claim 3 by induction on  $\ell$ . First for  $0 \leq \ell \leq m$ , the claim holds due to Claim 1; the claim also holds for  $\ell = m + 1$  because of Claim 2. In fact, we can say more

$$e_i^* T_\ell e_1 = e_i^* \tilde{T}_\ell e_1, \quad \text{for } 0 \leq \ell \leq m + 1 \text{ and } 1 \leq i \leq 2^m.$$

We now prove Claim 5 for the rest of  $\ell$ :

$$e_i^* T_\ell e_1 = e_i^* \tilde{T}_\ell e_1, \quad \text{for } m \leq \ell \leq 2m \text{ and } 1 \leq i \leq 2^{2m-\ell}, \tag{4.12}$$

by induction on  $\ell$ . This claim holds for  $\ell = m, m + 1$ . Suppose that  $m + 2 \leq \ell \leq 2m$ ,  $i \leq 2^{2m-\ell}$ , and that (4.12) holds for  $m, m + 1, \dots, \ell - 1$ . Notice that  $2i \leq 2^{2m-(\ell-1)}$  and  $2i + 1 < 2^{2m-(\ell-2)}$ . Then

$$\begin{aligned} e_i^* T_\ell e_1 &= e_i^* (T_a T_{\ell-1} + T_b T_{\ell-2}) e_1 \\ &= \left( \sum_{j=\lfloor (i+1)/2 \rfloor}^{2i} t_{a;ij} e_j^* \right) T_{\ell-1} e_1 + \left( \sum_{j=\max\{1, \lfloor i/2 \rfloor\}}^{2i+1} t_{b;ij} e_j^* \right) T_{\ell-2} e_1 \\ &= \sum_{j=\lfloor (i+1)/2 \rfloor}^{2i} t_{a;ij} (e_j^* \tilde{T}_{\ell-1} e_1) + \sum_{j=\max\{1, \lfloor i/2 \rfloor\}}^{2i+1} t_{b;ij} (e_j^* \tilde{T}_{\ell-2} e_1), \end{aligned}$$

on using the induction hypothesis. This can be simplified to

$$\begin{aligned} e_i^* T_\ell e_1 &= e_i^* T_{a(1:N, 1:N)} \tilde{T}_{\ell-1} e_1 + e_i^* T_{b(1:N, 1:N)} \tilde{T}_{\ell-2} e_1 \\ &= e_i^* \tilde{T}_\ell e_1, \end{aligned}$$

as expected.  $\square$

### 5. Quadratic eigenvalue problems

The Krylov type methods that we have derived can also be used to compute eigenvalues and eigenvectors of  $I\lambda^2 - A\lambda - B$  as follows:

- For the Arnoldi type process, if Algorithm 2.1 produces  $Q_{(:, 1:N)}$ ,  $H_{a(1:N, 1:N)}$ , and  $H_{b(1:N, 1:N)}$  and  $\theta_i$  is an eigenvalue and  $u_i$  is a right eigenvector of

$$I\lambda^2 - H_{a(1:N, 1:N)}\lambda - H_{b(1:N, 1:N)}, \tag{5.1}$$

then we use  $(\theta_i, x_i)$  as an approximate eigenpair for the original problem, where

$$x_i = Q_{(:,1:N)}u_i. \quad (5.2)$$

- For the Lanczos type process, if Algorithm 3.1 runs to its completion without breakdowns and produces  $V_{(:,1:N)}$ ,  $W_{(:,1:N)}$ ,  $T_{a(1:N,1:N)}$ , and  $T_{b(1:N,1:N)}$ , and if  $\theta_i$  is an eigenvalue and  $u_i$  (and  $v_i$ ) is a right (left, resp.) eigenvector of

$$I\lambda^2 - T_{a(1:N,1:N)}\lambda - T_{b(1:N,1:N)}, \quad (5.3)$$

then we use  $\theta_i$  as an approximate eigenvalue with  $x_i$  (and  $y_i$ ) as an approximate right (left, resp.) eigenvector for the original problem, where

$$x_i = V_{(:,1:N)}u_i, \quad y_i = W_{(:,1:N)}v_i. \quad (5.4)$$

The above processes approximate a quadratic eigenvalue problem with a projected quadratic eigenvalue problem.

