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A PERTURBATION BOUND FOR THE GENERALIZED POLAR DECOMPOSITION

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Abstract.

Let A be an $m \times n$ complex matrix. A decomposition $A = QH$ is termed a *generalized polar decomposition* of A if Q is an $m \times n$ subunitary matrix (sometimes also called a partial isometry) and H a positive semidefinite Hermitian matrix. It was proved that a nonzero matrix $A \in \mathbb{C}^{m \times n}$ has a unique generalized polar decomposition $A = QH$ with the property $\mathscr{R}(Q^H) = \mathscr{R}(H)$, where Q^H denotes the conjugate transpose of Q and $\mathscr{R}(H)$ the column space of H. The main result of this note is a perturbation bound for Q when A is perturbed.

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Throughout the paper, we use the following notation. $\mathbb{C}^{m \times n}$ is the set of m by n complex matrices; $\mathbb{C}_r^{m \times n} \subset \mathbb{C}^{m \times n}$ is the set of $m \times n$ complex matrices having rank r; and $\mathscr{U}_n \subset \mathbb{C}^{n \times n}$ is the set of $n \times n$ unitary matrices. For $A \in \mathbb{C}^{m \times n}$, $\mathscr{R}(A)$ denotes the column space of A. Further A^H and A^+ denote the conjugate transpose and Moore-Penrose inverse of A, respectively. $I^{(n)}$ is the $n \times n$ unit matrix and $I_{m,n}^{(r)} \in \mathbb{C}_r^{m \times n}$ is defined by

$$I_{m,n}^{(r)} \equiv \begin{pmatrix} I^{(r)} & 0 \\ 0 & 0 \end{pmatrix}.$$

$\|\cdot\|_2$ is used to denote the Euclidean length of a vector or the spectral norm of a matrix and $\|\cdot\|_F$ the Frobenius norm of a matrix.

First of all, let us summarize some essential definitions and properties concerning a generalized polar decomposition (for details, see Sun & Chen [6]).

$Q \in \mathbb{C}^{m \times n}$ is an $m \times n$ *unitary matrix* if $Q^H Q = I^{(n)}$ $(m \geq n)$ or $QQ^H = I^{(m)}$ $(m \leq n)$; and $Q \in \mathbb{C}^{m \times n}$ an $m \times n$ *subunitary matrix* (sometimes also called an $m \times n$ *partial isometry*) if $\|Qx\|_2 = \|x\|_2$ for all $x \in \mathscr{R}(Q^H)$. Every matrix $A \times \mathbb{C}^{m \times n}$ can be decomposed as $A = QH$, where $Q \in \mathbb{C}^{m \times n}$ is unitary or subunitary and $H \in \mathbb{C}^{n \times n}$ Hermitian positive semidefinite. The decomposition such that Q is unitary is called a *polar*

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decomposition of A, otherwise a *generalized polar decomposition*. With the help of singular value decomposition (SVD), we can construct a few generalized polar decompositions of a matrix. Let

$$A = U\Sigma V^H, \quad \Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

be a SVD of $A \in \mathbb{C}_r^{m\times n}$, where $r \le \min\{m,n\}$, $U \in \mathscr{U}_m$ and $V \in \mathscr{U}_n$, $\Omega = \operatorname{diag}(\sigma_1,\dots,\sigma_r)$, $\sigma_i > 0$, $i = 1,\dots,r$, then for any integer $p \ge r$, we have

$$A = U\Sigma V^H = (UI^{(p)}_{m,n}V^H)\left[V\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}V^H\right] \equiv Q_p H, \tag{2}$$

where

$$Q_p \equiv UI^{(p)}_{m,n}V^H \quad \text{and} \quad H \equiv V\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}V^H, \tag{3}$$

p running from r to $\min\{m,n\}$. It was proved that the decomposition $A = QH$ is unique under the condition:

$$\mathscr{R}(Q^H) = \mathscr{R}(H), \tag{4}$$

where $Q \in \mathbb{C}^{m\times n}$ is subunitary and $H \in \mathbb{C}^{n\times n}$ positive semidefinite Hermitian (see, e.g., Ben-Israel & Greville [1, p. 255]). Thus the unique decomposition $A = QH$ satisfying (4) can be given by (2) and (3) with $p = r$.

Perturbation bounds associated with the polar decomposition of a nonsingular matrix can be found in [2, 3, 4, 5]. In the following we will present a perturbation bound for the subunitary factor Q in the generalized polar decomposition.

