

## ASYMPTOTICALLY OPTIMAL LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL VANDERMONDE MATRIX\*

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**Abstract.** Lower bounds on the condition number  $\min \kappa_p(V)$  of a real Vandermonde matrix  $V$  are established in terms of the dimension  $n$  or  $n$  and the largest absolute value among all nodes that define the Vandermonde matrix. All bounds here are asymptotically sharp, similar to those in Beckermann (*Numer. Math.*, 85 (2000), pp. 553–577), but bounds here are sharper and cover more cases. Also, qualitative behaviors of  $\min \kappa_p(V)$ , as well as nearly optimally conditioned real Vandermonde matrices, as functions of the largest absolute value among all nodes are obtained.

**Key words.** optimal condition number, Vandermonde matrix, Chebyshev polynomials

**AMS subject classifications.** 15A12, 65F35

**DOI.** 10.1137/060652737

**1. Introduction.** Given  $n$  numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$  called *nodes*, the associated *Vandermonde matrix* is defined as

$$(1.1) \quad V \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

It is perhaps one of the best known structured matrices, arising from polynomial interpolation and others [3]. It is invertible if all nodes  $\alpha_j$  are distinct, i.e.,  $\alpha_i \neq \alpha_j$  for  $i \neq j$  (Vandermonde matrices are also notoriously known to be ill-conditioned [13, p. 428], [10]). Its condition number can become arbitrarily large, even for modest  $n$ . This is not surprising because moving one node arbitrarily close to another will make  $V$  arbitrarily close to a singular matrix. Therefore the question of importance about  $V$  is not *how bad a Vandermonde matrix  $V$  can be* but rather *what one can hope for at best from  $V$*  as far as its condition number is concerned.

Although  $V$  is well defined no matter if all or some of  $\alpha_j$  are real or complex, this paper is confined to real Vandermonde matrix  $V$  only, i.e., *all  $\alpha_j$  are real*. Throughout this paper, some notation is *exclusively* reserved for one assignment, including  $V$  and its nodes  $\alpha_j$  and  $\alpha_{\max} \stackrel{\text{def}}{=} \max_j |\alpha_j|$ , along with many others in Table 1.1.  $V_{\text{sym}}$  is one of those  $V$  whose nodes are real symmetric with respect to 0, i.e.,  $\alpha_i + \alpha_{n-i+1} = 0$ .

The major objective of this paper is to bound the  $\ell_p$ -condition number  $\kappa_p(V) = \|V\|_p \|V^{-1}\|_p$  from below in terms of  $n$  or  $n$  and  $\alpha_{\max}$ . Asymptotically optimal bounds have been established. By *asymptotically optimal bounds* we mean those that will give

$$(1.2) \quad \rho \equiv \text{asymptotic speed} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} [\min \kappa_p(V)]^{1/n}$$

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\*Received by the editors February 22, 2006; accepted for publication (in revised form) by J. H. Brandts April 18, 2006; published electronically October 4, 2006. Supported in part by the National Science Foundation CAREER award under grant CCR-9875201 and by the National Science Foundation under grant DMS-0510664.

<http://www.siam.org/journals/simax/28-3/65273.html>

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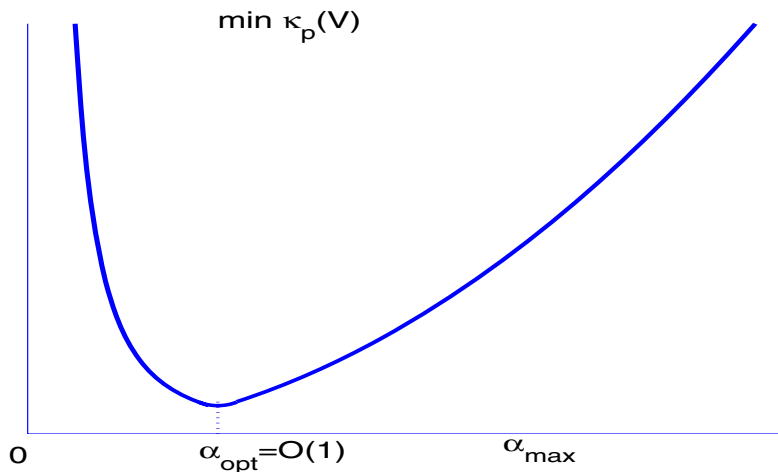


FIG. 1.1. Qualitative behaviors of  $\min_{\alpha_j} \kappa_p(V)$  and  $\min_{\alpha_j \geq 0} \kappa_p(V)$  as  $\alpha_{\max}$  varies.

exactly, where  $\min$  is taken over some prescribed subset or the entire set of real Vandermonde matrices. This is done through establishing bounds like

$$(1.3) \quad c_1 n^{d_1} \leq \min \kappa_p(V) / \rho^n \leq c_2 n^{d_2},$$

written for short as  $\min \kappa_p(V) = \mathcal{O}_n(\rho^n)$ , where  $c_1$ ,  $c_2$ ,  $d_1$ , and  $d_2$  are constants. Particular attention will be given to the case  $p = \infty$ . In a sense, considering  $p = \infty$  is sufficient because of the exponential growth of  $\kappa_\infty(V)$  and because

$$(1.4) \quad n^{-2/p} \kappa_p(V) \leq \kappa_\infty(V) \leq n^{2/p} \kappa_p(V),$$

and thus all  $\kappa_p(V)$  have the same asymptotic speed. Nonetheless, whenever it is possible to establish sharper bounds on  $\kappa_p(V)$  *directly* instead of *indirectly* through bounds on  $\kappa_\infty(V)$  combined with (1.4), we shall go for the sharper ones.

In the past, Gautschi and his coauthor had systematically studied the condition number estimations in [6, 7, 8, 9, 11], where various condition number bounds in terms of the nodes  $\alpha_j$  have been established, as well as bounds in terms of the dimension  $n$  only. In [11] two lower bounds in terms of  $n$  were obtained for positive nodes ( $\alpha_j \geq 0$ ) and real symmetric nodes ( $\alpha_j + \alpha_{n+1-j} = 0$ ). However, bounds in [11] are far from asymptotically optimal. It is Beckermann [2] in 2000 (see also [1]) who obtained asymptotically optimal condition number estimations for all real Vandermonde matrices for the first time.

This paper is based on the technical report [17] which was written before the author came across Beckermann's landmark paper [2]. But we have more detailed and refined analysis and cover more cases, and tighter lower and upper bounds, too. Specifically, the major differences are as follows.

1. We obtain a qualitative plot in Figure 1.1 which shows how  $\min_{\alpha_j} \kappa_p(V)$  and  $\min_{\alpha_j \geq 0} \kappa_p(V)$  subject to a fixed  $\alpha_{\max}$  behave qualitatively as functions of  $\alpha_{\max}$ . What Figure 1.1 says is that initially as  $\alpha_{\max}$  increases, both  $\min_{\alpha_j} \kappa_p(V)$  and  $\min_{\alpha_j \geq 0} \kappa_p(V)$  decrease until at  $\alpha_{\max} = \alpha_{\text{opt}}$  when global minimums of  $\kappa_p(V)$  are reached, and then they start climbing again. Notice  $\alpha_{\text{opt}}$  may be different for the two cases, but  $\alpha_{\text{opt}} = \mathcal{O}(1)$  in both cases.

