

PERTURBATION BOUNDS FOR GENERALIZED EIGENVALUES. I†

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ABSTRACT

For the perturbation of generalized eigenvalues of general regular matrix pairs, it is difficult to estimate perturbation bounds in spite of the existing results. In this paper, some bounds to redeem this defect are developed. To this end, we prove another version of Henrici type theorem, from which we can deduce two upper bounds which could not be deduced from the previous one.

§1. INTRODUCTION

So far, several bounds are available for the perturbation of the generalized eigenvalue problem $Ax = \lambda Bx$ (see Sun[1,3] and other references contained therein). However, it is very difficult to compute some of these bounds concerning general regular matrix pairs. In this paper, our attention is on the derivation of perturbation bounds which may be computed easily. We begin with definitions of some numbers related closely to the sensitivity of generalized eigenvalues of the problem $Ax = \lambda Bx$, and then develop our bounds for the perturbation of general regular matrix pairs.

We use the following notation. $\mathbb{C}^{m \times n}$ is for the set of m by n complex matrices, $\mathbb{C}^m = \mathbb{C}^{m \times 1}$, $\mathbb{C} = \mathbb{C}^1$; and \mathbb{U}_n for the set of $n \times n$ unitary matrices. For $A \in \mathbb{C}^{m \times n}$, A^T and A^H denote the transpose, conjugate transpose of A , respectively.

DEFINITION 1.1. Let $A, B \in \mathbb{C}^{n \times n}$. Matrix pair $\{A, B\}$ is called a regular matrix pair of order n , if

$$\det(A + \lambda B) \neq 0, \lambda \in \mathbb{C}. \quad (1.1)$$

Denote by $\mathcal{R}(n)$ the set of regular matrix pairs of order n , and

$$\mathbb{G}_{1,2} = \{(\alpha, \beta) \neq (0, 0) : \alpha, \beta \in \mathbb{C}\}. \quad (1.2)$$

$(\alpha, \beta) \in \mathbb{G}_{1,2}$ is a generalized eigenvalue, if

$$\det(\beta A - \alpha B) = 0. \quad (1.3)$$

the set of the generalized eigenvalues of $\{A, B\}$ is called the spectrum of $\{A, B\}$ denoted by $\lambda(A, B)$, i.e.,

$$\lambda(A, B) = \{(\alpha, \beta) \in \mathbb{G}_{1,2} : \det(\beta A - \alpha B) = 0\}. \quad (1.4)$$

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THEOREM 1.1(STEWART[7]). Let $\{A, B\} \in \mathcal{R}(n)$. Then there exist $U, V \in \mathbb{U}_n$, such that

$$\begin{aligned} U^H A V &= \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \equiv T_A \equiv \Lambda_A + M_A \\ U^H B V &= \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ & & \beta_n \end{pmatrix} \equiv T_B \equiv \Omega_B + M_B \end{aligned}, \quad (1.5)$$

where $(\alpha_i, \beta_i) \in \lambda(A, B)$ ($1 \leq i \leq n$), $\Lambda_A = \text{diag}(\alpha_1, \dots, \alpha_n)$ and $\Omega_B = \text{diag}(\beta_1, \dots, \beta_n)$.

It follows from Theorem 1.1 that

COROLLARY 1.1. Let $\{A, B\} \in \mathcal{R}(n)$. Then there exist a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{U}_n$ such that

$$A = P \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ & & \alpha_n \end{pmatrix} V^H, \quad B = P \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ & & \beta_n \end{pmatrix} V^H, \quad (1.6)$$

where $|\alpha_i|^2 + |\beta_i|^2 = 1$, $i = 1, \dots, n$. We call $\{A, B\}$ a normal matrix pair of order n , if the two upper triangular matrices appearing in (1.6) become two diagonal matrices.

The derivation of Henrici type theorem in Elsner & Sun[2] and Theorem 1.3 in Sun[3, Chapter 4] are on the base of decomposition (1.6) duo to using *the Chordal metric* on Riemann sphere

$$\rho((\alpha, \beta), (\gamma, \delta)) = \frac{|\delta\alpha - \beta\gamma|}{\sqrt{|\alpha|^2 + |\beta|^2} \sqrt{|\gamma|^2 + |\delta|^2}}. \quad (1.7)$$

Therefore, perturbation bounds derived in such a way contain implicitly or even explicitly the matrix P , and thus become difficult to compute out.

