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Spectral variations and Hadamard products: Some problems [1]

Ren-Cang Li [2]

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

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Abstract

Many perturbation problems in matrix theory are related to three matrix minimization problems involving Hadamard products. Possible solutions to the problems are conjectured. The problems, if solved, may imply significant advances in eigenvalue, generalized eigenvalue variations, and perhaps in other aspects in matrix theory, too. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

The spectral variation of a matrix has recently been a very active research subject in both matrix theory and numerical linear algebra. Over the last couple of decades significant progress has been made in partially extending the classical Weyl and Lidskii theory [15,24] to normal matrices and even to diagonalizable matrices for example. Recently these theories have been established for relative perturbations. This note will show how certain perturbation

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[2] E-mail: na.rcli@na-net.ornl. gov.

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problems can be reformulated as simple matrix optimization problems involving *Hadamard products*. These reformulations are not our main goal. What we hope is to reveal a possibly unifying theory that is behind these perturbation theorems.

We shall start with explaining the idea using normal matrices. Many other connections will be discussed in the subsequent sections. Suppose that A and $\tilde{A}$ are *normal*, and have eigendecompositions

$$A = U\Lambda U^* \quad \text{and} \quad \tilde{A} = \tilde{U}\tilde{\Lambda}\tilde{U}^*, \tag{1.1}$$

where U and $\tilde{U}$ are unitary, "*" denotes the conjugate transpose, and

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \text{and} \quad \tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n). \tag{1.2}$$

Let $\|X\|_{\mathrm{F}} = \left(\sum_{i,j} |x_{ij}|^2\right)^{1/2}$ be the Frobenius norm and let $X \circ Y \stackrel{\text{def}}{=} (x_{ij}y_{ij})$ be the *Hadamard Product* of $X = (x_{ij})$ and $Y = (y_{ij})$. We have

$$\|A - \tilde{A}\|_{\mathrm{F}} = \|\Lambda U^*\tilde{U} - U^*\tilde{U}\tilde{\Lambda}\|_{\mathrm{F}} = \|Q \circ Z_1\|_{\mathrm{F}}, \tag{1.3}$$

where $Q = U^*\tilde{U}$ and

$$Z_1 \stackrel{\text{def}}{=} \left(\lambda_i - \tilde{\lambda}_j\right)_{i,j=1}^n. \tag{1.4}$$

One part of spectral perturbation theory is to bound the differences $|\lambda_i - \tilde{\lambda}_{\tau(i)}|$ by the norm of $A - \tilde{A}$, where τ is a permutation of $\{1, 2, \ldots, n\}$. For example the Hoffman–Wielandt theorem [8] says that for some τ

$$\|A - \tilde{A}\|_{\mathrm{F}} \geqslant \sqrt{\sum_{i=1}^n |\lambda_i - \tilde{\lambda}_{\tau(i)}|^2}. \tag{1.5}$$

If we take τ to be the permutation that minimizes the right-hand side of (1.5) among all possible permutations, then

$$\sqrt{\sum_{i=1}^n |\lambda_i - \tilde{\lambda}_{\tau(i)}|^2} = \min_{P \text{ permutation}} \|P \circ Z_1\|_{\mathrm{F}}$$

which is independent of Q and thus the left-hand side is less than $\|A - \tilde{A}\|_{\mathrm{F}} = \|Q \circ Z_1\|_{\mathrm{F}}$ for any unitary Q by (1.3) and (1.5). Hence

$$\min_{Q \text{ unitary}} \|Q \circ Z_1\|_{\mathrm{F}} \geqslant \min_{P \text{ permutation}} \|P \circ Z_1\|_{\mathrm{F}}. \tag{1.6}$$

On the other hand, it is trivial to see that the left-hand side of (1.6) is no bigger than its right-hand side because of the permutation matrices are a subset of the unitary ones. Thus we have

$$\min_{Q \text{ unitary}} \|Q \circ Z_1\|_{\mathrm{F}} = \min_{P \text{ permutation}} \|P \circ Z_1\|_{\mathrm{F}}. \tag{1.7}$$

As (1.5) and (1.7) are equivalent and the equivalence yields no new result. However, the reformulation (1.7) provides room for generalizations and reveals a general result that would be otherwise difficult, if at all possible, to see from (1.5). That is, that Eq. (1.7) *holds with* Z_1 *replaced by a general matrix, not just the one defined so specially by* Eq. (1.4); see Proposition 3.1. The result encompasses other perturbation theorems for diagonalizable matrices and in relative perturbation theory, too.