2. We consider  $\min \kappa_p(V)$  under various constraints: (1) all  $\alpha_j \in \mathbb{R}$ , (2) all  $\alpha_j \geq 0$ , (3)  $\alpha_{\max} = \delta$  or  $\alpha_{\max} \leq \delta$  ( $\delta \leq 1$  or  $\delta > 1$ ), with or without assuming all  $\alpha_j \geq 0$ . Essentially only the first two cases were considered in [2], but not the third one which itself has many subcases and is conceivably important in practice. Suppose that we seek polynomial approximations to functions by interpolation on  $[\alpha, \beta]$ . For the approximations to be any good, most likely, the nodes must be distributed over the entire interval, and in particular  $\min \alpha_j \approx \alpha$  and  $\beta \approx \max_j \alpha_j$ . This will make  $\alpha_{\max} \approx \max\{|\alpha|, |\beta|\}$ .
3. Our lower and upper bounds are tighter: we have  $d_2 - d_1 = 1$  always in (1.3) for  $\min \kappa_p(V)$  over real  $\alpha_j$  or nonnegative  $\alpha_j \geq 0$  (see Remarks 5.1). Although Theorem 4.1 for the same purpose in [2] is for  $p = 2$ , it was remarked that bounds for the  $\ell_p$ -condition number can also be achieved similarly with<sup>1</sup>  $d_2 - d_1 = 2 - 1/p$ . Our bounds for  $p = \infty$  can be even tighter. In fact, for  $p = \infty$  the approach by Beckermann [2] would give  $d_2 - d_1 = 2$ , while our best results in later sections give  $d_2 - d_1 = \sqrt{2}/4$ , and therefore smaller upper over low bound ratios for large  $n$ ; see Tables 5.1 and 6.1.
4. Also for  $p = \infty$ , we have results that give  $d_1 = d_2 = 0$  for  $\alpha_{\max} \leq \delta$  or for  $\alpha_{\max} \leq \delta$  and all  $\alpha_i \geq 0$ , where  $\delta \leq 1$  is given, while no results as such<sup>2</sup> were presented in [2]; see Tables 5.1 and 6.1. Both in [2] and here it is obtained exactly

$$\rho = 1 + \sqrt{2} \quad \text{for } \min_{\alpha_i} \kappa_p(V), \text{ and } \quad \rho = (1 + \sqrt{2})^2 \quad \text{for } \min_{\alpha_i \geq 0} \kappa_p(V).$$

It is worth mentioning that despite its notorious ill-conditioning, there is a way to compute its singular value decomposition to highly relative accuracy [5, 15], and sometimes very accurate solutions to Vandermonde linear systems [3, 13].

Although our study here does not yield optimally conditioned  $V$ , i.e.,  $V$  that achieve  $\min \kappa_p(V)$ , it does, however, conclude what nearly optimally conditioned  $V$  are for various cases:

1. For nodes in  $[-\beta, \beta]$  or for nodes in  $[\alpha, \beta]$  with  $0 = \alpha < \beta$  (also true for  $0 < \alpha$ ; see [17]), subject to  $\alpha_{\max} = \beta$ , a nearly optimally conditioned  $V$  is the one defined with the translated Chebyshev nodes in a slightly larger interval (so that  $\alpha_{\max} = \beta$ ).
2. If all  $\alpha_j$  are allowed to vary freely along the entire real line, a nearly optimally conditioned  $V$  is the one defined with Chebyshev nodes (for which  $\alpha_{\max} = \cos \frac{\pi}{2n} \approx 1$ ).
3. If all  $\alpha_j$  are forced nonnegative but otherwise free, a nearly optimally conditioned  $V$  is the one defined with the translated Chebyshev nodes in the interval  $[\alpha, \beta] = [0, 1]$ .

Those nearly optimally conditioned  $V$  are truly by the word “nearly.” That is to say they are just nearly optimal but may not be optimal, according to those few optimally conditioned  $V$  computed in [8] under the condition that the optimal  $V$  is unique (for any fixed  $n$ ). Beckermann [2, Theorem 4.1] also implied other nearly optimal conditioned  $V$  for the case  $\alpha_i \in \mathbb{R}$  or the case  $\alpha_i \geq 0$ . In particular, Beckermann [1, Theorem 5.9] established that the optimal nodes for  $\min_{\alpha_j \geq 0} \kappa_1(V)$  subject to  $\alpha_{\max} = \gamma$  are  $\alpha_{j+1} = (1 + \cos \frac{j\pi}{n-1})\gamma/2$  for  $0 \leq j \leq n - 1$ .

<sup>1</sup> $V$  in [2] is  $V^T$  here.

<sup>2</sup>As pointed out by an anonymous referee, it is possible to derive asymptotically optimal lower bounds for  $\min \kappa_2(V)$  for  $-1 \leq \alpha \leq \alpha_j < \beta \leq 1$ , using the result about Krylov matrices given in [2, Remarks 3.4 and 3.5], but it was not done explicitly there.

TABLE 1.1  
Special notation.

$V, \alpha_j, \alpha_{\max}$	Vandermonde matrix $V$ , its $n$ nodes, and $\alpha_{\max} = \max_j  \alpha_j $ ;
$V_{\text{sym}}$	$V$ with symmetric nodes: $\alpha_j + \alpha_{n+1-j} = 0$ ;
$[\alpha, \beta]$	the interval that contains all nodes $\alpha_j$ ; see (3.1);
$\omega, \tau$	real parameters and whenever there is $[\alpha, \beta]$ in the context, they are defined by (3.2);
$T_n(t), T_n(x; \omega, \tau)$	Chebyshev polynomial, its translation $T_n(x/\omega + \tau)$ ;
$\theta_j, t_j$	$\theta_j = \frac{2j-1}{2n}\pi$ , and $t_j = \cos \theta_j$ : zeros of $T_n(t)$ , defined by (4.1);
$x_j$	$x_j = \omega(t_j - \tau)$ : zeros of $T_n(x; \omega, \tau)$ , defined by (4.2);
$a_{jn} \equiv a_{jn}(\omega, \tau)$	coefficients of $T_n(x; \omega, \tau)$ defined by (2.4);
$S_{n,p}(\omega, \tau)$	$\left(\sum_{j=0}^n  a_{jn} ^p\right)^{1/p}$ defined by (2.5).

The rest of this paper is organized as follows. A cornerstone of our study is the use of the absolute sums of coefficients of translated Chebyshev polynomials of the first kind. They are defined and computed for a symmetric interval or a nonnegative interval in section 2. Section 3 proves a general lower bound on  $\kappa_p(V)$  with nodes restricted to a given interval  $[\alpha, \beta]$ . Upper bounds on  $\min \kappa_p(V)$  are obtained by the computations for  $V$  with the translated Chebyshev nodes. This is done in section 4. Section 5 derives various asymptotically optimal bounds with or without fixing  $\alpha_{\max}$ , while section 6 considers the case when all  $\alpha_j \geq 0$ . Finally, section 7 draws a few concluding remarks.