In passing (1.5) to (1.6), it is natural to premultiply both sides of (1.5) by

$$\text{diag} \left(\frac{1}{\sqrt{|\alpha_1|^2 + |\beta_1|^2}}, \dots, \frac{1}{\sqrt{|\alpha_n|^2 + |\beta_n|^2}} \right). \quad (1.8)$$

this is feasible, since $(\alpha_i, \beta_i) \neq (0, 0)$ ($1 \leq i \leq n$). Hence P is closely related to $\sqrt{|\alpha_i|^2 + |\beta_i|^2}$ ($1 \leq i \leq n$). In §2 we will define several numbers to reflect this relation and study them. These numbers are proved to be useful in §3 and §4 below for drawing perturbation bounds.

§2. NUMBERS

Let $\{A, B\} \in \mathcal{R}(n)$. We denote by $\mathcal{M}_{A,B}$ the set of unitary matrix pairs $\{U, V\}$ appearing in decomposition (1.5), and by

$$\rho(A, B) = \{(\alpha, \beta) \in \mathbb{G}_{1,2} : \det(\beta A - \alpha B) \neq 0, |\alpha|^2 + |\beta|^2 = 1\}, \quad (2.1)$$

the set of the normalized regular points of $\{A, B\}$. Set

$$\sigma(A, B) = \sup_{\{U, V\} \in \mathcal{M}_{A,B}} \min_{1 \leq i \leq n} \sqrt{|\alpha_i|^2 + |\beta_i|^2}. \quad (2.2)$$

It is easy to see that

$$\sigma(A, B) \leq \|Z\|_2, \tag{2.3}$$

where $Z = (A, B) \in \mathbb{C}^{n \times 2n}$ and $\|\cdot\|_2$ is the spectral norm for matrices.

To give a lower bound of $\sigma(A, B)$ and to develop Henrici type theorem, we define also

$$\underline{\sigma}(A, B) = \inf_{\{U, V\} \in \mathcal{M}_{A, B}} \min_{1 \leq i \leq n} \sqrt{|\alpha_i|^2 + |\beta_i|^2}, \tag{2.4}$$

$$\omega(A, B) = \inf_{(\alpha, \beta) \in \rho(A, B)} \|(\beta A - \alpha B)^{-1}\|_2. \tag{2.5}$$

THEOREM 2.1. Suppose $\{A, B\} \in \mathcal{R}(n)$, the numbers defined above satisfy

$$\sigma(A, B) \geq \underline{\sigma}(A, B) \geq [\omega(A, B)]^{-1}. \tag{2.6}$$

PROOF: That the first \geq in (2.6) holds is obvious. We prove the second one. From (1.5) and unitary invariance of $\|\cdot\|_2$, we have for any $(\alpha, \beta) \in \rho(A, B)$

$$\|(\beta A - \alpha B)^{-1}\|_2 = \|(\beta T_A - \alpha T_B)^{-1}\|_2.$$

Since $(\beta\alpha_i - \beta_i\alpha)^{-1}$ ($i = 1, \dots, n$) are the eigenvalues of $(\beta T_A - \alpha T_B)^{-1}$, so

$$\|(\beta A - \alpha B)^{-1}\|_2 \geq \max_{1 \leq i \leq n} \frac{1}{|\beta\alpha_i - \beta_i\alpha|},$$

therefore

$$\begin{aligned} \|(\beta A - \alpha B)^{-1}\|_2^{-1} &\leq \min_{1 \leq i \leq n} |\beta\alpha_i - \beta_i\alpha| \\ &\leq \min_{1 \leq i \leq n} \sqrt{|\alpha|^2 + |\beta|^2} \sqrt{|\alpha_i|^2 + |\beta_i|^2} \\ &= \min_{1 \leq i \leq n} \sqrt{|\alpha_i|^2 + |\beta_i|^2}. \end{aligned}$$

Since (α, β) is arbitrary, we have

$$\left[\inf_{(\alpha, \beta) \in \rho(A, B)} \|(\beta A - \alpha B)^{-1}\|_2 \right]^{-1} \leq \min_{1 \leq i \leq n} \sqrt{|\alpha_i|^2 + |\beta_i|^2}.$$