LEMMA 1. *Let* $U \in \mathscr{U}_n$ *and*

$$\Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}_r^{m\times n}, \quad \tilde\Sigma = \begin{pmatrix} \tilde\Omega & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}_r^{m\times n}, \tag{5a}$$

$$\Omega = \operatorname{diag}(\sigma_1,\dots,\sigma_r), \quad \tilde\Omega = \operatorname{diag}(\tilde\sigma_1,\dots,\tilde\sigma_r). \tag{5b}$$

Then

$$\|U\Sigma - \tilde\Sigma V\|_F \ge \min_{1\le i,j\le r}\{\sigma_i,\tilde\sigma_j\}\,\|UI^{(r)}_{m,n} - I^{(r)}_{m,n}V\|_F. \tag{6}$$

PROOF. Without loss of generality, we prove this lemma for the case $m = n$; otherwise by considering suitably augmenting matrices the case $m \ne n$ degenerates to square one.

Assume that $m = n$ and set $\sigma = \min_{1\le i,j\le r}\{\sigma_i,\tilde\sigma_j\}$,

$$\Gamma = \Sigma - \sigma I^{(r)}_{m,n}, \quad \tilde\Gamma = \tilde\Sigma - \sigma I^{(r)}_{m,n}. \tag{7}$$

Γ and $\tilde{\Gamma}$ are two diagonal matrices with diagonal elements nonnegative. It is easy to verify that

$$\begin{aligned}(8)\quad \|U\Sigma - \tilde{\Sigma}V\|_F^2 &= \|U(\Gamma + \sigma I^{(r)}_{m,n}) - (\tilde{\Gamma} + \sigma I^{(r)}_{m,n})V\|_F^2 \\ &= \|\sigma(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V) + (U\Gamma - \tilde{\Gamma}V)\|_F^2 \\ &= \sigma^2\,\|UI^{(r)}_{m,n} - I^{(r)}_{m,n}V\|_F^2 \\ &\quad + 2\sigma\mathscr{R}\mathrm{tr}\,[(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)(U\Gamma - \tilde{\Gamma}V)^H] + \|U\Gamma - \tilde{\Gamma}V\|_F^2.\end{aligned}$$

Here tr denotes the trace of a matrix, $\mathscr{R}$ the real part of a complex number. We claim that

$$(9)\qquad \mathscr{R}\mathrm{tr}\,[(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)(U\Gamma - \tilde{\Gamma}V)^H] \geq 0.$$

So (8), together with (9), leads to

$$\|U\Sigma - \tilde{\Sigma}V\|_F^2 \geq \sigma^2\,\|UI^{(r)}_{m,n} - I^{(r)}_{m,n}V\|_F^2,$$

which proves (6). We have to prove (9).

$$\begin{aligned}(10)\quad & 2\mathscr{R}\mathrm{tr}\,[(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)(U\Gamma - \tilde{\Gamma}V)^H] \\ &= \mathrm{tr}\,[(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)(U\Gamma - \tilde{\Gamma}V)^H + (U\Gamma - \tilde{\Gamma}V)(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)^H] \\ &= \mathrm{tr}\,[(U^H(UI^{(r)}_{m,n} - I^{(r)}_{m,n}V) + (UI^{(r)}_{m,n} - I^{(r)}_{m,n}V)^H U)\Gamma] \\ &\quad + \mathrm{tr}\,[((I^{(r)}_{m,n}V - UI^{(r)}_{m,n})V^H + V(I^{(r)}_{m,n}V - UI^{(r)}_{m,n})^H)\tilde{\Gamma}] \\ &= \mathrm{tr}\,[(2I^{(r)}_{m,n} - U^H I^{(r)}_{m,n}V - V^H I^{(r)}_{m,n}U)\Gamma] \\ &\quad + \mathrm{tr}\,[(2I^{(r)}_{m,n} - UI^{(r)}_{m,n}V^H - VI^{(r)}_{m,n}U^H)\tilde{\Gamma}].\end{aligned}$$

We have employed a well-known property of the matrix trace: $\mathrm{tr}\,MN = \mathrm{tr}\,NM$ for two matrices M and N with suitable dimensions. Partition

$$U^H I^{(r)}_{m,n}V + V^H I^{(r)}_{m,n}U = \begin{pmatrix} M & * \\ * & * \end{pmatrix}.$$