**Notation.** We shall stick to the global assignments in Table 1.1, unless otherwise explicitly stated.  $1 \leq p \leq +\infty$  and  $p'$  is defined by  $1/p + 1/p' = 1$ .  $\mathbb{R}$  is the set of real numbers.  $\lceil \xi \rceil$  is the smallest integer that is no less than  $\xi$ . For two sequences of numbers  $a_n$  and  $b_n$ :  $a_n \sim b_n$  means  $a_n/b_n \rightarrow 1$  as  $n \rightarrow +\infty$ ;  $a_n = \mathcal{O}(b_n)$  means  $c_1 \leq a_n/b_n \leq c_2$  for constants  $c_1$  and  $c_2$ ;  $a_n = \mathcal{O}_n(b_n)$  means  $c_1 n^{d_1} \leq a_n/b_n \leq c_2 n^{d_2}$  for constants  $c_1, c_2, d_1$ , and  $d_2$ . In this paper, both  $a_n$  and  $b_n$  grow exponentially in  $n$ , and thus the hidden factors  $n^{d_i}$  in  $a_n = \mathcal{O}_n(b_n)$  are less significant compared to the exponential growth. For notational convenience, by  $\min_j$ , and  $\min_{\alpha_j}$  or  $\min$  over some constraints on  $\alpha_j$ , we mean that  $j$  runs from 1 to  $n$ .

**2. Coefficients of Chebyshev polynomials.** The  $n$ th Chebyshev polynomial of the first kind is

$$(2.1) \quad T_n(t) = \cos(n \arccos t) \quad \text{for } |t| \leq 1,$$

$$(2.2) \quad = \frac{1}{2} \left(t + \sqrt{t^2 - 1}\right)^n + \frac{1}{2} \left(t - \sqrt{t^2 - 1}\right)^n \quad \text{for } |t| \geq 1.$$

Given real parameters  $\omega$  and  $\tau$ , the  $n$ th translated Chebyshev polynomial is defined by

$$(2.3) \quad T_n(x; \omega, \tau) \stackrel{\text{def}}{=} T_n(x/\omega + \tau).$$

Here and in the rest of this paper  $T_n$  is overloaded with distinctions according to its argument(s). It can be seen that  $T_n(x; \omega, \tau)$  is a polynomial of degree  $n$  in  $x$ . Write

$$(2.4) \quad T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1n}x^{n-1} + \cdots + a_{1n}x + a_{0n},$$

where  $a_{jn} \equiv a_{jn}(\omega, \tau)$  are functions of  $\omega$  and  $\tau$  which, wherever referenced, are all either clear from the context or explicitly stated. Define

$$(2.5) \quad S_{n,p}(\omega, \tau) \stackrel{\text{def}}{=} \left(\sum_{j=0}^n |a_{jn}|^p\right)^{1/p},$$

a function of  $\omega$  and  $\tau$ , too. Successful computation of  $S_{n,p}(\omega, \tau)$  is crucial to our later development. But, in its generality, an explicit formula for  $S_{n,p}(\omega, \tau)$  is hard to find. Nevertheless, we still manage to find formulas for  $S_{n,1}(\omega, \tau)$  for two different cases  $\tau = 0$  and  $|\tau| \geq 1$ .

THEOREM 2.1.

1.  $S_{n,1}(\omega, 0) = |T_n(\iota/\omega)|$ , where  $\iota = \sqrt{-1}$  is the imaginary unit. Thus<sup>3</sup>

$$S_{n,1}(\omega, 0) = |T_n(\iota/\omega)| \sim \frac{1}{2} \left( \frac{1}{|\omega|} + \sqrt{1 + \frac{1}{|\omega|^2}} \right)^n.$$

2. For  $|\tau| \geq 1$ ,

$$S_{n,1}(\omega, \tau) = T_n \left( \frac{1}{|\omega|} + |\tau| \right) \sim \frac{1}{2} \left[ \left( \frac{1}{|\omega|} + |\tau| \right) + \sqrt{\left( \frac{1}{|\omega|} + |\tau| \right)^2 - 1} \right]^n.$$

For any other  $p$ , we may use the inequalities

$$(2.6) \quad (n + 1)^{-1/p'} S_{n,1}(\omega, \tau) \leq S_{n,p}(\omega, \tau) \leq S_{n,1}(\omega, \tau),$$

$$(2.7) \quad [(n + 1)/2]^{-1/p'} S_{n,1}(\omega, 0) \leq S_{n,p}(\omega, 0) \leq S_{n,1}(\omega, 0),$$

to get bounds on  $S_{n,p}$ . Both (2.6) and (2.7) can be proved by using Hölder inequality

$$(2.8) \quad \sum_{j=1}^m |\xi_j \zeta_j| \leq \left( \sum_{j=1}^m |\xi_j|^p \right)^{1/p} \left( \sum_{j=1}^m |\zeta_j|^{p'} \right)^{1/p'}$$

and the fact that  $\left( \sum_{j=1}^m |\xi_j|^p \right)^{1/p}$  is decreasing in  $p$  [12, Lemma 1.1].

THEOREM 2.2. Let  $a_n > 0$  and  $\delta = a_n^{1/n}$ . If  $a_n \sim cn^\mu$  for constant  $c > 0$  and  $\mu$ , then

$$(2.9) \quad 2S_{n,1}(\delta, 0) \sim \frac{(1 + \sqrt{2})^n}{(cn^\mu)^{1/\sqrt{2}}}, \quad 2S_{n,1}(\delta/2, 1) \sim \frac{(1 + \sqrt{2})^{2n}}{(cn^\mu)^{1/\sqrt{2}}}.$$

*Proof.* We will prove more general results: if  $(\ln a_n)/n \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$(2.10) \quad 2S_{n,1}(\delta, 0) \sim \frac{(1 + \sqrt{2})^n}{a_n^{1/\sqrt{2}}}, \quad 2S_{n,1}(\delta/2, 1) \sim \frac{(1 + \sqrt{2})^{2n}}{a_n^{1/\sqrt{2}}}.$$

Since  $a_n \sim cn^\mu$  implies  $(\ln a_n)/n \rightarrow 0$  as  $n \rightarrow \infty$ , we have (2.9) from (2.10).

The second asymptotical relation in (2.10) follows from the first one because

$$S_{n,1}(\delta/2, 1) = |T_n(1 + 2\delta^{-1})| = |T_{2n}(\iota/\sqrt{\delta})| = S_{2n}(\sqrt{\delta}, 0)$$

upon noticing that  $T_n(2t^2 - 1) = T_{2n}(t)$ , and  $\sqrt{\delta} = a_n^{1/(2n)}$ . We shall now prove the first relation in (2.10). Notice that  $\ln \delta^{-1} \sim -(\ln a_n)/n \equiv \epsilon \Rightarrow \delta^{-1} \sim 1 + \epsilon$  to get

$$\delta^{-1} + \sqrt{1 + \delta^{-2}} \sim 1 + \epsilon + \sqrt{2}(1 + \epsilon/2) = (1 + \sqrt{2})(1 + \epsilon/\sqrt{2}).$$

<sup>3</sup>Going through the proofs in [17], one may see that Theorem 2.1 is valid for complex  $\omega$  as well. But for the purpose of this paper,  $\omega$  is real.