This inequality holds for all decompositions of form (1.5), thus leads to (2.6). **Q.E.D.**

THEOREM 2.2. Let $\{A, B\} \in \mathcal{R}(n)$. There exists a decomposition of form (1.6) such that nonsingular matrix $P \in \mathbb{C}^{n \times n}$ satisfies

$$|\det P| \geq [\sigma(A, B)]^n \geq [\underline{\sigma}(A, B)]^n \geq [\omega(A, B)]^{-n}, \tag{2.7}$$

$$\begin{aligned} |\det P| &\geq \sigma_{\min}(Z) [\sigma(A, B)]^{n-1} \geq \sigma_{\min}(Z) [\underline{\sigma}(A, B)]^{n-1} \\ &\geq \sigma_{\min}(Z) [\omega(A, B)]^{-n+1}, \end{aligned} \tag{2.8}$$

where $Z = (A, B)$, $\sigma_{\min}(Z)$ stands for the smallest singular value of Z .

PROOF: It is easy to verify that $\mathcal{M}_{A, B}$ is a bounded closed set in $\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$, hence the supremum value in (2.2) can be attained for some $\{U, V\}$, which together with (1.8) and (2.5) give (2.7). To prove (2.8), we note that from decomposition (1.5),

$$\begin{aligned} |\alpha_n|^2 + |\beta_n|^2 &\geq \sigma_{\min}(ZZ^H) \\ \Rightarrow \sqrt{|\alpha_n|^2 + |\beta_n|^2} &\geq \sigma_{\min}(Z), \end{aligned}$$

the consequence of which and (1.8) and (2.6) is (2.8). **Q.E.D.**

In Bauer-Fike type theorems (see [2], [8]), the condition numbers of strictly equivalent transformation matrices are employed. In the following, we derive some relations among the above defined numbers and these condition numbers.

DEFINITION 2.1. $\{A, B\} \in \mathcal{R}(n)$ is called *diagonalizable*, or *normalizable*, if there exist invertible matrices $P, Q \in \mathbb{C}^{n \times n}$ such that

$$\begin{cases} A = P\Lambda Q \equiv P \operatorname{diag}(\alpha_1, \dots, \alpha_n)Q \\ B = P\Omega Q \equiv P \operatorname{diag}(\beta_1, \dots, \beta_n)Q \end{cases} \quad (2.9)$$

THEOREM 2.3. Suppose that $\{A, B\} \in \mathcal{R}(n)$ is diagonalizable, and admits decomposition (2.9). Set $Z = (A, B)$, $\kappa_2(Q) = \|Q\|_2 \|Q^{-1}\|_2$ and

$$s(A, B) \triangleq \sup_{(\alpha, \beta) \in \mathbb{G}_{1,2}} \min_{(\gamma, \delta) \in \lambda(A, B)} \rho((\alpha, \beta), (\gamma, \delta)). \quad (2.10)$$

Then

$$\omega(A, B) \leq \kappa_2(Q) \|Z^\dagger\|_2 / s(A, B), \quad (2.11)$$

where Z^\dagger is the Moore-Penrose inverse of Z , i.e.,

$$Z^\dagger = Z^H (ZZ^H)^{-1}. \quad (2.12)$$

PROOF: For any $(\alpha, \beta) \in \rho(A, B)$, we have

$$\begin{aligned} \|(\beta A - \alpha B)^{-1}\|_2 &= \|Q^{-1}(\beta\Lambda - \alpha\Omega)^{-1}P^{-1}ZZ^\dagger\|_2 \\ &= \left\| Q^{-1}(\beta\Lambda - \alpha\Omega)^{-1}(\Lambda, \Omega) \begin{pmatrix} Q & \\ & Q \end{pmatrix} Z^\dagger \right\|_2 \\ &\leq \kappa_2(Q) \|Z^\dagger\|_2 \|(\beta\Lambda - \alpha\Omega)^{-1}(\Lambda, \Omega)\|_2. \end{aligned}$$