We now discuss convergence properties for the Lanczos type process only. Corresponding results for the Arnoldi type process can be obtained similarly. We first present a residual bound for the Ritz values and vectors.

**Theorem 5.1.** *For the Ritz values and Ritz vector obtained by the Lanczos type process (Algorithm 3.1), we have*

$$\begin{aligned} & \| (I\theta_i^2 - A\theta_i - B)x_i \| \\ & \leq \|V\| \left( \|\theta_i T_{a(N+1:\alpha_N, p:N)}\| + \|T_{b(N+1:\beta_N, p:N)}\| \right) \|u_{i,(p:N)}\| \end{aligned}$$

and

$$\begin{aligned} & \|y_i^* (I\theta_i^2 - A\theta_i - B)\| \\ & \leq \|W\| \left( \|\theta_i T_{a(p:N, N+1:\alpha_N)}\| + \|T_{b(p:N, N+1:\beta_N)}\| \right) \|v_{i,(p:N)}\|, \end{aligned}$$

where  $p$  is the smallest integer such that  $\beta_p > N$  and is equal to the value of  $\ell_b$  at step  $N + 1$ .

**Proof.** First, from (3.21) and (3.22), we have

$$\begin{aligned} AV_{(:,1:N)} &= V_{(:,1:N)}T_{a(1:N,1:N)} + V_{(:,N+1:\alpha_N)}T_{a(N+1:\alpha_N,1:N)}, \\ BV_{(:,1:N)} &= V_{(:,1:N)}T_{b(1:N,1:N)} + V_{(:,N+1:\beta_N)}T_{b(N+1:\beta_N,1:N)} \end{aligned}$$

Then

$$(I\theta_i^2 - A\theta_i - B)x_i = (\theta_i^2 V_{(:,1:N)} - \theta_i AV_{(:,1:N)} - BV_{(:,1:N)})u_i$$

$$\begin{aligned}
 &= V_{(:,1:N)}(I\theta_i^2 - T_{a(1:N,1:N)}\theta_i - T_{b(1:N,1:N)})u_i \\
 &\quad - \theta_i V_{(:,N+1:\alpha_N)}T_{a(N+1:\alpha_N,1:N)}u_i - V_{(:,N+1:\beta_N)}T_{b(N+1:\beta_N,1:N)}u_i \\
 &= -\theta_i V_{(:,N+1:\alpha_N)}T_{a(N+1:\alpha_N,p:N)}u_{i,(p:N)} \\
 &\quad - V_{(:,N+1:\beta_N)}T_{b(N+1:\beta_N,p:N)}u_{i,(p:N)},
 \end{aligned}$$

where we note that the first  $p - 1$  columns of  $T_{b(N+1:\beta_N,1:N)}$  and  $T_{a(N+1:\alpha_N,1:N)}$  are zeros. Taking the norm of above, we obtain the bound.  $\square$

As in the standard Lanczos algorithm, bottom end elements of eigenvectors  $u_{i,(p:N)}$  and  $v_{i,(p:N)}$  typically become small for Ritz values in the extreme part of the spectrum and that leads to small residuals. In particular, the above bounds can be used to estimate the residual norms without actually computing them.

We next derive an error bound similar to the one for the nonsymmetric Lanczos algorithm derived in [30]. Let

$$L = \begin{pmatrix} 0 & I \\ T_b & T_a \end{pmatrix} \quad \text{and} \quad L_N = \begin{pmatrix} 0 & I \\ T_{b(1:N,1:N)} & T_{a(1:N,1:N)} \end{pmatrix}.$$