Therefore

$$\begin{aligned} \ln [2S_{n,1}(\delta, 0)] &\sim n \ln \left( \delta^{-1} + \sqrt{1 + \delta^{-2}} \right) \\ &\sim n \left[ \ln(1 + \sqrt{2}) + \epsilon/\sqrt{2} \right] \\ &= \ln(1 + \sqrt{2})^n - (\ln a_n)/\sqrt{2}, \end{aligned}$$

which gives the first asymptotical relation in (2.10).  $\square$

**3. A general lower bound on condition numbers of Vandermonde matrices.** Given  $1 \leq p \leq \infty$ , the  $\ell_p$ -norm of vector  $u = (\mu_1, \mu_2, \dots, \mu_n)^T$  is defined as  $\|u\|_p = \left( \sum_{j=1}^n |\mu_j|^p \right)^{1/p}$ , and  $\|u\|_\infty = \lim_{p \rightarrow \infty} \|u\|_p = \max_j |\mu_j|$ . The associated  $\ell_p$ -operator norm of an  $m \times n$  matrix  $A$  is defined as  $\|A\|_p = \max_{u \neq 0} \|Au\|_p / \|u\|_p$ . It can be proved that  $\|A\|_p = \|A^T\|_{p'}$ , upon noticing that

$$\|A\|_p = \max_{u \neq 0, v \neq 0} \frac{|v^T Au|}{\|v\|_{p'} \|u\|_p},$$

where  $1/p + 1/p' = 1$  (see also [16]). Superscript “ $\cdot^T$ ” takes the transpose of a matrix or a vector.

We shall start by establishing a general lower bound on  $\kappa_p(V)$  for

$$(3.1) \quad \alpha \leq \min_j \alpha_j \leq \max_j \alpha_j \leq \beta.$$

The case  $\alpha = \beta$  is of no interest because then  $V$  is of rank 1 and thus  $\kappa_p(V) = +\infty$  (unless  $n = 1$ ). There are many ways to realize (3.1), and it is tempting to always let  $\alpha = \min_j \alpha_j$  and  $\beta = \max_j \alpha_j$ , but that may not always be possible for theorems that require  $-\alpha = \beta$ . Recall  $\omega$  and  $\tau$  defined by (3.2), and let  $\alpha_{\max} \stackrel{\text{def}}{=} \max_j |\alpha_j|$ . Set

$$(3.2) \quad \omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = -\frac{\beta + \alpha}{\beta - \alpha}.$$

The linear transformation  $t = x/\omega + \tau$  maps  $x \in [\alpha, \beta]$  one-to-one and onto  $t \in [-1, 1]$ .

LEMMA 3.1.

$$(3.3) \quad \max\{n, n\alpha_{\max}^{n-1}\} \geq \|V\|_p \geq \max\{n^{1/p'}, \alpha_{\max}^{n-1}\},$$

$$(3.4) \quad \|V^{-1}\|_p \geq \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}} \geq \begin{cases} \left( \frac{n}{\lceil n/2 \rceil} \right)^{1/p} \frac{S_{n-1,1}(\omega, 0)}{n} & \text{if } -\alpha = \beta, \\ \frac{S_{n-1,1}(\omega, \tau)}{n} & \text{always.} \end{cases}$$

*Proof.* Let  $e_j$  be the  $j$ th column of the  $n \times n$  identity matrix. Then

$$\|V\|_p = \|V^T\|_{p'} \geq \begin{cases} \|V^T e_1\|_{p'} = n^{1/p'}, \\ \|V^T e_n\|_{p'} \geq \alpha_{\max}^{n-1}. \end{cases}$$

This yields the second inequality in (3.3). The known formulas for  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  [4, page 22] yield  $\|V\|_1, \|V\|_\infty \leq \max\{n, n\alpha_{\max}^{n-1}\}$  and now use [14, page 29]

$$\|V\|_p \leq \|V\|_\infty^{1/p'} \|V\|_1^{1/p}$$

to arrive at the first inequality in (3.3).

We now show (3.4). Let  $v$  be the vector of the coefficients of  $T_{n-1}(x; \omega, \tau) \equiv T_{n-1}(x/\omega + \tau)$ , i.e.,  $v = (a_{0,n-1} \ a_{1,n-1} \ \cdots \ a_{n-1,n-1})^T$ . Then

$$V^T v = (T_{n-1}(\alpha_1/\omega + \tau) \ T_{n-1}(\alpha_2/\omega + \tau) \ \cdots \ T_{n-1}(\alpha_n/\omega + \tau))^T,$$

which yields  $\|V^T v\|_{p'} \leq n^{1/p'}$  because  $|T_{n-1}(x/\omega + \tau)| \leq 1$  for  $x \in [\alpha, \beta]$ . We therefore have

$$\|V^{-1}\|_p = \|V^{-T}\|_{p'} \geq \frac{\|v\|_{p'}}{\|V^T v\|_{p'}} \geq \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}.$$

This is the first inequality in (3.4). Use it, together with (2.6) and (2.7), to get the second inequality.  $\square$

**THEOREM 3.2.**

$$(3.5) \quad \kappa_p(V) \geq \max \left\{ S_{n-1,p'}(\omega, \tau), \frac{\alpha_{\max}^{n-1} S_{n-1,p'}(\omega, \tau)}{n^{1/p'}} \right\}$$

$$(3.6) \quad \geq \max \left\{ \frac{S_{n-1,1}(\omega, \tau)}{n^{1/p}}, \frac{\alpha_{\max}^{n-1} S_{n-1,1}(\omega, \tau)}{n} \right\}.$$

*Proof.* This theorem is an immediate consequence of Lemma 3.1.  $\square$

This is the most general theorem of this paper for a lower bound on  $\kappa_p(V)$ . It is its various applications combined with results in section 4 that lead to many interesting asymptotically optimal lower bounds. There are at least two different ways to apply Theorem 3.2 to any given  $V$ :

1. Take  $\alpha = \min_j \alpha_j$  and  $\beta = \max_j \alpha_j$  and then compute the right-hand side of (3.5) or (3.6). But unless  $\alpha \geq 0$  or  $-\alpha = \beta$ , we may have to compute  $S_{n-1,1}(\omega, \tau)$  by its definition (2.5) because no explicit formula has yet been found. In this case, both  $\alpha$  and  $\beta$  are nodes of  $V$ .
2. Take  $-\alpha = \beta = \alpha_{\max}$  (and thus  $\omega = \alpha_{\max}$  and  $\tau = 0$ ) and then use the explicit formula for  $S_{n-1,1}(\alpha_{\max}, 0)$  to compute the right-hand side of (3.6). In this case, one of  $\alpha$  and  $\beta$  is guaranteed to be a node for  $V$ .

*Remark 3.1.* The lower bounds in [2] were essentially obtained as follows. Let  $\omega = \eta \alpha_{\max}$ . It follows from  $\|V\|_p \geq \max_j \|V e_j\|_p = \left( \sum_{j=0}^{n-1} \alpha_{\max}^{jp} \right)^{1/p}$  and (3.4) that

$$n^{1/p'} \kappa_p(V) \geq \left( \sum_{j=0}^{n-1} \alpha_{\max}^{jp} \right)^{1/p} S_{n-1,p'}(\omega, \tau).$$

But  $S_{n-1,p'}(\omega, \tau) = \left( \sum_j |\omega^{-j} a_{j,n-1}(1, \tau)|^{p'} \right)^{1/p'}$ . By Hölder inequality (2.8), we have  $n^{1/p'} \kappa_p(V) \geq \sum_j \eta^{-j} |a_{j,n-1}(1, \tau)| = S_{n-1,1}(\eta, \tau)$  which gives

$$(3.7) \quad \kappa_p(V) \geq S_{n-1,1}(\eta, \tau) / n^{1/p'}.$$

In the case of [2],  $p = p' = 2$ , either  $\eta = 1$  and  $\tau = 0$  or  $\eta = 1/2$  and  $\tau = -1$ . This is a pretty decent bound, but it partially collapses the interval information, unlike (3.5) and (3.6) which form the basis for us to eventually arrive at the qualitative behaviors in Figure 1.1.

