

PERTURBATION BOUNDS FOR GENERALIZED EIGENVALUES. II†

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ABSTRACT

As a continuation of Li[4], we still study the perturbation bounds for generalized eigenvalues of general regular matrix pairs, but with a different method. Through estimating the perturbation bounds of the coefficients of the generalized eigenpolynomials, we obtain two perturbation theorems for the generalized eigenvalue problem $Ax = \lambda Bx$.

§4. UPPER BOUNDS FOR $S_{\{A,B\}}\{C, D\}$ II

Notation in Li[4] is followed throughout this part.

In this section, we study perturbation bounds for generalized eigenvalues through the investigation of the perturbation of generalized eigenpolynomials.

We define first a sequence of operators

$$T_k : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \mapsto \mathbb{C}, \quad k = 0, 1, \dots, n \quad (4.1)$$

as: let $\{A, B\} \in \mathcal{R}(n)$, $T_k(A, B)$ is the sum of $\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!}$ determinants each of which is constructed by replacing some k columns of B by the corresponding columns of A .

PROPERTY 4.1.

$$T_0(A, B) = \det B, \quad T_n(A, B) = \det A. \quad (4.2)$$

DEFINITION 4.1. Let $(\alpha, \beta) \in \mathbb{G}_{1,2}$. We call

$$f(\alpha, \beta) = \det(\beta A - \alpha B) \quad (4.3)$$

the generalized eigenpolynomial of $\{A, B\} \in \mathcal{R}(n)$.

By the properties of determinant, one can verify easily that

THEOREM 4.1.

$$f(\alpha, \beta) = \sum_{i=0}^n \alpha^i \beta^{n-i} (-1)^i T_{n-i}(A, B). \quad (4.4)$$

PROPERTY 4.2 ("SYMMETRY").

$$T_k(A, B) = T_{n-k}(B, A). \quad (4.5)$$

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PROPERTY 4.3.

$$T_k(A, B) = \frac{1}{k!} \frac{d^k}{dt^k} \det(tA + B) \Big|_{t=0}. \quad (4.6)$$

We adopt the following compact notation. For $1 \leq k \leq n$, let $\Omega_{k,n}$ denote the collection of strictly increasing sequences of k integers chosen from $1, 2, \dots, n$. In other words,

$$\Omega_{k,n} \equiv \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}.$$

Set $\alpha_I = (1, 2, \dots, n)$. For $\alpha \in \Omega_{k,n}$, α_I/α is for the complementary of α in α_I , i.e., $\alpha_I/\alpha \in \Omega_{n-k,n}$ consists of a strictly increasing sequence of $n-k$ integers which doesn't appear in α . We order $\Omega_{k,n}$ by the usual lexicographic ordering. We shall denote by $A[\alpha|\beta]$ or by $A[(\alpha_1, \dots, \alpha_k)|(\beta_1, \dots, \beta_k)]$ the submatrix of A constructed from the rows $\alpha_1, \dots, \alpha_k$ and the columns β_1, \dots, β_k , i.e., this is the submatrix whose ij th entry is $a_{\alpha_i \beta_j}$, where $A = (a_{ij})_{n \times n}$.

Remark 4.1. Denote by Λ^k the operator

$$\Lambda^k : A \mapsto \Lambda^k A, \quad \text{for } A \in \mathbb{C}^{n \times n}, \quad (4.7)$$

where $\Lambda^k A$ is the k th compound of A , i.e.,

$$\Lambda^k A = (\det A[\alpha|\beta])_{\binom{n}{k} \times \binom{n}{k}}.$$

We see if $B = I$, then

$$T_k(A, B) = \text{tr } \Lambda^k A,$$

here and later tr stands for trace. Therefore, we can symbolically denote

$$T_k(A, B) \triangleq \text{tr}_B \Lambda^k A \triangleq \text{tr}_A \Lambda^{n-k} B. \quad (4.8)$$

For the case when A or B is diagonal, (4.8) is natural and thus reasonable, because now tr_A or tr_B can be regarded a weighted trace of matrices. (see Property 4.4 below)

PROPERTY 4.4. *Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$, then*

$$\text{tr}_A \Lambda^{n-k} B = \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \lambda_{\alpha_i} \det B[\alpha_I/\alpha|\alpha_I/\alpha]. \quad (4.9)$$