**Lemma 5.1.** *Let  $S_\ell$  and  $T_\ell$  be recursively defined by*

$$\begin{aligned}
 S_0 &= 0, & T_0 &= I_n, \\
 S_1 &= T_b, & T_1 &= T_a, \\
 S_\ell &= T_a S_{\ell-1} + T_b S_{\ell-2} T_\ell = T_a T_{\ell-1} + T_b T_{\ell-2}, & \text{for } \ell \geq 2.
 \end{aligned}$$

Then

$$L^\ell = \begin{pmatrix} S_{\ell-1} & T_{\ell-1} \\ S_\ell & T_\ell \end{pmatrix}.$$

Similarly, let  $\tilde{S}_\ell$  and  $\tilde{T}_\ell$  be recursively defined from  $L_N$  as in  $S_\ell$  and  $T_\ell$  above. Then,

$$L_N^\ell = \begin{pmatrix} \tilde{S}_{\ell-1} & \tilde{T}_{\ell-1} \\ \tilde{S}_\ell & \tilde{T}_\ell \end{pmatrix}.$$

**Proof.** It can be verified by induction.  $\square$

It can be seen that the conclusion of Lemma 4.5 holds for  $\tilde{T}_\ell$  and  $T_\ell$  as well as for  $\tilde{S}_\ell$  and  $S_\ell$ . Thus we have

$$e_1^* S_\ell e_1 = e_1^* \tilde{S}_\ell e_1,$$

for  $\ell = 0, 1, \dots, 2m$  ( $m = \lfloor \log_2 N \rfloor$ ), which implies

$$e_1^* L^{\ell+1} e_1 = e_1^* L_N^{\ell+1} e_1.$$

Therefore, for any polynomial  $f$  of degree  $2m + 1$ ,

$$e_1^* f(L) e_1 = e_1^* f(L_N) e_1. \tag{5.5}$$

There are other similar results. They include, for example,

$$e_{n+1}^* L^\ell e_1 = e_{k+1}^* L_N^\ell e_1. \tag{5.6}$$

For the sake of simplicity, we assume now that  $L$  and  $L_N$  are diagonalizable, and write

$$L_N = U^* \Theta V \quad \text{and} \quad L = X^* \Lambda Y, \tag{5.7}$$

where  $\Theta = \text{diag}(\theta_1, \dots, \theta_m)$ ,  $U^* V = I$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ ,  $X^* Y = I$ . Write  $U = (u_{ij})$ ,  $V = (v_{ij})$ ,  $X = (x_{ij})$  and  $Y = (y_{ij})$ . Now substituting (5.7) into (5.5), we obtain

$$e_1^* X^* f(\Lambda) Y e_1 = e_1^* U^* f(\Theta) V e_1.$$

Thus

$$\sum_{i=1}^n f(\lambda_i) \bar{x}_{i1} y_{i1} = \sum_{i=1}^m f(\theta_i) \bar{u}_{i1} v_{i1}.$$

In particular, using  $f(x) = (x - \theta_1)p(x)$ , we have

$$\begin{aligned} & \lambda_1 - \theta_1 \\ &= \frac{1}{p(\lambda_1) \bar{x}_{11} y_{11}} \left[ - \sum_{i=2}^n (\lambda_i - \theta_1) p(\lambda_i) \bar{x}_{i1} y_{i1} + \sum_{i=2}^m (\theta_i - \theta_1) p(\theta_i) \bar{u}_{i1} v_{i1} \right]. \end{aligned}$$

This leads to the following theorem.

**Theorem 5.2.** *Let  $|\lambda_1 - \theta_1| = \min_j |\lambda_1 - \theta_j|$ . Then we have*

$$|\lambda_1 - \theta_1| \leq K \epsilon_{2m} \frac{(\sum_{i \neq 1} |x_{i1}|^2 + \sum_{i \neq 1} |u_{i1}|^2)^{1/2}}{|x_{11}|} \frac{(\sum_{i \neq 1} |y_{i1}|^2 + \sum_{i \neq 1} |v_{i1}|^2)^{1/2}}{|y_{11}|},$$

where

$$\epsilon_\ell = \min_{\deg(p)=\ell, p(\lambda_1)=1} \max_{i \neq 1} \{|p(\lambda_i)|, |p(\theta_i)|\}$$

and  $K = \max_{i \neq 1} \{|\lambda_i - \theta_1|, |\theta_i - \theta_1|\}$ .