**4. Vandermonde matrices with translated Chebyshev nodes.** The zeros of  $T_n(t)$  are called

$$(4.1) \quad \text{Chebyshev nodes: } t_j = \cos \theta_j, \theta_j = \frac{2j-1}{2n}\pi \quad (1 \leq j \leq n),$$

and the zeros of the translated Chebyshev polynomial  $T_n(x; \omega, \tau)$  as in (2.3) are called

$$(4.2) \quad \text{translated Chebyshev nodes: } x_j = \omega(t_j - \tau) \quad (1 \leq j \leq n).$$

This section, inspired by Gautschi [7], computes  $\kappa_\infty(V)$  for  $V$  with the translated Chebyshev nodes for the case  $-\alpha = \beta$  and the case  $0 \leq \alpha < \beta$ . But we are still unsure how to deal with the general case  $\alpha < 0 < \beta$ ,  $-\alpha \neq \beta$ . Recall  $\omega$  and  $\tau$  defined in (3.2).

First we compute  $\|V\|_\infty$  for  $V$  with  $\alpha_j = x_j = \omega(\cos \theta_j - \tau)$ . This is relatively easy. By [8, Theorem 2.1],

$$(4.3) \quad \|V\|_\infty = \max \left\{ n, \sum_{j=1}^n |\alpha_j|^{n-1} \right\} = \max \{ n, \omega^{n-1} \Lambda_n(\tau) \},$$

where

$$(4.4) \quad \Lambda_n(\tau) \stackrel{\text{def}}{=} \sum_{j=1}^n |\cos \theta_j - \tau|^{n-1}.$$

It can be seen that  $\Lambda_n(-\tau) = \Lambda_n(\tau)$ . In [17, Appendix B], the following asymptotical behaviors

$$(4.5) \quad \Lambda_n(0) \sim \sqrt{\frac{2n}{\pi}}, \quad \Lambda_n(1) \sim \sqrt{\frac{n}{\pi}} 2^{n-1}.$$

were obtained. With (4.5), we have the following theorem.

**THEOREM 4.1.** *Let  $\alpha_j = x_j$  ( $1 \leq j \leq n$ ) as in (4.2) with (3.2). Then*

$$\begin{aligned} \|V\|_\infty &\sim \max \left\{ n, \sqrt{\frac{2n}{\pi}} \omega^{n-1} \right\} \sim \max \left\{ n, \sqrt{\frac{2n}{\pi}} \alpha_{\max}^{n-1} \right\} \quad \text{for } -\alpha = \beta > 0, \\ \|V\|_\infty &\sim \max \left\{ n, \sqrt{\frac{n}{\pi}} \beta^{n-1} \right\} \sim \max \left\{ n, \sqrt{\frac{n}{\pi}} \alpha_{\max}^{n-1} \right\} \quad \text{for } 0 = \alpha < \beta. \end{aligned}$$

In both cases  $-\alpha = \beta$  or  $0 = \alpha < \beta$ ,  $\sum_{j=1}^n |x_j|^{n-1} = \mathcal{O}(\sqrt{n} \alpha_{\max}^{n-1})$ . But will this also be true for arbitrary interval  $[\alpha, \beta]$ ? We do not know.

We now estimate  $\|V^{-1}\|_\infty$  with translated Chebyshev nodes. It is made possible by Gautschi's formulas for  $\|V^{-1}\|_\infty$  for  $V$  with symmetric nodes or with nonnegative nodes [7]. We have the following theorem.

**THEOREM 4.2** (see [17]). *Let  $\alpha_j = x_j$  ( $1 \leq j \leq n$ ) as in (4.2) with (3.2). Then*

$$(4.6) \quad \omega \min \left\{ 1, \frac{1 + \omega}{1 + \omega^2} \right\} \frac{1}{n} \leq \frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega, 0)} \leq \omega \max \left\{ 1, \frac{1 + \omega}{1 + \omega^2} \right\} \frac{3^{3/4}}{2n} \quad \text{for } -\alpha = \beta > 0,$$

$$(4.7) \quad \frac{\frac{\beta - \alpha}{2} \cos \frac{\pi}{2n}}{n \left( 1 + \frac{\beta + \alpha}{2} \right)} \leq \frac{\|V^{-1}\|_\infty}{S_{n,1}(\omega, \tau)} \leq \frac{\beta - \alpha}{2n \sqrt{(1 + \beta)(1 + \alpha)}} \quad \text{for } 0 = \alpha < \beta,$$

where the first inequality in (4.6) is valid for  $n \geq 3$  only. Note also that  $V$  is a  $V_{\text{sym}}$  when  $-\alpha = \beta$ .

Theorem 4.2 says that for  $V$  with translated Chebyshev nodes on  $[\alpha, \beta]$ , if  $-\alpha = \beta$  or  $0 \leq \alpha < \beta$  or  $\alpha < \beta \leq 0$ , then

$$(4.8) \quad \frac{n\|V^{-1}\|_\infty}{S_{n,1}(\omega, \tau)} = \mathcal{O}(1).$$

(The case  $[\alpha, \beta]$  for  $\alpha < \beta \leq 0$  can be turned into  $[-\beta, -\alpha]$ , a case that is covered by Theorem 4.2.) But what happens when  $\alpha < 0 < \beta$  and  $-\alpha \neq \beta$ ? Is (4.8) still true? We conjecture it would be, but do not have any proof for now.

**THEOREM 4.3.** *Let  $\alpha_j = x_j$  ( $1 \leq j \leq n$ ) as in (4.2) with (3.2). Then*

$$\begin{aligned} \min_{-\alpha=\beta} \kappa_\infty(V) &\leq \frac{3^{3/4}}{2} \beta_{\text{opt}} S_{n,1}(\beta_{\text{opt}}, 0) \sim \frac{3^{3/4}}{2} \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \frac{(1 + \sqrt{2})^n}{2n^{\sqrt{2}/4}} \text{ for } -\alpha = \beta > 0, \\ \min_{0=\alpha<\beta} \kappa_\infty(V) &\leq \frac{\beta_{\text{opt}}^+}{2\sqrt{1 + \beta_{\text{opt}}^+}} S_{n,1}(\beta_{\text{opt}}^+/2, 1) \sim \frac{\sqrt{2}(1 + \sqrt{2})^{2n}}{4(n\pi)^{\sqrt{2}/4}} \text{ for } 0 = \alpha < \beta, \end{aligned}$$

where  $\beta_{\text{opt}} \equiv \omega_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)} \sim 1$  and  $\beta_{\text{opt}}^+/2 \equiv \omega_{\text{opt}}^+ = (n/\Lambda_n(1))^{1/(n-1)} \sim 1/2$ .