What we have shown above is just one of many spectral perturbation theorems that can be interpreted as bounding the norms of $Q \circ Z$ from below by the norms of $P \circ Z$, where Q is unitary or more generally nonsingular, P is a permutation matrix, and Z is a special matrix defined by the eigenvalues. The goal of this note is to address the question whether there is a unifying theory behind these types of perturbation theorems.

The rest of this paper is organized as follows. Section 2 reveals more connections between matrix minimization problems involving Hadamard products to matrix spectral perturbation theories. Section 3 puts forward the problems that are distilled from our discussion in Section 2; we also present conjectures regarding the proposed problems and solutions for the Frobenius norm and for the spectral norm in the case of 2 × 2 matrices. Finally we discuss the significance of the conjecture, if it were proved.

2. Relevant matrix perturbation theory

We shall use $|||\cdot|||$ to denote a unitarily invariant norm [3]. $\|X\|_2$, the largest singular value of X, and $\|X\|_{\mathrm{F}}$ are two frequently used unitarily invariant norms. " $^{\mathrm{T}}$" denotes the transpose, and I denotes the identity matrix.

Let A and $\tilde{A}$ be two $n \times n$ matrices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$, respectively.

2.1. Normal matrices

For normal matrices A and $\tilde{A}$ having eigen-decompositions (1.1) and (1.2), we have

$$|||A - \tilde{A}||| = |||AU^*\tilde{U} - U^*\tilde{U}\tilde{A}||| = |||Q \circ Z_1||| \tag{2.1}$$

[3] A matrix norm on the space of $n \times n$ complex matrices is called *unitarily invariant* if it satisfies, besides the usual properties of any norm, also (see [21], pp. 74–88).

1. $|||UXV||| = |||X|||$ for all unitary matrices U and V; and
2. $|||X||| = \|X\|_2$ for any X of rank one.

for any unitarily invariant norm $|||\cdot|||$, where $Q = U^*\tilde{U}$ and Z_1 is defined in Eq. (1.4).

2.2. Diagonalizable matrices

Suppose A and $\tilde{A}$ are *diagonalizable*:

$$A = X\Lambda X^{-1} \quad \text{and} \quad \tilde{A} = \tilde{X}\tilde{\Lambda}\tilde{X}^{-1}, \tag{2.2}$$

where Λ and $\tilde{\Lambda}$ are as in Eq. (1.2). Then one has

$$\|X^{-1}\|_2 \left|\left|\left| A - \tilde{A} \right|\right|\right| \|\tilde{X}\|_2 \geqslant \left|\left|\left| \Lambda X^{-1}\tilde{X} - X^{-1}\tilde{X}\tilde{\Lambda} \right|\right|\right| = |||W \circ Z_1|||,$$

where $W = X^{-1}\tilde{X}$ and Z_1 is as in Eq. (1.4).

Bhatia et al. [5] study diagonalizable matrix eigenvalue variations from a different angle by looking into

$$\left|\left|\left| A\Gamma - \Gamma\tilde{A} \right|\right|\right| \; \left|\left|\left| \Gamma^{-1}A - \tilde{A}\Gamma^{-1} \right|\right|\right|, \tag{2.3}$$

where Γ is nonsingular, and A and $\tilde{A}$ are both *normal.* Using Eq. (1.1), we can turn Eq. (2.3) into

$$|||W \circ Z_1||| \; |||W^{-1} \circ Z_1^{\mathrm{T}}|||, \quad W = U^*\Gamma\tilde{U}.$$

2.3. Relative perturbation theory

In at least four instances in developing a relative eigenvalue perturbation theory [13,14], matrix Hadamard products are involved.

Instance 1. A and $\tilde{A}$ are *non-negative definite* and $\tilde{A} = D^*AD$, where D is nonsingular. Let A and $\tilde{A}$ have eigen-decompositions (1.1) and (1.2). Write $A = B^*B$ and $\tilde{A} = \tilde{B}^*\tilde{B}$, where $\tilde{B} = BD$, and let the singular value decompositions of B and $\tilde{B}$ be $B = V\Lambda^{1/2}U^*$ and $\tilde{B} = \tilde{V}\tilde{\Lambda}^{1/2}\tilde{U}^*$. Noticing that $BB^* - \tilde{B}\tilde{B}^* = B(D^{-*} - D)\tilde{B}^*$, we have [13]

$$\Lambda V^*\tilde{V} - V^*\tilde{V}\tilde{\Lambda} = \Lambda^{1/2}U^*(D^{-*} - D)\tilde{U}\tilde{\Lambda}^{1/2}. \tag{2.4}$$