We note that, if  $\lambda_1$  is an extreme eigenvalue, an appropriate polynomial can be chosen to make  $\epsilon_\ell$  small. Therefore, smaller bounds are obtained for the extreme part of the spectrum and for larger  $m$ . The last two terms in the bound depend on the angles between the initial vectors  $v_1$  and  $w_1$  and the right and left eigenvectors, respectively.

### 6. Numerical example

We present a numerical example in this section. Our example comes from a finite element discretization of the following problem from dissipative acoustics [6,28].

Let  $\Omega \subset \mathbb{R}^2$  be a rectangular cavity filled with an acoustic fluid (such as air), with one absorbing wall  $\Gamma_A$  and three reflecting walls  $\Gamma_R$ . Let  $P(x, t)$  and  $U(x, t)$  be the acoustic pressure and the fluid displacement, respectively. Also let  $\rho$  be the density of the fluid, and  $c$  the speed at which the fluid conducts sound. Then the behavior of the fluid satisfies the equations

$$\rho \frac{\partial^2 U}{\partial t^2} + \nabla P = 0, \tag{6.1}$$

$$-\rho c^2 \operatorname{div} U = P, \tag{6.2}$$

with boundary conditions

$$U \cdot \nu = 0 \quad \text{on } \Gamma_R, \tag{6.3}$$

$$\alpha U \cdot \nu + \beta \frac{\partial U}{\partial t} \cdot \nu = P \quad \text{on } \Gamma_A, \tag{6.4}$$

where the scalar constants  $\alpha, \beta$  are related to the impedance of the absorbing material. As in [6], we choose  $\rho = 1 \text{ kg/m}^3$ ,  $c = 340 \text{ m/s}$ ,  $\alpha = 5 \times 10^4 \text{ N/m}^3$ ,  $\beta = 200 \text{ N s/m}^3$  for our test; this choice of  $\alpha$  and  $\beta$  models a very viscous absorbing material.

We are interested in finding the damped vibration modes of the fluid, which are solutions of the form  $U(x, t) = e^{\lambda t} u(x)$ ,  $P(x, t) = e^{\lambda t} p(x)$ . Then, Eqs. (6.1)–(6.4) reduce to finding  $\lambda, p, u$  satisfying

$$\begin{aligned} \rho \lambda^2 u + \nabla p &= 0 \quad \text{in } \Omega, \\ p &= -\rho c^2 \operatorname{div} u \quad \text{in } \Omega, \\ p &= (\alpha + \lambda \beta) u \cdot \nu \quad \text{on } \Gamma_A, \\ u \cdot \nu &= 0 \quad \text{on } \Gamma_R. \end{aligned}$$

This system can be converted to a variational formulation. Let  $\mathcal{V} = \{v \in H(\operatorname{div}, \Omega) : v \cdot \nu \in L^2(\partial\Omega) \text{ and } v \cdot \nu = 0 \text{ on } \Gamma_R\}$ . The problem is equivalent to finding  $\lambda \in \mathbb{C}$ , nonzero  $u \in \mathcal{V}$  so that

$$\lambda^2 \int_{\Omega} \rho u \cdot v + \lambda \int_{\Gamma_A} \beta u \cdot \nu v \cdot \nu + \int_{\Gamma_A} \alpha u \cdot \nu v \cdot \nu + \int_{\Omega} \rho c^2 \operatorname{div} u \operatorname{div} v = 0 \tag{6.5}$$

for all  $v \in \mathcal{V}$ . Using finite elements to approximate  $\mathcal{V}$  by  $\mathcal{V}_h = \operatorname{span}\{\phi_1, \dots, \phi_n\}$  yields the  $n \times n$  quadratic matrix eigenvalue problem

$$\lambda^2 Mx + \lambda \beta Fx + (\alpha F + K)x = 0, \tag{6.6}$$

where

$$M_{ij} = \int_{\Omega} \rho \phi_i \cdot \phi_j, \quad K_{ij} = \int_{\Omega} \rho c^2 \operatorname{div} \phi_i \operatorname{div} \phi_j, \quad F_{ij} = \int_{\Gamma_A} \phi_i \cdot \nu \phi_j \cdot \nu.$$