Because (see Li[8, Lemma 2.1])

$$\|(\beta\Lambda - \alpha\Omega)^{-1}(\Lambda, \Omega)\|_2 = \max_{1 \leq i \leq n} \frac{\sqrt{|\alpha_i|^2 + |\beta_i|^2}}{|\beta\alpha_i - \alpha\beta_i|}, \quad (2.13)$$

and (α, β) is arbitrary, we have (2.11). **Q.E.D.**

THEOREM 2.4. Suppose $\{A, B\} \in \mathcal{R}(n)$, and

$$A = PJ_A Q, \quad B = PJ_B Q, \quad (2.14)$$

where $\{J_A, J_B\}$ is the the Weierstrass-Jordan canonical form (see [5], [3] or [8]). Then

$$\omega(A, B) \leq \sqrt{2}\kappa_2(Q) \|Z^\dagger\|_2 \sum_{i=1}^{\ell} \frac{1}{[s(A, B)]^i}, \quad (2.15)$$

where ℓ is the maximum dimension of the Weierstrass-Jordan block pairs.

The proof of Theorem 2.4 is similar to that of Theorem 2.3. Instead of (2.13), at present time Lemma 2.2 in [8] must be used in the proof.

Remark 2.1. In Theorems 2.3–2.4, $s(A, B)$ appears. It depends only on the distribution of the generalized eigenvalues on Riemann sphere. Generally,

$$0 < s(A, B) \leq 1 \text{ and } s(A, B) = 1 \Leftrightarrow \lambda(A, B) \text{ contains only one element.} \quad (2.16)$$

Obviously, computing $\omega(A, B)$ accurately is very difficult like the accurate computation of $\sigma(A, B)$ except in few special cases. However, giving a rough estimate of it may be possible since the elements of $\lambda(A, B)$ is finite (at most n), and finding a point $(\alpha_0, \beta_0) \in \rho(A, B)$ is likely easy. Once such a point is available, an estimate of $\omega(A, B)$ can be made by Theorem 2.5 below. Theorems 2.3 and 2.4 shed light on the choice of (α_0, β_0) as the one for which the supremum value in (2.10) is attained. We can not expect that (α_0, β_0) chosen in this way is best (in fact it may be far from the best one and even very bad).

THEOREM 2.5. Let $\{A, B\} \in \mathcal{R}(n)$, $(\alpha_0, \beta_0) \neq (0, 0)$, $|\alpha_0|^2 + |\beta_0|^2 = 1$ and $\det(\beta_0 A - \alpha_0 B) \neq 0$. Then

$$\omega(A, B) \leq \|(\beta_0 A - \alpha_0 B)^{-1}\|_2. \tag{2.17}$$

§3. UPPER BOUNDS FOR $S_{\{A,B\}}\{C, D\}$ I

DEFINITION 3.1. Let $\{A, B\} \in \mathcal{R}(n)$. We call (see (1.5))

$$\Delta_\nu(A, B) = \inf_{\{U,V\} \in \mathcal{M}_{A,B}} \nu(M_A, M_B) \tag{3.1}$$

the ν -departure from normality of $\{A, B\}$.

It follows from the definition of a normal pair (refer Corollary 1.1) that if in (1.5) $M_A = M_B = 0$ then $\{A, B\}$ must be a normal pair, whereas that $\{A, B\}$ is a normal pair doesn't guarantee a decomposition of form (1.5) with $M_A = M_B = 0$. In view of this, Definition 3.1 has its own drawback. Nevertheless, we argue that the drawback gives no negative effect to our estimates for perturbation bounds. As a matter of fact, Theorems 3.2-3.3 below shows that Definition 3.1 is more convenient than a definition of departure from normality for matrix pairs used and introduced by Elsner & Sun[2] in the study of perturbations for matrix pairs.

Besides (3.1), we also introduce

$$\bar{\Delta}_\nu(A, B) = \sup_{\{U,V\} \in \mathcal{M}_{A,B}} \nu(M_A, M_B). \tag{3.2}$$

LEMMA 3.1. Let $R = D - M \in \mathbb{C}^{n \times n}$ be invertible, where $D = \text{diag}(\delta_1, \dots, \delta_n)$, M is strictly upper triangular. If

$$m = \|M\|_2 \neq 0, \|R^{-1}\|_2 \leq \epsilon, \tag{3.3}$$

then

$$\min_{1 \leq i \leq n} |\delta_i| \leq m / g\left(\frac{m}{\epsilon}\right), \tag{3.4}$$

where $g(\eta)$ is the unique solution of equation $g + g^2 + \dots + g^n = \eta$ ($\eta > 0$) (see Sun[3, pp.156-157]).