Define [4]

$$Z_2 \stackrel{\text{def}}{=} \left(\frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{|\lambda_i\tilde{\lambda}_j|}} \right)_{i,j=1}^{n}. \tag{2.5}$$

[4] This involves a nonstandard measure $\chi(\xi,\zeta) \stackrel{\text{def}}{=} |\xi - \zeta|/\sqrt{|\xi\zeta|}$ first used by Barlow and Demmel [2]. It is topologically equivalent to the standard relative measure $|\xi - \zeta|/|\zeta|$. The interested reader is referred to Li [13] for a detailed study of various relative measures.

Then we have $U^*(D^{-*} - D)\tilde{U} = Q \circ Z_2$, where $Q = V^*\tilde{V}$. Thus for any unitarily invariant norm $|||\cdot|||$,

$$|||D^{-*} - D||| \;=\; |||Q \circ Z_2|||. \tag{2.6}$$

We wonder whether Eq. (2.6) holds for indefinite Hermitian (or even more generally for normal) matrix pair A and $\tilde{A} = D^*AD$?

Instance 2. $A = S^*HS$ and $\tilde{A} = S^*\tilde{H}S$ are both *non-negative definite*, where S is a scaling matrix. The significance of having a matrix S to scale A is that there are cases, e.g., for graded matrices, where H is much better-conditioned than A is. Assume $\|H^{-1/2}(\tilde{H} - H)H^{-1/2}\|_2 < 1$. Write $A = BB^*$ and $\tilde{A} = \tilde{B}\tilde{B}^*$, where $B = S^*H^{1/2}, \tilde{B} = BD$ and $D = \left(I + H^{-1/2}(\tilde{H} - H)H^{-1/2}\right)^{1/2}$. Let the SVDs of B and $\tilde{B}$ be $B = V\Lambda^{1/2}U^*$ and $\tilde{B} = \tilde{V}\tilde{\Lambda}^{1/2}\tilde{U}^*$. Both Eqs. (2.4) and (2.6) hold [13].

Instance 3. Slapničar [20] proposed to use $|A|^{-1/2}$ to scale A in the case of *indefinite Hermitian* matrices, assuming $|A|^{-1/2}(\tilde{A} - A)|A|^{-1/2}$ is small even though $\tilde{A} - A$ may *not be*, where $|A| \stackrel{\text{def}}{=} (A^*A)^{1/2}$. It is argued by Li [14] that much nicer and cleaner bounds can be obtained if relative differences in eigenvalues are bounded in terms of the norms of $\tilde{\delta}A \stackrel{\text{def}}{=} |A|^{-1/2}(\tilde{A} - A)|\tilde{A}|^{-1/2}$. The following arguments also apply to normal pair A and $\tilde{A}$. Let A and $\tilde{A}$ have eigendecompositions (1.1) and (1.2). Noticing that $A - \tilde{A} = |A|^{1/2}(\tilde{\delta}A)|\tilde{A}|^{1/2}$, we have [14]

$$\Lambda U^*\tilde{U} - U^*\tilde{U}\tilde{\Lambda} = |\Lambda|^{1/2}U^*(\tilde{\delta}A)\tilde{U}|\tilde{\Lambda}|^{1/2}. \tag{2.7}$$

Then $U^*(\tilde{\delta}A)\tilde{U} = Q \circ Z_2$, where $Q = U^*\tilde{U}$. Thus for any unitarily invariant norm $|||\cdot|||$,

$$\left|\left|\left|\tilde{\delta}A\right|\right|\right| = |||Q \circ Z_2|||. \tag{2.8}$$

Instance 4. A and $\tilde{A}$ are *nonsingular diagonalizable matrices.* Of interest is the case when $\hat{\delta}A \stackrel{\text{def}}{=} A^{-1/2}(\tilde{A} - A)\tilde{A}^{-1/2}$ is small, even though $\tilde{A} - A$ may *not* be small. Here the matrix square roots $A^{1/2}$ and $\tilde{A}^{1/2}$ may not be uniquely defined, in which case any one of them is as good as any other. Let A and $\tilde{A}$ have eigendecompositions (2.2) and (1.2). Noticing that $A - \tilde{A} = A^{1/2}(\hat{\delta}A)\tilde{A}^{1/2}$, we have [14]

$$\Lambda X^{-1}\tilde{X} - X^{-1}\tilde{X}\tilde{\Lambda} = \Lambda^{1/2}X^{-1}(\hat{\delta}A)\tilde{X}\tilde{\Lambda}^{1/2}. \tag{2.9}$$