To avoid spurious eigenvalues caused by discretization, it is suggested in [6] to use lowest order Raviart–Thomas finite elements [22]. Each basis element  $\phi_i$  is a vector-valued function with piecewise constant divergence on each triangle of the mesh and  $\phi_i \cdot \nu$  constant along each edge. With a natural choice of the basis, each finite element corresponds to an edge in the interior or on the absorbing boundary  $\Gamma_A$ . We use a triangulation of  $\Omega$  with  $6N$  edges along the vertical sides and  $8N$  edges along the horizontal sides; a model with 9168 degrees of freedom is obtained with the choice of the parameter  $N = 8$  (Fig. 2).

Let  $A = M$ ,  $B = \beta F$ ,  $C = \alpha F + K$  and write (6.6) as the symmetric quadratic eigenvalue problem

$$(\lambda^2 A + \lambda B + C)x = 0. \quad (6.7)$$

Note that  $A$  is symmetric positive definite, and  $B$ ,  $C$  are positive semidefinite matrices. The eigenvalues of interest are those with smaller imaginary parts, corresponding to lower-frequency vibration modes. They have real parts between  $-250$  and  $-320$ . Substituting  $\hat{A} = \sigma^2 A + \sigma B + C$ ,  $\hat{B} = 2\sigma A + B$ ,  $\hat{C} = A$ , and  $\mu = (\lambda - \sigma)^{-1}$  gives a shifted and inverted problem

$$(\mu^2 \hat{A} + \mu \hat{B} + \hat{C})x = 0;$$

for the choice of  $\sigma = -253$ ,  $\hat{A}$  remains positive definite. Therefore, we can take the Cholesky decomposition  $\hat{A} = LL^T$  and construct the equivalent monic problem

$$(\mu^2 I + \mu(L^{-1} \hat{B} L^{-T}) + (L^{-1} \hat{C} L^{-T}))u = 0. \quad (6.8)$$

Algorithm 3.1 is now applied to (6.8) in a symmetric Lanczos type process to get a basis  $V_k$  and banded  $k \times k$  matrices  $T_a, T_b$ . It follows that if  $(\theta_i, u_i)$  is an eigenpair to the projected problem

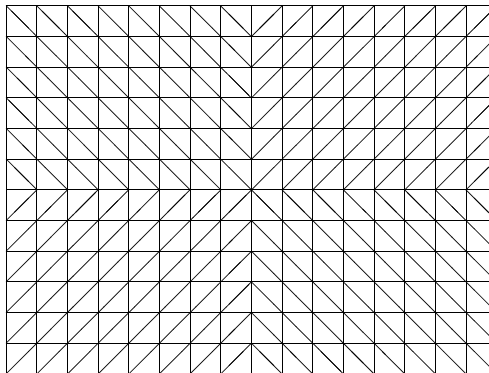


Fig. 2. Triangulation of  $\Omega$  with  $N = 2$ .

$$(\theta^2 I_k + \theta T_{a(1:k,1:k)} + T_{b(1:k,1:k)})u = 0, \tag{6.9}$$

then  $(\lambda_i, x_i) = (\sigma + 1/\theta_i, z_i / \|z_i\|)$  is an approximate eigenpair to the original problem (6.7), where  $z_i = L^{-T} V_{k(:,1:i)} u_i$ .