THEOREM 3.1 (HENRICI TYPE). Suppose $\{A, B\}, \{C, D\} \in \mathcal{R}(n)$, and $Z(P_{Z^H} - P_{W^H}) \neq 0$, where $Z = (A, B)$, $W = (C, D)$, $P_{Z^H} = Z^\dagger Z$ and $P_{W^H} = W^\dagger W$. If ν is any norm majorizing the spectral norm, i.e., $\nu(\cdot) \geq \|\cdot\|_2$, and if

$$\eta = \frac{\tilde{\Delta}_\nu(A, B)}{\nu(Z(P_{Z^H} - P_{W^H}))},$$

then

$$S_{\{A,B\}}\{C, D\} \leq \frac{1}{\tilde{\sigma}(A, B)} \frac{\eta}{g(\eta)} \nu(Z(P_{Z^H} - P_{W^H})), \tag{3.5}$$

where $g(\eta)$ as described in Lemma 3.1 and

$$\begin{cases} \tilde{\sigma}(A, B) = \sigma(A, B) \\ \tilde{\Delta}_\nu(A, B) = \bar{\Delta}_\nu(A, B) \end{cases} \quad \text{or} \quad \begin{cases} \tilde{\sigma}(A, B) = \underline{\sigma}(A, B) \\ \tilde{\Delta}_\nu(A, B) = \Delta_\nu(A, B) \end{cases}, \tag{3.6}$$

$$S_{\{A,B\}}\{C, D\} \triangleq \max_{(\gamma, \delta) \in \lambda(C, D)} \min_{(\alpha, \beta) \in \lambda(A, B)} \rho((\alpha, \beta), (\gamma, \delta)). \tag{3.7}$$

If, in addition, $\tilde{\Delta}_\nu(A, B) = 0$, we assume the right-hand side of (3.5) is equal to

$$\frac{\nu(Z(P_{Z^H} - P_{W^H}))}{\tilde{\sigma}(A, B)},$$

which coincides with the limit of the right side of (3.5) as $\tilde{\Delta}_\nu(A, B) \rightarrow 0$.

PROOF: (3.6) gives two possible way for assigning the values of $\tilde{\sigma}(A, B)$ and $\tilde{\Delta}_\nu(A, B)$. In the following, we prove (3.5) only when they are assigned by the second pair of equations in (3.6) and we omit the proof for the other case since the proof is similar.

We have noted that $\mathcal{M}_{A, B}$ is a bounded closed set. Hence there exists a decomposition of form (1.5) with $\|(M_A, M_B)\|_2 = \Delta_2(A, B)$. For any $(\gamma, \delta) \in \lambda(C, D)$, assume first that $(\gamma, \delta) \notin \lambda(A, B)$ and $|\gamma|^2 + |\delta|^2 = 1$. Set

$$\begin{aligned} R &= \delta(\Lambda_A + M_A) - \gamma(\Omega_B + M_B) \\ &= (\delta\Lambda_A - \gamma\Omega_B) + (\delta M_A - \gamma M_B), \end{aligned}$$

thus

$$\|R^{-1}\|_2 = \|(URV^H)^{-1}\|_2 = \|(\delta A - \gamma B)^{-1}\|_2^{-1} \leq \|\delta Ax - \gamma Bx\|_2,$$

where x is a generalized eigenvector of $\{C, D\}$ corresponding to (γ, δ) with $\|x\|_2 = 1$, i.e., $\delta Cx = \gamma Dx$. Note

$$\delta Ax - \gamma Bx = Z(P_{Z^H} - P_{W^H}) \begin{pmatrix} \delta x \\ -\gamma x \end{pmatrix}, \quad (3.8)$$

from which follows

$$\|R^{-1}\|_2^{-1} \leq \|Z(P_{Z^H} - P_{W^H})\|_2. \quad (3.9)$$

On the other hand, the strictly upper triangular part of R satisfies

$$\|(\delta M_A - \gamma M_B)\|_2 \leq \|(M_A, M_B)\|_2 = \Delta_2(A, B). \quad (3.10)$$