Let $W = X^{-1}\tilde{X}$, and define

$$Z_3 \stackrel{\text{def}}{=} \left(\frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{\lambda_i}\sqrt{\tilde{\lambda}_j}}\right)_{i,j=1}^{n}, \tag{2.10}$$

where we may have a choice in selecting either one of the two square root signs; for our purposes the choice is not critical. Then $X^{-1}(\hat{\delta} A)\tilde{X} = W \circ Z_3$. Thus for any unitarily invariant norm $|||\cdot|||$,

$$\left|\left|\left|X^{-1}(\hat{\delta} A)\tilde{X}\right|\right|\right| = |||W \circ Z_3|||. \tag{2.11}$$

2.4. Generalized eigenvalue and singular value problems

Li [12] shows the equivalence between a perturbation theory for the generalized eigenvalue problem of *diagonalizable matrix pencils* and minimizing (see also [6,11])

$$\|W^{-1}\|_2 \left|\left|\left|\Lambda W\tilde{\Omega} - \Omega W\tilde{\Lambda}\right|\right|\right| = \|W^{-1}\|_2 |||W \circ Z_4|||$$

over all possible nonsingular matrices W, where $\Lambda = \mathrm{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$, and analogously for $\tilde{\Lambda}$ and $\tilde{\Omega}$, and

$$Z_4 \stackrel{\mathrm{def}}{=} \left(\alpha_i \tilde{\beta}_j - \beta_i \tilde{\alpha}_j\right)_{i,j=1}^{n}. \tag{2.12}$$

2.5. Singular value and generalized singular value problems

Analogous formulations apply to singular value and generalized singular value variations.

3. The problems, conjectures, and some solutions

Let G be a given arbitrary $n \times n$ matrix. We have made clear that many well-known spectral perturbation theorems deal with special cases of the following three problems.

Problem 1. Relate $\min\limits_{Q \text{ unitary}} |||Q \circ G|||$ to $\min\limits_{P \text{ permutation}} |||P \circ G|||$.

Problem 2. Relate $\min\limits_{W \text{ nonsingular}} \|W^{-1}\|_2 |||W \circ G|||$ to $\min\limits_{P \text{ permutation}} |||P \circ G|||$.

Problem 3. Relate $\min\limits_{W \text{ nonsingular}} |||W \circ G||| \, \left|\left|\left|W^{-1} \circ G^T\right|\right|\right|$ to $\min\limits_{P \text{ permutation}} |||P \circ G|||^2$.

The singular values, and thus the norms, of Hadamard products are frequently studied objects. Past research has been more or less along the lines of bounding certain norms of a Hadamard product from *above*; see, e.g., [1,16–18,23] and the references therein.

A trivial relation between the two expressions in each of the above problems is that the first expression is always no bigger than the second one. But as far as spectral variations are concerned, we would also like to bound the first expression in each of the above problems from below by the second expression. In view of our previous discussion and known results from matrix perturbation theory [3,21], we conjecture the following possible solutions to Problems 1–3.

Conjecture. *There are constants* $c_1 \geqslant 1$ *and* $c_2 \geqslant 1$, *both possibly dependent on the norm* $\|\!| \cdot |\!\|$, *but not on the matrix dimensions, such that*

$$c_1 \min_{W \text{ nonsingular}} \|W^{-1}\|_2 \|\!|W \circ G|\!\| \geqslant \min_{P \text{ permutation}} \|\!|P \circ G|\!\|, \tag{3.1}$$

$$c_2 \min_{W \text{ nonsingular}} \|\!|W \circ G|\!\| \; \|\!|W^{-1} \circ G^T|\!\| \geqslant \min_{P \text{ permutation}} \|\!|P \circ G|\!\|^2. \tag{3.2}$$

Now we state two special cases of (3.1). When $\|\!| \cdot |\!\| = \|\!| \cdot |\!\|_F$, Conjecture (3.1) with $c_1 = 1$ can be solved beautifully.