We applied Algorithm 3.1 to compute two conjugate pairs and two real eigenvalues of (6.7). As both the full-scale eigenvalue problem and its projection are real symmetric, all of their eigenvalues are real or appear in conjugate pairs. Thus, both eigenvalues of (6.7) in a conjugate pair converge simultaneously. For each selected eigenvalue and each iteration  $i = 1, \dots, 500$ , we compute the corresponding approximate eigenpair  $(\lambda_i, x_i)$  by linearizing and solving (6.9) with the MATLAB `eigs` function. The resulting relative residual norms  $r_i = \frac{\|(\lambda_i^2 A + \lambda_i B + C)x_i\|}{|\lambda_i|^2 \|A\| + |\lambda_i| \|B\| + \|C\|}$  are plotted in Fig. 3.

The following table lists the values of the four selected eigenvalues. For each eigenvalue  $\hat{\lambda}_i$  (as computed by MATLAB), the table gives the type of line used in Fig. 3 to plot the corresponding residual norms, the number of matrix–vector products required to obtain a relative residual norm  $r_i < 10^{-8}$ , and the error of the corresponding eigenvalue  $\lambda_i$  at that point.

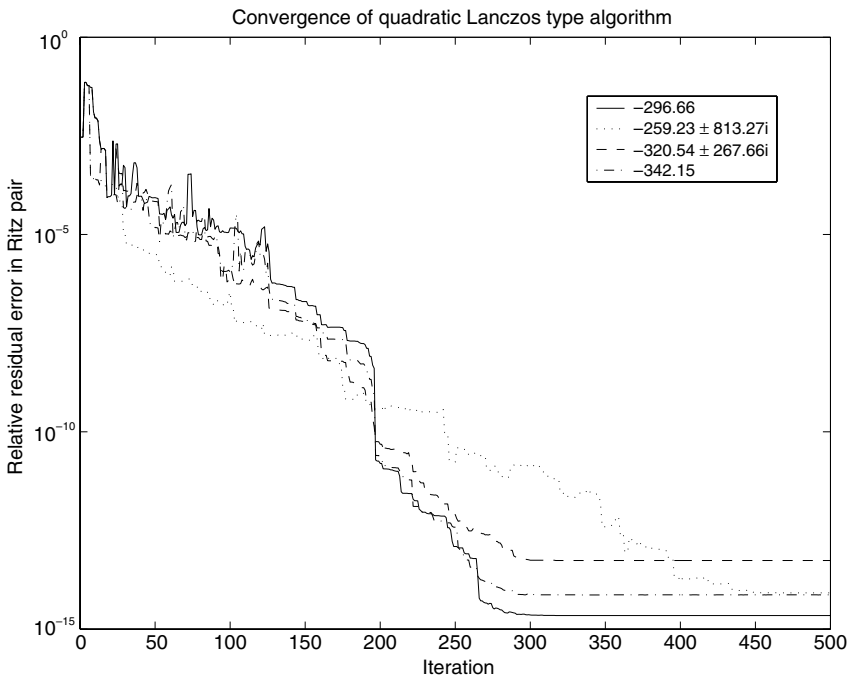


Fig. 3. Relative residual norms for selected eigenvalues.

Eigenvalue $\lambda_i$	Plot line	Matrix–vector products	$ \lambda_i - \hat{\lambda}_i $
$-259.23 \pm 813.27 i$	Dotted	318	$1.7 \times 10^{-8}$
$-320.54 \pm 267.66 i$	Dashed	322	$1.0 \times 10^{-8}$
$-342.15$	Dash-dot	356	$8.8 \times 10^{-9}$
$-296.66$	Solid	386	$3.8 \times 10^{-9}$

## 7. Conclusions

We have presented basic properties for some Arnoldi and Lanczos type processes for a monic matrix polynomial with large and sparse coefficient matrices. These processes operate on the same space as the matrix polynomial live and the reduced problem are a matrix polynomial itself. This has the advantage of preserving certain properties of the original system in the reduced systems and the process could hold the key on practical applications where such a feature is a necessity.

What we have presented here are basic ideas. Robust implementations of these new algorithms will require carefully dealing with many (subtle) technical details. Bearing similarity in nature to the standard Arnoldi and Lanczos processes, these new Arnoldi and Lanczos type processes could incorporate many proven techniques developed over the years for the former. We shall leave these matters to future investigations.

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