Hence tr_A is an operator mapping $\binom{n}{n-k}^2$ -dimensional linear space to \mathbb{C} : For $C \in \mathbb{C}^{\binom{n}{n-k} \times \binom{n}{n-k}}$ and $C = (c_{\alpha\beta})$, $\alpha, \beta \in \Omega_{n-k,n}$,

$$\text{tr}_A C \triangleq \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \lambda_{\alpha_i} c_{\alpha_I/\alpha \alpha_I/\alpha}. \quad (4.10)$$

Moreover, we have

$$\begin{aligned} \left\| (\text{tr}_A)_{\binom{n}{n-k}} \right\|_2 &= \sup_{\substack{\|C\|_2=1 \\ C \in \mathbb{C}^{\binom{n}{n-k} \times \binom{n}{n-k}}}} |\text{tr}_A C| \\ &= \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k |\lambda_{\alpha_i}|. \end{aligned} \quad (4.11)$$

PROOF: By the properties of determinant, (4.9) is obvious. Now, we prove (4.11). It follows from $\|C\|_2 = 1$ that

$$|c_{\alpha_I/\alpha} \alpha_I/\alpha| \leq 1, \alpha \in \Omega_{k,n},$$

which yields

$$\left\| (\text{tr } A)_{\binom{n}{n-k}} \right\|_2 \leq \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k |\lambda_{\alpha_i}|.$$

On the other hand, for $C = \text{diag} \{ \text{sign} (\prod_{i=1}^k \lambda_{n-k+i}), \dots, \text{sign} (\prod_{i=1}^k \lambda_i) \}$, we have

$$\begin{aligned} |\text{tr } {}_A C| &= \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k |\lambda_{\alpha_i}| \\ \Rightarrow \left\| (\text{tr } A)_{\binom{n}{n-k}} \right\|_2 &\geq \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k |\lambda_{\alpha_i}|, \end{aligned}$$

where $\text{sign}(\cdot)$ is the sign function. (4.11) is proved. **Q.E.D.**

4.1 Bounds by $\|\cdot\|_2$. In this and the next subsection, the derivative of a map at a point is employed.

Let $A, B, C \in \mathbb{C}^{n \times n}$,

$$(\mathcal{D}\Lambda^k A)(C) = \left. \frac{d}{dt} \Lambda^k(A + tC) \right|_{t=0}, \quad 1 \leq k \leq n, \quad (4.12)$$

$$(\mathcal{D}_A T_k(A, B))(C) = \left. \frac{d}{dt} T_k(A + tC, B) \right|_{t=0}, \quad 0 \leq k \leq n, \quad (4.13)$$

$$(\mathcal{D}_B T_k(A, B))(C) = \left. \frac{d}{dt} T_k(A, B + tC) \right|_{t=0}, \quad 0 \leq k \leq n \quad (4.14)$$

are respectively the *directional*, the *partial directional derivatives* of operators Λ^k, T_k at points $A, (A, B)$ in the direction of C . We shall find bounds for the operator norm of $\mathcal{D}\Lambda^k A, \mathcal{D}_A T_k(A, B)$ and $\mathcal{D}_B T_k(A, B)$ in the subsection, and for the Frobenius norm in the next subsection:

$$\|\mathcal{D}\Lambda^k A\|_2 \triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_2 = 1}} \|(\mathcal{D}\Lambda^k A)(C)\|_2, \quad 1 \leq k \leq n, \quad (4.15)$$

$$\|\mathcal{D}_A T_k(A, B)\|_2 \triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_2 = 1}} |(\mathcal{D}_A T_k(A, B))(C)|, \quad 0 \leq k \leq n, \quad (4.16)$$

$$\|\mathcal{D}_B T_k(A, B)\|_2 \triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_2 = 1}} |(\mathcal{D}_B T_k(A, B))(C)|, \quad 0 \leq k \leq n. \quad (4.17)$$

LEMMA 4.1 (SVD). Let $A \in \mathbb{C}^{n \times n}$. There are $U, V \in \mathbb{U}_n$ such that

$$A = U \Sigma V \equiv U \text{diag}(\nu_1, \dots, \nu_n) V, \quad (4.18)$$

where $\nu_i = \nu_i(A) \geq 0$ ($1 \leq i \leq n$) are singular values of A .