Now, if $\Delta_2(A, B) = 0$ (hence $\tilde{\Delta}_\nu(A, B) = 0$), then (3.5) is a straightforward consequence of (3.9). Assume that $\Delta_\nu(A, B) > 0$, then from Lemma 3.1, (3.9) and (3.10), we have

$$\min_{1 \leq i \leq n} |\delta\alpha_i - \gamma\beta_i| \leq \frac{\Delta_2(A, B)}{g\left(\frac{\Delta_2(A, B)}{\|Z(P_{Z^H} - P_{W^H})\|_2}\right)}. \quad (3.11)$$

This inequality is valid even for $(\gamma, \delta) \in \lambda(A, B)$, hence (3.11) holds for any $(\gamma, \delta) \in \lambda(C, D)$. So (3.11) and (2.4) lead to

$$S_{\{A, B\}}\{C, D\} \leq \frac{1}{\underline{\sigma}(A, B)} \frac{\Delta_2(A, B)}{g\left(\frac{\Delta_2(A, B)}{\|Z(P_{Z^H} - P_{W^H})\|_2}\right)}.$$

Now from the monotoneity of $g(\eta)$ as a function of $\eta > 0$, we have for any norm ν majorizing $\|\cdot\|_2$

$$\frac{\Delta_2(A, B)}{g\left(\frac{\Delta_2(A, B)}{\|Z(P_{Z^H} - P_{W^H})\|_2}\right)} \leq \frac{\eta_1}{g(\eta_1)} \nu(Z(P_{Z^H} - P_{W^H})),$$

where $\eta_1 = \Delta_2(A, B) / \nu(Z(P_{Z^H} - P_{W^H}))$. Set $\eta = \Delta_\nu(A, B) / \nu(Z(P_{Z^H} - P_{W^H}))$, for which $0 < \eta_1 \leq \eta$, therefore by the monotoneity of $\eta/g(\eta) = 1 + g(\eta) + \dots + [g(\eta)]^{n-1}$ and the continuity of $g(\eta)$, we conclude (3.5) for the case $\tilde{\sigma}(A, B)$ and $\tilde{\Delta}_\nu(A, B)$ taking the second value pair of (3.6).

Q.E.D.

Remark 3.1. The appearance of $\underline{\sigma}(A, B)$ and $\tilde{\Delta}_\nu(A, B)$ in (3.5) make the conclusion of Theorem 3.1 less satisfying. In many cases, this doesn't lessen the role of Theorem 3.1. In fact, the estimations of a lower bound for $\Delta_\nu(A, B)$ and an upper bound for $\Delta_\nu(A, B)$ are often done for any decomposition of form (1.5), therefore the obtained bounds is applicable to $\underline{\sigma}(A, B)$ and $\tilde{\Delta}_\nu(A, B)$ just like Theorem 2.1.

The following corollary is a consequence of (3.8) in the above proof.

COROLLARY 3.1. Notation is as described in Theorem 3.1. If $Z(P_{Z^H} - P_{W^H}) = 0$, then the spectrum of $\{A, B\}$ is the same as that of $\{C, D\}$, i.e., $\lambda(A, B) = \lambda(C, D)$.

From (see Henrici[4] or Sun[3, pp.155-156])

$$\frac{\eta}{g(\eta)} \leq \begin{cases} n, & \text{if } 0 \leq \eta \leq n, \\ n^{\frac{1}{n}} \eta^{1-\frac{1}{n}}, & \text{if } \eta > n, \end{cases} \quad (3.12)$$

we deduce from Theorem 3.1 the following corollary.

COROLLARY 3.2. The condition is as described in Theorem 3.1, then

$$S_{\{A,B\}}\{C, D\} \leq \frac{1}{\bar{\sigma}(A, B)} \max\{n\nu(Z(P_{Z^H} - P_{W^H})), n^{\frac{1}{n}}[\bar{\Delta}_\nu(A, B)]^{1-\frac{1}{n}}[\nu(Z(P_{Z^H} - P_{W^H}))]^{\frac{1}{n}}\}. \quad (3.13)$$

Now we are in the position to deduce two more concise perturbation bounds. Let

$$S(\Delta, r) = \begin{cases} \frac{\eta}{g(\eta)}, & \eta = \frac{\Delta}{r} \quad \text{for } r > 0, \\ 0 & \text{for } r = 0. \end{cases}$$

LEMMA 3.2(ELSNER[6]). $S(\Delta, r)$ has the following property

- (i) $S(\Delta, r)$ is strictly monotone in Δ and r ,
- (ii) $r^{-1/n}S(\Delta, r)$ is strictly monotone in Δ and r .