Proposition 3.1. *For an arbitrary complex matrix G, we have*

$$\min_{W \text{ nonsingular}} \|W^{-1}\|_2 \|W \circ G\|_F = \min_{P \text{ permutation}} \|P \circ G\|_F. \tag{3.3}$$

Proof. The main result of Elsner and Friedland [7] says that there exists a doubly stochastic matrix S such that for the (i,j) entry $|(W)_{ij}|^2 \geqslant \|W^{-1}\|_2^{-2}(S)_{ij}$. Employing the technique used by Hoffman and Wielandt [8] completes the proof. □

Perturbation theorems for $\|\cdot\|_F$ in [8,12,13,22,25] are special cases of this proposition. In connection to Conjecture (3.1) for $\|\!| \cdot |\!\| = \|\cdot\|_2$, we have

Proposition 3.2. *For* 2×2 *matrices G,*

$$\min_{Q \text{ unitary}} \|Q \circ G\|_2 = \min_{P \text{ permutation}} \|G \circ P\|_2.$$

A proof may be given by straightforward computations.

4. Possible implications of the conjecture

Any solutions to Problems 1–3 in the general setting as in the conjecture would certainly be a significant progress in *Matrix Theory*. In what follows we shall provide two examples on their implications to the studies in matrix perturbation theory.

It is conjectured by Mirsky [19] that for normal matrices A and $\tilde{A}$

$$\max_{1 \leqslant i \leqslant n} |\lambda_i - \tilde{\lambda}_{\tau(i)}| \leqslant \|A - \tilde{A}\|_2, \tag{4.1}$$

where τ is a permutation of $1, 2, \ldots, n$. The best result so far in this vein is

$$\max_{1 \leqslant i \leqslant n} |\lambda_i - \lambda_{\tau(i)}| \leqslant c\|A - \tilde{A}\|_2 \tag{4.2}$$

for $c < 2.91$, due to Bhatia et al. [4], and that $c > 1$ due to Holbrook [9].

Could (4.2) be extended to a unitarily invariant norm $|||\cdot|||$ other than $\|\cdot\|_2$ and $\|\cdot\|_{\mathrm{F}}$?

The answer to this is contained in the solution of (3.1)

The second example is from the relative perturbation theory. Let A and $\tilde{A}$ be as in Instance 1 of Section 2.3, and let the λ_i's and $\tilde{\lambda}_j$'s be in ascending order. It is proved in [13] that

$$\max_{1 \leqslant i \leqslant n} \chi(\lambda_i, \tilde{\lambda}_i) \leqslant \|D^{-*} - D\|_2 \equiv \|Q \circ Z_2\|_2, \tag{4.3}$$

$$\sqrt{\sum_{i=1}^{n} \left[\chi(\lambda_i, \tilde{\lambda}_i)\right]^2} \leqslant \|D^{-*} - D\|_F \equiv \|Q \circ Z_2\|_{\mathrm{F}}, \tag{4.4}$$

where $\chi(\xi, \zeta) \stackrel{\mathrm{def}}{=} |\xi - \zeta| / \sqrt{|\xi\zeta|}$. Recently, Li and Mathias [10] showed that

$$\left|\left|\left|\mathrm{diag}\left(\chi(\lambda_1, \tilde{\lambda}_1), \ldots, \chi(\lambda_n, \tilde{\lambda}_n)\right)\right|\right|\right| \leqslant |||D^{-*} - D||| \tag{4.5}$$

for real λ_i's and $\tilde{\lambda}_j$'s. Inequalities Eqs. (4.5) and (2.6) [5] imply

$$\min_{Q \text{ unitary}} |||Q \circ Z_2||| = |||I \circ Z_2||| \quad \text{when all } \lambda_i, \tilde{\lambda}_j \geqslant 0. \tag{4.6}$$

This solves partially a special case of (3.1). Inequality (4.4), leaving out the middle term $\|D^{-*} - D\|_{\mathrm{F}}$, can be extended to complex λ_i's and $\tilde{\lambda}_j$'s as proved in Proposition 3.1. As a part of (3.1), we ask

Could (4.6), perhaps within some constant factors, be extended to real or complex λ_i's and λ_j's for norms other than $\|.\|_{\mathrm{F}}$?

Eq. (4.6) together with Instance 3 of Section 2.3 imply the following proposition.

Proposition 4.1. *Let A and $\tilde{A}$ be positive definite. Then*

$$\left|\left|\left|\mathrm{diag}\left(\chi(\lambda_1, \tilde{\lambda}_1), \ldots, \chi(\lambda_n, \tilde{\lambda}_n)\right)\right|\right|\right| \leqslant \left|\left|\left||A|^{-1/2}(\tilde{A} - A)|\tilde{A}|^{-1/2}\right|\right|\right|.$$

[5] It is proved for nonnegative definite A and $\tilde{A}$. For this reason, Eq. (4.6) requires that all $\lambda_i, \tilde{\lambda}_j \geqslant 0$, even though (4.5) holds for real $\lambda_i, \tilde{\lambda}_j$.