LEMMA 4.2 (BHATIA-FRIEDLAND[1]). Let $A \in \mathbb{C}^{n \times n}$ with singular values are $\nu_1 \geq \dots \geq \nu_n \geq 0$. Then for $1 \leq k \leq n$,

$$\|\mathcal{D}\Lambda^k A\|_2 = \sum_{i=1}^k \prod_{\substack{j=1 \\ j \neq i}}^k \nu_j. \quad (4.19)$$

In particular, this implies

$$\|\mathcal{D}\Lambda^k A\|_2 \leq k\|A\|_2^{k-1}, \quad 1 \leq k \leq n. \quad (4.20)$$

LEMMA 4.3. Let $A, B \in \mathbb{C}^{n \times n}$. Then

$$\|\mathcal{D}_A T_k(A, B)\|_2 \leq k\|A\|_2^{k-1} \binom{n}{n-k} \|B\|_2^{n-k}, \quad 0 \leq k \leq n, \quad (4.21)$$

$$\|\mathcal{D}_B T_k(A, B)\|_2 \leq (n-k)\|B\|_2^{n-k-1} \binom{n}{k} \|A\|_2^k, \quad 0 \leq k \leq n. \quad (4.22)$$

PROOF: By Lemma 4.1, A has a singular value decomposition (SVD)

$$A = U\Sigma V, \quad U, V \in \mathbb{U}_n,$$

thus from Property 4.3, we have

$$\begin{aligned} T_k(A, B) &= \frac{1}{k!} \det U \det V \left. \frac{d^k}{dt^k} \det(t\Sigma + U^H B V^H) \right|_{t=0} \\ &= \det U \det V \operatorname{tr}_{\Sigma} \Lambda^{n-k} (U^H B V^H). \end{aligned}$$

The chain rule of differentiation produces

$$\begin{aligned} \mathcal{D}_B T_k(A, B) &= \det U \det V \mathcal{D}_B \operatorname{tr}_{\Sigma} \Lambda^{n-k} (U^H B V^H) \\ &= \det U \det V \mathcal{D} \operatorname{tr}_{\Sigma} \mathcal{D}_{U^H B V^H} \Lambda^{n-k} (U^H B V^H) \mathcal{D}_B (U^H B V^H). \end{aligned}$$

So by Property 4.4, Lemma 4.2 and $\|\mathcal{D}_B (U^H B V^H)\|_2 = 1$ (induced norm of operator), we get

$$\begin{aligned} \|\mathcal{D}_B T_k(A, B)\|_2 &= \left\| (\operatorname{tr}_{\Sigma})_{\binom{n}{n-k}} \right\|_2 \|\mathcal{D}_{U^H B V^H} \Lambda^{n-k} (U^H B V^H)\|_2 \|\mathcal{D}_B (U^H B V^H)\|_2 \\ &\leq \binom{n}{k} \|A\|_2^k (n-k) \|U^H B V^H\|_2^{n-k-1} \\ &= (n-k) \|B\|_2^{n-k-1} \binom{n}{k} \|A\|_2^k, \end{aligned}$$

which proves (4.22). Inequality (4.21) is a consequence of Property 4.2 and (4.22). **Q.E.D.**

LEMMA 4.4. Let $A_i, B_i \in \mathbb{C}^{n \times n}$, $i = 1, 2$. Denote by

$$M_{21} = \max\{\|A_i\|_2, i = 1, 2.\}, \quad M_{22} = \max\{\|B_i\|_2, i = 1, 2.\}. \quad (4.23)$$

Then for $0 \leq k \leq n$

$$\begin{aligned} &|T_k(A_1, B_1) - T_k(A_2, B_2)| \\ &\leq k M_{21}^{k-1} \binom{n}{n-k} \|B_1\|_2^{n-k} \|A_2 - A_1\|_2 + (n-k) M_{22}^{n-k-1} \binom{n}{k} \|A_2\|_2^k \|B_2 - B_1\|_2, \end{aligned} \quad (4.24)$$

$$\begin{aligned} &|T_k(A_1, B_1) - T_k(A_2, B_2)| \\ &\leq (n-k) M_{22}^{n-k-1} \binom{n}{k} \|A_1\|_2^k \|B_2 - B_1\|_2 + k M_{21}^{k-1} \binom{n}{n-k} \|B_2\|_2^{n-k} \|A_2 - A_1\|_2. \end{aligned} \quad (4.24')$$