*Proof.* A proof can be found in [17], and the asymptotic relations can be achieved by applying Theorem 2.2.  $\square$

**5. Condition numbers for  $V$  with  $\alpha_i \in [\alpha, \beta]$  and  $-\alpha = \beta$ .** In this section, we shall establish lower and upper bounds on

min $\kappa_p(V)$ subject to $\dots$	
$\alpha_j \in \mathbb{R}$	$\alpha_{\text{max}} \leq \delta$ or $\alpha_{\text{max}} = \delta$
Theorems 5.3, 5.3'	Theorems 5.4, 5.4'

where for each type of minimization we have two versions of bounds—one for all  $p$  (Theorems 5.3 and 5.4) and one just for  $p = \infty$  (Theorems 5.3' and 5.4', sharper at least asymptotically than by just setting  $p = \infty$  in the other version).

**LEMMA 5.1.**

1. In  $|\omega|$ ,  $S_{n,p}(\omega, \tau)$  is decreasing, while  $|\omega|^n S_{n,p}(\omega, \tau)$  is increasing.
2.  $\omega S_{n,1}(\omega, 0)$  is decreasing in  $\omega$  if  $\omega \leq \max\{\sqrt{n-1}, \sqrt{2}\}$  or  $n$  is odd.

*Proof.* For item 1, we notice that  $[S_{n,p}(\omega, \tau)]^p$  is a polynomial in  $|\omega|^{-p}$  while  $[|\omega|^n S_{n,p}(\omega, \tau)]^p$  is a polynomial in  $|\omega|^p$ . Item 2 is proved in [17].  $\square$

**LEMMA 5.2.**

$$(5.1) \quad \kappa_p(V) \geq \max \left\{ S_{n-1,p'}(\alpha_{\text{max}}, 0), \frac{\alpha_{\text{max}}^{n-1} S_{n-1,p'}(\alpha_{\text{max}}, 0)}{n^{1/p'}} \right\}$$

$$(5.2) \quad = \begin{cases} S_{n-1,p'}(\alpha_{\text{max}}, 0) & \text{if } \alpha_{\text{max}} \leq n^{1/[p'(n-1)]}, \\ \alpha_{\text{max}}^{n-1} S_{n-1,p'}(\alpha_{\text{max}}, 0) / n^{1/p'} & \text{if } \alpha_{\text{max}} > n^{1/[p'(n-1)]}. \end{cases}$$

*Proof.* Apply Theorem 3.2 to the case  $-\alpha = \beta = \alpha_{\text{max}}$  (and thus  $\omega = \alpha_{\text{max}}$  and  $\tau = 0$ ) to get (5.1). By Lemma 5.1, the first quantity within  $\max\{\dots\}$  in (5.1) is decreasing in  $\alpha_{\text{max}}$ , while the second one is increasing in  $\alpha_{\text{max}}$ . Therefore the right-hand side of (5.1) achieves its minimum when the two are equal, i.e.,  $n^{1/p'} = \alpha_{\text{max}}^{n-1}$ , which yields (5.2).  $\square$

THEOREM 5.3.

$$(5.3) \quad S_{n-1,p'}(n^{1/[p'(n-1)]}, 0) \leq \min_{\alpha_j} \kappa_p(V) \leq \min_{\alpha_j} \kappa_p(V_{\text{sym}}) \leq n^{1/p} \frac{3^{3/4}}{2} S_{n,1}(1, 0).$$

*Proof.* The right-hand side of (5.1), as a function of  $\alpha_{\max}$ , achieves its minimum at  $\alpha_{\max} = n^{1/[p'(n-1)]}$ . That gives the first inequality. The second inequality is true because  $\{V_{\text{sym}}\}$  is a subset of all Vandermonde matrices. We now prove the third one. To this end, consider  $V = V_{\text{sym}}$  with Chebyshev nodes  $t_j$  as in (4.1). Then  $\|V\|_p \leq n$  by (3.3). Apply Theorem 4.2 to the case  $-\alpha = \beta = 1 = \omega$  to get  $\|V^{-1}\|_{\infty} \leq \frac{3^{3/4}}{2} \cdot n^{-1} S_{n,1}(1, 0)$  and then to get

$$\|V^{-1}\|_p \leq n^{1/p} \|V^{-1}\|_{\infty} \leq n^{1/p} \frac{3^{3/4}}{2} \cdot n^{-1} S_{n,1}(1, 0).$$

So for this  $V_{\text{sym}}$ ,  $\kappa_p(V_{\text{sym}}) \leq n^{1/p} \frac{3^{3/4}}{2} \cdot S_{n,1}(1, 0)$ , as needed.  $\square$

We include  $\min_{\alpha_j} \kappa_p(V_{\text{sym}})$  in (5.3) mainly because Vandermonde matrices with symmetric nodes were heavily studied by Gautschi [7, 8] and Gautschi and Ingese [11]. Moreover, assuming that the optimally conditioned  $V$  is unique, Gautschi [8] showed that the optimally conditioned  $V$  must have symmetric nodes.

*Remark 5.1.* Upon using (3.7) with  $\eta = 1$  and  $\tau = 0$ , we have

$$(5.4) \quad S_{n-1,1}(1, 0)/n^{1/p'} \leq \min_{\alpha_j} \kappa_p(V) \leq n^{1/p} \frac{3^{3/4}}{2} S_{n,1}(1, 0),$$

which differs from (5.3) only in the leftmost inequalities. The left inequality in (5.4) is due to [2] for  $p = 2$ , and it is less sharp than the left inequality in (5.3) at least for  $p = \infty$  because, by Theorem 2.2,

$$S_{n-1,1}(n^{1/(n-1)}, 0) \sim \frac{(1 + \sqrt{2})^{n-1}}{2n^{1/\sqrt{2}}}, \quad \frac{S_{n-1,1}(1, 0)}{n} \sim \frac{(1 + \sqrt{2})^{n-1}}{2n}.$$

Even so, for any  $1 \leq p \leq \infty$ , the ratio of the upper bound in (5.4) over the lower bound is  $n \frac{3^{3/4}}{2}$ , and it gives  $\min_{\alpha_j} \kappa_p(V)$  a lower and upper bound like (1.3) with  $d_2 - d_1 = 1$ , while similar lower and upper bounds in [2] for the same purpose are with  $d_2 - d_1 = 2 - 1/p$ .

The third inequality in (5.3) was proved by simply picking a special  $V$  with Chebyshev nodes. This turns out to be good enough, as we shall see later, in yielding the correct asymptotic speed in our notation  $\mathcal{O}_n$ , but it does not produce the best possible factor  $n^d$  hidden in the notation. For  $p = \infty$ , however, a tighter upper bound is possible by using the  $V$  with the translated Chebyshev nodes in  $[-\beta_{\text{opt}}, \beta_{\text{opt}}]$ , where  $\beta_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)}$  as in Theorem 4.3. Of course, one may use this  $V$  for all  $p$ , but doing so will not only lead to a more complicated bound but also the resulted bound may not be much better due to more complicated estimation of  $\|V\|_p$ . For this reason, we shall state a sharper version of Theorem 5.3 for  $p = \infty$  only as a consequence of Theorem 4.3. The upper bound in (5.5) is sharper because of  $1 \sim \beta_{\text{opt}} > 1$  and item 2 in Lemma 5.1.

THEOREM 5.3'. Let  $\beta_{\text{opt}} = (n/\Lambda_n(0))^{1/(n-1)}$  with  $\Lambda_n(1)$  defined by (4.4). Then

$$(5.5) \quad S_{n-1,1}(n^{1/(n-1)}, 0) \leq \min_{\alpha_j} \kappa_{\infty}(V) \leq \min_{\alpha_j} \kappa_{\infty}(V_{\text{sym}}) \leq \frac{3^{3/4}}{2} \beta_{\text{opt}} S_{n,1}(\beta_{\text{opt}}, 0).$$

In what follows, we shall establish theorems that are in the same spirit as Theorems 5.3 and 5.3', but with  $\alpha_{\max}$  subject to a constraint.