M is an $r \times r$ Hermitian matrix. Since

$$\|U^H I^{(r)}_{m,n}V + V^H I^{(r)}_{m,n}U\|_2 \leq \|U^H I^{(r)}_{m,n}V\|_2 + \|V^H I^{(r)}_{m,n}U\|_2 \leq 2,$$

every diagonal element of M is a real number less than or equal to 2. This leads to the conclusion that the diagonal elements of $2I^{(r)} - M$ are nonnegative. Thus from (5) and (7), it follows that

$$\mathrm{tr}\,[(2I^{(r)}_{m,n} - U^H I^{(r)}_{m,n}V - V^H I^{(r)}_{m,n}U)\Gamma] = \mathrm{tr}\,[(2I^{(r)} - M)(\Omega - \sigma I)] \geq 0.$$

Similar arguments also show

$$\mathrm{tr}\,[(2I^{(r)}_{m,n} - UI^{(r)}_{m,n}V^H - VI^{(r)}_{m,n}U^H)\tilde{\Gamma}] \geq 0.$$

The two inequalities above and (10) establish (9). ■

THEOREM 2. *Let* $U, \tilde{U} \in \mathcal{U}_m$, $V, \tilde{V} \in \mathcal{U}_n$ *and* Σ, $\tilde{\Sigma}$, Ω, $\tilde{\Omega}$ *be of forms* (5). *Then*

$$\|U\Sigma V^H - \tilde{U}\tilde{\Sigma}\tilde{V}^H\|_F \geq \min_{1 \leq i,j \leq r} \{\sigma_i, \tilde{\sigma}_j\} \|UI^{(r)}_{m,n}V^H - \tilde{U}I^{(r)}_{m,n}\tilde{V}^H\|_F. \tag{11}$$

PROOF. It follows from the unitary invariance of $\|\cdot\|_F$ and Lemma 1 that

$$\begin{aligned}\|U\Sigma V^H - \tilde{U}\tilde{\Sigma}\tilde{V}^H\|_F &= \|\tilde{U}^H U\Sigma - \tilde{\Sigma}\tilde{V}^H V\|_F \\ &\geq \min_{i,j} \{\sigma_i, \tilde{\sigma}_j\} \|\tilde{U}^H U I^{(r)}_{m,n} - I^{(r)}_{m,n}\tilde{V}^H V\|_F \\ &\geq \min_{i,j} \{\sigma_i, \tilde{\sigma}_j\} \|UI^{(r)}_{m,n}V^H - \tilde{U}I^{(r)}_{m,n}\tilde{V}^H\|_F.\end{aligned}$$

A restatement of Theorem 2 is as follows.

THEOREM 2A. *Let* $A, \tilde{A} \in \mathbb{C}^{m\times n}_r$,

$$A = QH \quad \text{and} \quad \tilde{A} = \tilde{Q}\tilde{H}$$

be generalized polar decompositions of A *and* $\tilde{A}$ *respectively satisfying* (4). *Then*

$$\|A - \tilde{A}\|_F \geq \min\{\|A^+\|_2^{-1}, \|\tilde{A}^+\|_2^{-1}\} \|Q - \tilde{Q}\|_F. \tag{12}$$

PROOF. Suppose that A and $\tilde{A}$ admit SVDs

$$A = U\Sigma V^H \quad \text{and} \quad \tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^H,$$

where U, $\tilde{U}$, V, $\tilde{V}$, Σ, $\tilde{\Sigma}$ are as described in Theorem 2. As we remarked before,

$$Q = UI^{(r)}_{m,n}V^H \quad \text{and} \quad \tilde{Q} = \tilde{U}I^{(r)}_{m,n}\tilde{V}^H.$$

On the other hand, note that

$$\min_{1 \leq i \leq r} \sigma_i = \|A^+\|_2^{-1}, \quad \min_{1 \leq j \leq r} \tilde{\sigma}_j = \|\tilde{A}\|_2^{-1}.$$

Thus (12) is nothing but (11). ■

Sun and Chen [6] prove that under the conditions of Theorem 2a

$$2\|A - \tilde{A}\|_F \geq \|A^+\|_2^{-1} \|Q - \tilde{Q}\|_F. \tag{13}$$

From the point of view of numerical analysis, when A is perturbed to $\tilde{A}$, $\|A - \tilde{A}\|_F$ may be sufficiently small, thus ignoring the difficulty in determining the numerical rank, we have approximately

$$\begin{aligned}\|A^+\|_2^{-1} = \min_{1 \leq i \leq r} \sigma_i &\approx \min_{1 \leq j \leq r} \tilde{\sigma}_j = \|\tilde{A}^+\|_2^{-1} \\ &\approx \min\{\|A^+\|_2^{-1}, \|\tilde{A}^+\|_2^{-1}\} = \min_{1 \leq i,j \leq r} \{\sigma_i, \tilde{\sigma}_j\}.\end{aligned}$$

This suggests that inequality (12) improves inequality (13) by a factor 2.

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