PROOF: We bound $|T_k(A_1, B_1) - T_k(A_2, B_2)|$ by

$$\begin{aligned} & |T_k(A_1, B_1) - T_k(A_2, B_2)| \\ & \leq |T_k(A_1, B_1) - T_k(A_2, B_1)| + |T_k(A_2, B_1) - T_k(A_2, B_2)|. \end{aligned} \tag{4.25}$$

Consider the map $f : [0, 1] \mapsto \mathbb{C}^{n \times n}$ defined as $f(t) = (1-t)A_1 + tA_2$. By the mean-value theorem (8.5.4 in [3]), we get

$$\begin{aligned} & |T_k(A_1, B_1) - T_k(A_2, B_1)| \\ & \leq \sup_{0 \leq t \leq 1} \|\mathcal{D}_{f(t)} T_k(f(t), B_1)\|_2 \|\mathcal{D}f(t)\|_2 \\ & \leq \sup_{0 \leq t \leq 1} k \|f(t)\|_2^{k-1} \binom{n}{n-k} \|B_1\|_2 \|A_2 - A_1\|_2 \quad (\text{By Lemma 4.3}) \\ & \leq k M_{21}^{k-1} \binom{n}{n-k} \|B_1\|_2^{n-k} \|A_2 - A_1\|_2. \end{aligned} \tag{4.26}$$

On the other hand, Property 4.2 and (4.26) yield

$$\begin{aligned} |T_k(A_2, B_1) - T_k(A_2, B_2)| &= |T_{n-k}(B_1, A_2) - T_k(B_2, A_2)| \\ &\leq (n-k) M_{22}^{n-k-1} \binom{n}{k} \|A_2\|_2^k \|B_2 - B_1\|_2. \end{aligned} \tag{4.27}$$

(4.24) is a consequence of inequalities (4.25)–(4.27). Similarly,

$$\begin{aligned} & |T_k(A_1, B_1) - T_k(A_2, B_2)| \\ & \leq |T_k(A_1, B_1) - T_k(A_1, B_2)| + |T_k(A_1, B_2) - T_k(A_2, B_2)| \end{aligned}$$

yields (4.24'). **Q.E.D.**

THEOREM 4.2. Let $\{A, B\}, \{C, D\} \in \mathcal{R}(n)$. Denote by

$$M_{21} = \max\{\|A\|_2, \|C\|_2\}, \quad M_{22} = \max\{\|B\|_2, \|D\|_2\}. \tag{4.28}$$

Then

$$S_{\{A,B\}}\{C, D\} \leq \frac{n^{\frac{1}{n}}}{\sigma(A, B)} [(M_{21} + \|B\|_2)^{n-1} \|C - A\|_2 + (M_{22} + \|C\|_2)^{n-1} \|D - B\|_2]^{\frac{1}{n}}, \tag{4.29}$$

$$S_{\{A,B\}}\{C, D\} \leq \frac{n^{\frac{1}{n}}}{\sigma(A, B)} [(M_{21} + \|D\|_2)^{n-1} \|C - A\|_2 + (M_{22} + \|A\|_2)^{n-1} \|D - B\|_2]^{\frac{1}{n}}. \tag{4.29'}$$

More concisely but less sharply, we have

$$S_{\{A,B\}}\{C, D\} \leq \frac{n^{\frac{1}{n}}}{\sigma(A, B)} [2 \max(M_{21}, M_{22})]^{1-\frac{1}{n}} (\|C - A\|_2 + \|D - B\|_2)^{\frac{1}{n}}. \tag{4.30}$$

PROOF: Suppose that $f(\alpha, \beta), g(\alpha, \beta)$ are the generalized eigenpolynomials of $\{A, B\}$ and $\{C, D\}$, respectively. It follows from (4.4), (4.24) and

$$\begin{aligned} k \binom{n}{k} &= n \binom{n-1}{k-1}, \quad 1 \leq k \leq n, \\ (n-k) \binom{n}{n-k} &= n \binom{n-1}{n-k-1} = n \binom{n-1}{k}, \quad 0 \leq k \leq n-1 \end{aligned}$$