THEOREM 5.4. *Let  $\delta > 0$  and set  $\delta' = \delta / \cos \frac{\pi}{2n}$ . If  $\delta \leq 1$ , then*

$$(5.6) S_{n-1,p'}(\delta, 0) \leq \min_{\alpha_{\max} \leq \delta} \kappa_p(V) \leq \min_{\alpha_{\max} = \delta} \kappa_p(V) \leq n^{1/p} \frac{(\sqrt{2} + 1)3^{3/4}}{4} \delta' S_{n,1}(\delta', 0).$$

If  $\delta > 1$ , then

$$(5.7) \frac{\delta^{n-1} S_{n-1,p'}(\delta, 0)}{n^{p'}} \leq \min_{\alpha_{\max} = \delta} \kappa_p(V) \leq n^{1/p} \frac{3^{3/4} [\cos \frac{\pi}{2n}]^{n-1}}{2} (\delta')^n S_{n,1}(\delta', 0),$$

$$(5.8) S_{n-1,p'}(n^{1/[p'(n-1)]}, 0) \leq \min_{\alpha_{\max} \leq \delta} \kappa_p(V) \leq n^{1/p} \frac{3^{3/4}}{2} \cdot S_{n,1}(1, 0).$$

Inequalities (5.6), (5.7), and (5.8) remain valid with  $V$  replaced by  $V_{\text{sym}}$ .

Proof. (1) Observe that  $\{V : \alpha_{\max} = \delta\} \subset \{V : \alpha_{\max} \leq \delta\}$  to get the middle inequality in (5.6).

(2) Lemma 3.1 also implies that

$$(5.9) \kappa_p(V) \geq \begin{cases} S_{n-1,p'}(\omega, \tau) & \text{if } \alpha_{\max} \leq 1, \\ \alpha_{\max}^{n-1} S_{n-1,p'}(\omega, \tau) / n^{1/p'} & \text{if } \alpha_{\max} > 1 \end{cases}$$

upon noticing that  $\|V\|_p \geq n^{1/p'}$  if  $\alpha_{\max} \leq 1$ , and  $\|V\|_p \geq \alpha_{\max}^{n-1}$  if  $\alpha_{\max} > 1$ . Apply it to the case  $-\alpha = \beta = \alpha_{\max} \leq \delta \leq 1$  (and thus  $\omega = \alpha_{\max}$  and  $\tau = 0$ ) to obtain  $\kappa_p(V) \geq S_{n-1,p'}(\alpha_{\max}, 0) \geq S_{n-1,p'}(\delta, 0)$  by Lemma 5.1. This gives the first inequality in (5.6).

(3) Apply (5.9) to the case  $-\alpha = \beta = \delta = \alpha_{\max}$  to obtain the first inequality in (5.7).

(4) Take  $-\alpha = \beta = \delta / \cos \frac{\pi}{2n} = \delta'$ , and  $\alpha_j = x_j$  ( $1 \leq j \leq n$ ), the translated Chebyshev nodes as in (4.2). Then

$$\tau = 0, \quad \alpha_{\max} = \max |\alpha_j| = \beta \cos \frac{\pi}{2n} = \delta, \quad \delta \leq \omega = \beta = \delta'.$$

Theorem 4.2 says that for the  $V$  with those nodes

$$\frac{\|V^{-1}\|_{\infty}}{S_{n,1}(\omega, 0)} \leq \omega \max \left\{ 1, \frac{1 + \omega}{1 + \omega^2} \right\} \frac{3^{3/4}}{2n} \leq \begin{cases} \delta' \frac{(\sqrt{2} + 1)3^{3/4}}{4n} & \text{if } \delta \leq 1, \\ \delta' \frac{3^{3/4}}{2n} & \text{if } \delta \geq 1, \end{cases}$$

where we have used

$$(5.10) \max_{\omega > 0} \left\{ 1, \frac{1 + \omega}{1 + \omega^2} \right\} = \frac{1 + \omega}{1 + \omega^2} \Big|_{\omega = \sqrt{2}-1} = \frac{\sqrt{2} + 1}{2}, \quad \max_{\omega \geq 1} \left\{ 1, \frac{1 + \omega}{1 + \omega^2} \right\} = 1.$$

Now employ  $\|V\|_p \leq n$  if  $\delta \leq 1$ ,  $\|V\|_p \leq n\delta^{n-1}$  if  $\delta \geq 1$ , and  $\|V^{-1}\|_p \leq n^{1/p} \|V^{-1}\|_{\infty}$  to get the last inequalities in (5.6) and in (5.7).

(5) A proof of (5.8) can be done in the same way as for Theorem 5.3.

(6) Finally, when  $V$  is replaced by  $V_{\text{sym}}$ , the first inequalities in (5.6), (5.7), and (5.8) still hold. The middle inequality in (5.6) also remains valid. The last inequalities in (5.6), (5.7), and (5.8) hold because they all were proved by bounding some  $\kappa_p(V_{\text{sym}})$ .  $\square$

There are stronger versions of (5.7) and (5.8) for  $p = \infty$ , too, just as we did for Theorem 5.3.

THEOREM 5.4'. *Let  $\delta > 1$  and set  $\delta' = \delta / \cos \frac{\pi}{2n}$ . Then*

$$(5.11) \quad \frac{\delta^{n-1} S_{n-1,1}(\delta, 0)}{n} \leq \min_{\alpha_{\max}=\delta} \kappa_{\infty}(V) \leq \frac{3^{3/4} \max\{n, \Lambda_n(0)(\delta')^{n-1}\}}{2} \delta' S_{n,1}(\delta', 0),$$

$$(5.12) \quad S_{n-1,1}(n^{1/(n-1)}, 0) \leq \min_{\alpha_{\max} \leq \delta} \kappa_{\infty}(V) \leq \frac{3^{3/4}}{2} \omega_1 S_{n,1}(\omega_1, 0),$$

where  $\omega_1 = \min\{\delta', (n/\Lambda_n(0))^{1/(n-1)}\}$ . It can be seen that  $\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}$  for  $n$  sufficiently large. Inequalities (5.11) and (5.12) remain valid with  $V$  replaced by  $V_{\text{sym}}$ .