Proof. It is a consequence of Eqs. (2.8) and (4.6). □

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References

[1] T. Ando, R.A. Horn, C.R. Johnson, The singular values of a Hadamard product: a basic inequality, Linear and Multilinear Algebra 87 (1987) 345–365.

[2] J. Barlow, J. Demmel, Computing accurate eigensystems of scaled diagonally dominant matrices, SIAM J. Numer. Anal. 27 (1990) 762–791.

[3] R. Bhatia, Matrix Analysis, Graduate Texts in Mathematics, vol. 169, Springer, New York, 1996.

[4] R. Bhatia, C. Davis, A. McIntosh, Perturbation of spectral subspaces and solution of linear operator equations, Linear Algebra Appl. 52/53 (1983) 45–67.

[5] R. Bhatia, F. Kittaneh, R.-C. Li, Some inequalities for commutators and an application to spectral variation II, Linear and Multilinear Algebra, 43 (1997) 207–219.

[6] R. Bhatia, R.-C. Li, On perturbations of matrix pencils with real spectra, II, Math. Comput. 65 (214) (1996) 637–645.

[7] L. Elsner, S. Friedland, Singular values, doubly stochastic matrices, and applications, Linear Algebra Appl. 220 (1995) 161–169.

[8] A.J. Hoffman, H.W. Wielandt, The variation of the spectrum of a normal matrix, Duke Math. J. 20 (1953) 37–39.

[9] J.A. Holbrook, Spectral variation of normal matrices, Linear Algebra Appl. 174 (1992) 131–144.

[10] C.-K. Li, R. Mathias, On the Lidskii–Mirsky–Wielandt theorem, Manuscript, Department of Mathematics, College of William and Mary, 1996.

[11] R.-C. Li, Norms of certain matrices with applications to variations of the spectra of matrices and matrix pencils, Linear Algebra Appl. 182 (1993) 199–234.

[12] R.-C. Li, On perturbations of matrix pencils with real spectra, Math. Comput. 62 (1994) 231–265.

[13] R.-C Li, Relative perturbation theory: (I) eigenvalue and singular value variations, Technical Report UCB//CSD-94-855, Computer Science Division, Department of EECS, University of California at Berkeley, 1994. Also LAPACK working notes #85 (revised January 1996, available at http://www.netlib.org/lapack/lawns/lawn84.ps).

[14] R.-C. Li, Thoughts about relative perturbation theory for indefinite Hermitian matrices, in preparation.

[15] V.B. Lidskii, The proper values of the sum and product of symmetric matrices, Dokl. Akad. Nauk SSSR 75 (1950) 769–772. [In Russian, Translation by C. Benster, available from the National Translation Center of the Library of Congress.]

[16] R. Mathias, The singular values of the Hadamard product of a positive semidefinite and a skewsymmetric matrix, Linear and Multilinear Algebra 31 (1992) 57–70.

[17] R. Mathias, The Hadamard operator norm of a circulant and applications, SIAM J. Matrix Anal. Appl. 14 (1993) 1152–1167.
[18] R. McEachin, Analyzing specific cases of an operator inequality, Linear Algebra Appl. 208/209 (1994) 343–365.
[19] L. Mirsky, Symmetric gauge functions and unitarily invariant norms, Quart. J. Math. 11 (1960) 50–59.
[20] I. Slapničar, Accurate Symmetric Eigenreduction By a Jacobi Method, Ph.D. Thesis, Fernuniversität, Gesamthochschule, Hagen, Facbereich Mathematik, 1992.
[21] G.W. Stewart, J.-G. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.
[22] J.-G. Sun, On the perturbation of the eigenvalues of a normal matrix (in Chinese), Math. Numer. Sinica 6 (1984) 334–336.
[23] G.A. Watson, Estimating Hadamard operator norms, with applications to triangular truncation, Linear Algebra Appl. 216 (1995) 97–110.
[24] H. Weyl, Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung), Math. Ann. 71 (1912) 441–479.
[25] Z. Zhang, On the perturbation of the eigenvalues of a non-defective matrix (in Chinese), Math. Numer. Sinica 6 (1986) 106–108.