that if $|\alpha|^2 + |\beta|^2 = 1$ then

$$\begin{aligned}
|f(\alpha, \beta) - g(\alpha, \beta)| &\leq \sum_{k=0}^n |T_k(A, B) - T_k(C, D)| \\
&\leq \sum_{k=0}^{n-1} (n-k) M_{22}^{n-k-1} \binom{n}{n-k} \|C\|_2^k \|D - B\|_2 \\
&\quad + \sum_{k=1}^n k M_{21}^{k-1} \binom{n}{k} \|B\|_2^{n-k} \|C - A\|_2 \quad (\text{by Lemma 4.4}) \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} M_{22}^{n-k-1} \|C\|_2^k \|D - B\|_2 \\
&\quad + n \sum_{k=1}^n \binom{n-1}{k-1} M_{21}^{k-1} \|B\|_2^{n-k} \|C - A\|_2 \\
&= n [(M_{22} + \|C\|_2)^{n-1} \|D - B\|_2 + (M_{21} + \|B\|_2)^{n-1} \|C - A\|_2].
\end{aligned} \tag{4.31}$$

Decompose $\{A, B\}$ as in (1.6) such that (2.7) holds, then for any $(\gamma, \delta) \in \lambda(C, D)$ with $|\gamma|^2 + |\delta|^2 = 1$, by (1.6) and $g(\gamma, \delta) = 0$ we have

$$\begin{aligned}
\prod_{i=1}^n |\delta\alpha_i - \gamma\beta_i| &= \frac{1}{|\det P|} |f(\gamma, \delta)| = \frac{1}{|\det P|} |f(\gamma, \delta) - g(\gamma, \delta)| \\
&\leq \frac{n}{[\sigma(A, B)]^n} [(M_{22} + \|C\|_2)^{n-1} \|D - B\|_2 + (M_{21} + \|B\|_2)^{n-1} \|C - A\|_2].
\end{aligned}$$

This produces

$$\begin{aligned}
&\prod_{i=1}^n \rho((\alpha_i, \beta_i), (\gamma, \delta)) \\
&\leq \frac{n}{[\sigma(A, B)]^n} [(M_{22} + \|C\|_2)^{n-1} \|D - B\|_2 + (M_{21} + \|B\|_2)^{n-1} \|C - A\|_2].
\end{aligned}$$

Therefore, there is at least one (α_i, β_i) satisfying

$$\begin{aligned}
&\rho((\alpha_i, \beta_i), (\gamma, \delta)) \\
&\leq \frac{n^{\frac{1}{n}}}{\sigma(A, B)^n} [(M_{22} + \|C\|_2)^{n-1} \|D - B\|_2 + (M_{21} + \|B\|_2)^{n-1} \|C - A\|_2]^{\frac{1}{n}},
\end{aligned}$$

which prove (4.29). Similarly, employing (4.24') instead of (4.24) leads to (4.29'). (4.30) is the straightforward corollary of (4.29) or (4.29'). **Q.E.D.**

Remark 4.2. In the above proof, if decomposition (1.6) employed is chosen such that (2.8), instead of (2.7), holds, then we will get upper bounds slightly different from (4.29), (4.29') and (4.30) in the way of replacing $\sigma(A, B)$ by

$$[\sigma_{\min}(Z)]^{\frac{1}{n}} [\sigma(A, B)]^{1-\frac{1}{n}}. \tag{4.32}$$

4.2 Bounds by $\|\cdot\|_F$. The subsection gives perturbation bounds in a similar way by attempting to bound

$$\begin{aligned} \|\mathcal{D}_A T_k(A, B)\|_F &\triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_F=1}} |(\mathcal{D}_A T_k(A, B))(C)|, \\ \|\mathcal{D}_B T_k(A, B)\|_F &\triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_F=1}} |(\mathcal{D}_B T_k(A, B))(C)|, \end{aligned} \tag{4.33}$$

for $0 \leq k \leq n$.

LEMMA 4.5 (BHATIA-MUKHERJEA[2]). Let $A \in \mathbb{C}^{n \times n}$, and

$$\det_n : \mathbb{C}^{n \times n} \mapsto \mathbb{C} \tag{4.34}$$

be the determinant map. Then

$$\|\mathcal{D} \det_n(A)\|_F \leq n^{1-\frac{n}{2}} \|A\|_F^{n-1}, \tag{4.35}$$

where

$$(\mathcal{D} \det_n(A))(C) = \left. \frac{d}{dt} \det_n(A + tC) \right|_{t=0}, \quad C \in \mathbb{C}^{n \times n}, \tag{4.36}$$

$$\|\mathcal{D} \det_n(A)\|_F \triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_F=1}} |(\mathcal{D} \det_n(A))(C)|. \tag{4.37}$$