LEMMA 3.3(ELSNER[6]). For given real $\tau \geq 0$, $\delta > 0$ and positive integer n define

$$\xi = (\delta^{n-1} + \delta^{n-2}\tau + \dots + \tau^{n-1})^{1/n}. \quad (3.14)$$

Then ξ is the minimal number such that

$$\min\{S(\tau M, r), \delta M\} \leq \xi M^{1-\frac{1}{n}} r^{\frac{1}{n}} \quad (3.15)$$

for all $M \geq 0$, $r \geq 0$.

THEOREM 3.2. Suppose $\{A, B\}, \{C, D\} \in \mathcal{R}(n)$, $Z = (A, B)$ and $W = (C, D)$. Then

$$S_{\{A,B\}}\{C, D\} \leq \frac{\|Z\|_2}{\sigma(A, B)} c_1\left(n, \frac{\sigma(A, B)}{\|Z\|_2}\right) \|P_{Z^H} - P_{W^H}\|_2^{\frac{1}{n}}, \quad (3.16)$$

where $c_1(n, x)$ is defined by

$$c_1(n, x) = \left(\sum_{i=0}^{n-1} x^i 2^{n-1-i}\right)^{\frac{1}{n}}. \quad (3.17)$$

Hence using (2.3), we get

$$c_1\left(n, \frac{\sigma(A, B)}{\|Z\|_2}\right) \leq (2^n - 1)^{\frac{1}{n}}. \quad (3.18)$$

PROOF: It follows from Theorem 3.1 that

$$S_{\{A,B\}}\{C, D\} \leq \frac{1}{\sigma(A, B)} S(\bar{\Delta}_2(A, B), \|Z\|_2 r_2),$$

where $r_2 = \|P_{Z^H} - P_{W^H}\|_2$. Note that for any decomposition (1.5), we have

$$\begin{aligned} \|(M_A, M_B)\|_2 &= \left\| U^H(A, B) \begin{pmatrix} V & \\ & V \end{pmatrix} - (\Lambda_A, \Omega_B) \right\|_2 \\ &\leq \|Z\|_2 + \|(\Lambda_A, \Omega_B)\|_2 \leq 2\|Z\|_2 \\ &\Rightarrow \bar{\Delta}_2(A, B) \leq 2\|Z\|_2. \end{aligned}$$

Therefore together with $S_{\{A, B\}}\{C, D\} \leq 1$ and by Lemmas 3.2–3.3 we have

$$\begin{aligned} S_{\{A, B\}}\{C, D\} &\leq \frac{1}{\sigma(A, B)} \min\{S(2\|Z\|_2, \|Z\|_2 r_2), \sigma(A, B)\} \\ &= \frac{\|Z\|_2}{\sigma(A, B)} \min\{S(2, r_2), \frac{\sigma(A, B)}{\|Z\|_2}\} \\ &\leq \frac{\|Z\|_2}{\sigma(A, B)} c_1 \left(n, \frac{\sigma(A, B)}{\|Z\|_2} \right) \|P_{Z^H} - P_{W^H}\|_2^{\frac{1}{2}}. \end{aligned}$$

Q.E.D.

THEOREM 3.3. *The condition and notation are as described in Theorem 3.2. Then*

$$S_{\{A, B\}}\{C, D\} \leq \frac{\|Z\|_F}{\sigma(A, B)} c_2 \left(n, \frac{\sigma(A, B)}{\|Z\|_F} \right) \|P_{Z^H} - P_{W^H}\|_F^{\frac{1}{n}}, \quad (3.19)$$

where $\|\cdot\|_F$ stands for Frobenius norm of a matrix, and $c_2(n, x)$ is defined by

$$c_2(n, x) = \left(\sum_{i=0}^{n-1} x^i \right)^{\frac{1}{n}}. \quad (3.20)$$

Again using (2.3), $\sigma(A, B) \leq \|Z\|_2 \leq \|Z\|_F$, we get

$$c_2 \left(n, \frac{\sigma(A, B)}{\|Z\|_F} \right) \leq n^{\frac{1}{n}}. \quad (3.21)$$

PROOF: It follows from Theorem 3.1 and $\|\cdot\|_F \geq \|\cdot\|_2$ that

$$S_{\{A, B\}}\{C, D\} \leq \frac{1}{\sigma(A, B)} S(\bar{\Delta}_F(A, B), \|Z\|_F r_F),$$

where $r_F = \|P_{Z^H} - P_{W^H}\|_F$. Note that for any decomposition (1.5), we have

$$\begin{aligned} \|(M_A, M_B)\|_F^2 &= \|Z\|_F^2 - \|(\Lambda_A, \Omega_B)\|_F^2 < \|Z\|_F^2 \\ &\Rightarrow \bar{\Delta}_F(A, B) \leq \|Z\|_F. \end{aligned}$$

Therefore together with $S_{\{A, B\}}\{C, D\} \leq 1$ and by Lemmas 3.2–3.3 show

$$\begin{aligned} S_{\{A, B\}}\{C, D\} &\leq \frac{1}{\sigma(A, B)} \min\{S(\|Z\|_F, \|Z\|_F r_F), \sigma(A, B)\} \\ &= \frac{\|Z\|_F}{\sigma(A, B)} \min\{S(1, r_F), \frac{\sigma(A, B)}{\|Z\|_F}\} \\ &\leq \frac{\|Z\|_F}{\sigma(A, B)} c_2 \left(n, \frac{\sigma(A, B)}{\|Z\|_F} \right) \|P_{Z^H} - P_{W^H}\|_F^{\frac{1}{n}}. \end{aligned}$$

Q.E.D.

Now, we present an example to illustrate the applications of Theorems 3.2–3.3 to give perturbation bounds.

Example 3.1. $n = 2$. Denote $i = \sqrt{-1}$, $Z = (A, B)$, $W = (C, D)$,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 10^{-4} & 1 \end{pmatrix}.$$

It is easy to compute out

$$\lambda(A, B) = \{(1, 1), (1, 1)\}, \lambda(C, D) = \{(1, 1 - \frac{1}{2} \times 10^{-4} \pm i \frac{1}{2} \cdot 10^{-2} \sqrt{4 - 10^{-4}})\},$$

$$S_{\{A, B\}}\{C, D\} = 0.005$$

$$\|Z\|_2 = \sqrt{1 + \frac{3 + \sqrt{5}}{2}}, \sigma_{\min}(Z) = \sqrt{1 + \frac{3 - \sqrt{5}}{2}}, \sigma(A, B) \geq \sqrt{2},$$

$$\omega(A, B) \leq \left\| \left(\frac{1}{\sqrt{2}}A + \frac{1}{\sqrt{2}}B \right)^{-1} \right\|_2 = \frac{1}{4} \sqrt{9 + \sqrt{17}}, \|Z\|_F = \sqrt{5}.$$

Utilizing

$$\|P_{Z^H} - P_{W^H}\|_d \leq \|Z - W\|_d / \sigma_{\min}(Z) \tag{3.22}$$

(see Sun[3, Chapter 4, Theorem 4.6]) to estimate $\|P_{Z^H} - P_{W^H}\|_d$ ($d = 2, F$), and (3.18) and (3.21) to estimate $c_1(\cdot, \cdot)$, $c_2(\cdot, \cdot)$, we have

$$S_{\{A, B\}}\{C, D\} \leq 0.021486 \quad (\text{by Theorem 3.2}),$$

$$S_{\{A, B\}}\{C, D\} \leq 0.020623 \quad (\text{by Theorem 3.3}).$$

If, in addition, $\sigma(A, B)$ is estimated by $[\omega(A, B)]^{-1}$ (refer Theorem 2.1), the computed bounds are as follows

$$S_{\{A, B\}}\{C, D\} \leq 0.033552 \quad (\text{by Theorem 3.2}),$$

$$S_{\{A, B\}}\{C, D\} \leq 0.032205 \quad (\text{by Theorem 3.3}).$$

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