LEMMA 4.6. For $A, B \in \mathbb{C}^{n \times n}$,

$$\|\mathcal{D}_A T_k(A, B)\|_F \leq n^{-\frac{n-k}{2}} \binom{n}{n-k} \|B\|_F^{n-k} k^{1-\frac{k}{2}} \|A\|_F^{k-1}, \quad 0 \leq k \leq n, \tag{4.38}$$

$$\|\mathcal{D}_B T_k(A, B)\|_F \leq n^{-\frac{k}{2}} \binom{n}{k} \|A\|_F^k (n-k)^{1-\frac{n-k}{2}} \|B\|_F^{n-k-1}, \quad 0 \leq k \leq n. \tag{4.39}$$

PROOF: Analogously to the proof of Lemma 4.3, we have

$$T_k(A, B) = \det U \det V \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \nu_{\alpha_i} \det_{n-k}(U^H B V^H)[\alpha_I / \alpha | \alpha_I / \alpha]. \tag{4.40}$$

Set

$$\varphi_\alpha(U^H B V^H) = (U^H B V^H)[\alpha_I / \alpha | \alpha_I / \alpha].$$

Obviously, $\varphi_\alpha : \mathbb{C}^{n \times n} \mapsto \mathbb{C}^{(n-k) \times (n-k)}$ is linear, and

$$\begin{aligned} \|\mathcal{D}_{U^H B V^H} \varphi_\alpha(U^H B V^H)\|_F &\triangleq \sup_{\substack{C \in \mathbb{C}^{n \times n} \\ \|C\|_F=1}} \left\| \left. \frac{d}{dt} \varphi_\alpha(U^H B V^H + tC) \right|_{t=0} \right\|_F \\ &\leq 1. \end{aligned} \tag{4.41}$$

Now, according to the chain rule of differentiation, we get

$$\mathcal{D}_B \det_{n-k} \varphi_\alpha(U^H B V^H) = \mathcal{D} \det_{n-k} \mathcal{D} \varphi_\alpha(U^H B V^H) \mathcal{D}_B (U^H B V^H).$$

So by the above equation, (4.40), (4.41) and Lemma 4.5,

$$\begin{aligned}
\|\mathcal{D}_B T_k(A, B)\|_F &\leq \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \nu_{\alpha_i} \|\mathcal{D}_B \det_{n-k} \varphi_\alpha(U^H B V^H)\|_F \\
&\leq \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \nu_{\alpha_i} (n-k)^{1-\frac{n-k}{2}} \|\varphi_\alpha(U^H B V^H)\|_F^{n-k-1} \\
&\leq \left(\sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k \nu_{\alpha_i} \right) (n-k)^{1-\frac{n-k}{2}} \|B\|_F^{n-k-1}. \tag{4.42}
\end{aligned}$$

Using Lagrange's method of undetermined multipliers, it can be seen that if y_1, \dots, y_n are n non-negative real variables subject to the constraint $\sum_{i=1}^n y_i^2 = M > 0$, and if

$$f(y_1, \dots, y_n) = \sum_{\alpha \in \Omega_{k,n}} \prod_{i=1}^k y_{\alpha_i},$$

then f attains a maximum value given by

$$f_{\max} = \binom{n}{k} \left(\sqrt{\frac{M}{n}} \right)^k.$$

Applying this fact to (4.42) gives

$$\|\mathcal{D}_B T_k(A, B)\|_F \leq (n-k)^{1-\frac{n-k}{2}} \|B\|_F^{n-k-1} \binom{n}{k} n^{-\frac{k}{2}} \|A\|_F^k,$$

which is nothing but (4.39). Inequality (4.38) is a consequence of (4.39) and Property 4.2.

Q.E.D.

LEMMA 4.7. Let $A_i, B_i \in \mathbb{C}^{n \times n}$, $i = 1, 2$. Denote by

$$M_{F1} = \max\{\|A_i\|_F, i = 1, 2\}, \quad M_{F2} = \max\{\|B_i\|_F, i = 1, 2\}. \tag{4.43}$$

Then for $0 \leq k \leq n$

$$\begin{aligned}
&|T_k(A_1, B_1) - T_k(A_2, B_2)| \\
&\leq k^{1-\frac{k}{2}} M_{F1}^{k-1} n^{-\frac{n-k}{2}} \binom{n}{n-k} \|B_1\|_F^{n-k} \|A_2 - A_1\|_F \\
&\quad + (n-k)^{1-\frac{n-k}{2}} M_{F2}^{n-k-1} n^{-\frac{k}{2}} \binom{n}{k} \|A_2\|_F^k \|B_2 - B_1\|_F, \tag{4.44}
\end{aligned}$$

$$\begin{aligned}
&|T_k(A_1, B_1) - T_k(A_2, B_2)| \\
&\leq k^{1-\frac{k}{2}} M_{F1}^{k-1} n^{-\frac{n-k}{2}} \binom{n}{n-k} \|B_2\|_F^{n-k} \|A_2 - A_1\|_F \\
&\quad + (n-k)^{1-\frac{n-k}{2}} M_{F2}^{n-k-1} n^{-\frac{k}{2}} \binom{n}{k} \|A_1\|_F^k \|B_2 - B_1\|_F. \tag{4.44'}
\end{aligned}$$

A proof of this lemma can be given in much the same way as for Lemma 4.4. The following theorem can be proved similarly to Theorem 4.2.

THEOREM 4.3. Let $\{A, B\}, \{C, D\} \in \mathcal{R}(n)$. Denote by

$$M_{F1} = \max\{\|A\|_F, \|C\|_F\}, M_{F2} = \max\{\|B\|_F, \|D\|_F\}. \tag{4.45}$$

Then

$$S_{\{A,B\}\{C,D\}} \leq \frac{1}{\sigma(A,B)} \left[\left(\sum_{k=1}^n k^{1-\frac{k}{2}} M_{F1}^{k-1} n^{-\frac{n-k}{2}} \binom{n}{n-k} \|B\|_F^{n-k} \right) \|C-A\|_F + \left(\sum_{k=0}^{n-1} (n-k)^{1-\frac{n-k}{2}} M_{F2}^{n-k-1} n^{-\frac{k}{2}} \binom{n}{k} \|C\|_F^k \right) \|D-B\|_F \right]^{\frac{1}{n}}, \tag{4.46}$$

$$S_{\{A,B\}\{C,D\}} \leq \frac{1}{\sigma(A,B)} \left[\left(\sum_{k=1}^n k^{1-\frac{k}{2}} M_{F1}^{k-1} n^{-\frac{n-k}{2}} \binom{n}{n-k} \|D\|_F^{n-k} \right) \|C-A\|_F + \left(\sum_{k=0}^{n-1} (n-k)^{1-\frac{n-k}{2}} M_{F2}^{n-k-1} n^{-\frac{k}{2}} \binom{n}{k} \|A\|_F^k \right) \|D-B\|_F \right]^{\frac{1}{n}}. \tag{4.46'}$$

More concisely but less sharply, we have

$$S_{\{A,B\}\{C,D\}} \leq \frac{c(n)}{\sigma(A,B)} [\max(M_{F1}, M_{F2})]^{1-\frac{1}{n}} (\|C-A\|_F + \|D-B\|_F)^{\frac{1}{n}}, \tag{4.47}$$

where

$$c(n) = \left(\sum_{k=1}^n k^{1-\frac{k}{2}} n^{-\frac{n-k}{2}} \binom{n}{k} \right)^{\frac{1}{n}}. \tag{4.48}$$

Remark 4.3 By the same arguments as in Remark 4.2, we see that $\sigma(A, B)$ appearing in (4.46), (4.46') and (4.47) can be replaced by

$$[\sigma_{\min}(Z)]^{\frac{1}{n}} [\sigma(A, B)]^{1-\frac{1}{n}}.$$

Example 4.1 $n = 2$. $\{A, B\}$ and $\{C, D\}$ are as described as in §3, Example 3.1. Table 4.1 displays the bounds of $S_{\{A,B\}\{C,D\}}$ according to our theorems, where the second line and the third line are computed with the estimations

$$\sigma(A, B) \geq \sqrt{2} \quad \text{and} \quad \sigma(A, B) \geq [\omega(A, B)]^{-1} \geq \frac{4}{\sqrt{9 + \sqrt{17}}}$$

for $\sigma(A, B)$, respectively.

Table 4.1

(4.29)	(4.29')	(4.30)	(4.46)	(4.46')	(4.47)
0.01618	0.01618	0.017989	0.013899	0.013899	0.014459
0.025267	0.025267	0.028091	0.021704	0.021704	0.022579

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