*Proof.* Only the second inequalities in (5.11) and (5.12) need proofs. For (5.11), it follows from the proof of Theorem 5.4, upon using  $\|V\|_{\infty} = \max\{n, \omega^{n-1} \Lambda_n(0)\}$  which for large  $n$  is proportional to  $\sqrt{n} \delta^{n-1}$ , better than  $\|V\|_{\infty} \leq n \delta^{n-1}$ . The second inequality in (5.12) is obtained by minimizing

$$\frac{3^{3/4} \max\{n \alpha'_{\max} S_{n,1}(\alpha'_{\max}, 0), \Lambda_n(0)(\alpha'_{\max})^n S_{n,1}(\alpha'_{\max}, 0)\}}{2}$$

subject to  $\alpha_{\max} \leq \delta$ , where  $\alpha'_{\max} = \alpha_{\max} / \cos \frac{\pi}{2n}$ . This minimization is solved by noticing Lemma 5.1, which says the first quantity within  $\max\{\dots\}$  is decreasing in  $\alpha_{\max}$  when  $\alpha_{\max} \leq \max\{\sqrt{n-1}, \sqrt{2}\}$ , while the second one is increasing in  $\alpha_{\max}$ . That  $\omega_1 = (n/\Lambda_n(0))^{1/(n-1)}$  for  $n$  sufficiently large is due to  $(n/\Lambda_n(0))^{1/(n-1)} \sim (2\pi/n)^{1/[2(n-1)]} \sim 1$ .  $\square$

We shall now investigate the tightness of the upper and the lower bounds we have established so far, as well as the asymptotical speeds of  $\kappa_p(V)$  minimized over a certain set of Vandermonde matrices. For this purpose, Li [17] obtained Table 5.1 for the asymptotical behaviors of the ratios of the upper bounds over the corresponding lower bounds. This table is for  $p = \infty$ . (For any other  $p$ ,  $S_{n-1,p'}$  in the lower bounds will have to be weakened by using (2.7) so as to apply the same lines of arguments in [17].)

Given that  $S_{n,1}(\delta, 0)$  goes to  $+\infty$  exponentially as  $n \rightarrow +\infty$ , our upper bounds and the lower bounds in Theorems 5.3, 5.3', 5.4, and 5.4' are very tight. These bounds, together with Lemma 5.1, lead to the qualitative behavior of  $\min_{\alpha_j} \kappa_p(V)$  as  $\alpha_{\max}$  varies, depicted in Figure 1.1. Examining how we got the upper bounds by these inequalities, we conclude that

For a fixed  $\alpha_{\max}$ , a nearly optimally conditioned  $V$  is the one with the translated Chebyshev nodes on the symmetric interval that is slightly larger than  $[-\alpha_{\max}, \alpha_{\max}]$  (so that  $\pm\alpha_{\max}$  are part of the nodes).

(5.13)

In addition to Table 5.1, Li [17] also obtained the following corollary on the asymptotical speeds of  $\min \kappa_{\infty}(V)$  as functions of  $n$  for various cases.

TABLE 5.1  
Ratios of the upper bounds over the lower bounds for  $p = \infty$ .

$\min \kappa_\infty(V)$ subject to $\dots$	Ratio (asymptotically dominant term)	Ineq.
$\alpha_j \in \mathbb{R}$	$\frac{(1+\sqrt{2})3^{3/4}}{2} \times n^{1/\sqrt{2}}$	(5.3)
	$\frac{(1+\sqrt{2})3^{3/4}}{2} \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \times n^{\sqrt{2}/4}$	(5.5)
$\alpha_{\max} \leq \delta$ or $\alpha_{\max} = \delta$ for $\delta \leq 1$	$\frac{(1+\sqrt{2})3^{3/4}}{2} (1 + \sqrt{\delta^2 + 1}) \times n^0$	(5.6)
$\alpha_{\max} = \delta$ for $\delta > 1$	$\frac{3^{3/4}}{2} (1 + \sqrt{\delta^2 + 1}) \times n^1$	(5.7)
	$\frac{3^{3/4}}{2} \sqrt{\frac{\pi}{2}} (1 + \sqrt{\delta^2 + 1}) \times n^{1/2}$	(5.11)
$\alpha_{\max} \leq \delta$ for $\delta > 1$	$\frac{(1+\sqrt{2})3^{3/4}}{2} \times n^{1/\sqrt{2}}$	(5.8)
	$\frac{(1+\sqrt{2})3^{3/4}}{2} \left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \times n^{\sqrt{2}/4}$	(5.12)

COROLLARY 5.5. Let  $\delta > 0$ . We have

$$(5.14) \quad \min_{\alpha_j} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{2})^n \right),$$

$$(5.15) \quad \min_{\alpha_{\max} \leq \delta} \kappa_\infty(V), \quad \min_{\alpha_{\max} = \delta} \kappa_\infty(V) = \mathcal{O} \left( (\delta^{-1} + \sqrt{1 + \delta^{-2}})^n \right) \quad \text{for } \delta \leq 1,$$

$$(5.16) \quad \min_{\alpha_{\max} = \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{1 + \delta^2})^n \right) \quad \text{for } \delta > 1,$$

$$(5.17) \quad \min_{\alpha_{\max} \leq \delta} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{2})^n \right) \quad \text{for } \delta > 1.$$

Equations (5.14)–(5.17) remain valid with  $V$  replaced by  $V_{\text{sym}}$ .

This is a very informative corollary; for example,

$$\min_{\alpha_{\max} \leq 1/2} \kappa_\infty(V), \quad \min_{\alpha_{\max} = 1/2} \kappa_\infty(V) = \mathcal{O} \left( (2 + \sqrt{5})^n \right),$$

$$\min_{\alpha_{\max} = 2} \kappa_\infty(V) = \mathcal{O}_n \left( (1 + \sqrt{5})^n \right).$$

It is worth mentioning that (5.15) is in terms of  $\mathcal{O}$ , while all other equations in Corollary 5.5 are in terms of  $\mathcal{O}_n$ .

**6. Condition numbers for  $V$  with  $\alpha_i \in [\alpha, \beta]$  and  $0 \leq \alpha < \beta$ .** Notice that the case  $\alpha < \beta \leq 0$  can be turned into this case by reversing the signs of all  $\alpha_j$  while leaving  $\|V\|_p$  and  $\|V^{-1}\|_p$  unchanged. So the results in what follows apply to the case  $\alpha < \beta \leq 0$  as well after minor modifications. We shall present lower and upper bounds on

$\min \kappa_p(V)$ subject to $\dots$	
$\alpha_j \geq 0$	$\alpha_j \geq 0, \alpha_{\max} \leq \delta$ or $\alpha_{\max} = \delta$
Theorems 6.2, 6.2'	Theorems 6.3, 6.3'

Most developments here are parallel to those in the previous section. Proofs share similar lines of arguments as well and thus will be omitted. Also omitted here are various results for the case  $0 < \alpha < \beta$ , except (6.1) below. The interested reader may find omitted proofs and results in [17].

LEMMA 6.1. *Suppose that all  $\alpha_j \geq 0$ , and let  $\alpha = \min_j \alpha_j$  and  $\beta = \max_j \alpha_j$ . Then*

$$(6.1) \quad \kappa_p(V) \geq \max \left\{ S_{n-1,p'}(\omega, \tau), \alpha_{\max}^{n-1} S_{n-1,p'}(\omega, \tau) / n^{1/p'} \right\}$$

$$(6.2) \quad \geq \max \left\{ S_{n-1,p'}(\alpha_{\max}/2, 1) \alpha_{\max}^{n-1} S_{n-1,p'}(\alpha_{\max}/2, 1) / n^{1/p'} \right\}$$

$$(6.3) \quad = \begin{cases} S_{n-1,p'}(\alpha_{\max}/2, 1) & \text{if } \alpha_{\max} \leq n^{1/[p'(n-1)]}, \\ \alpha_{\max}^{n-1} S_{n-1,p'}(\alpha_{\max}/2, 1) / n^{1/p'} & \text{if } \alpha_{\max} > n^{1/[p'(n-1)]}. \end{cases}$$

THEOREM 6.2.

$$(6.4) \quad S_{n-1,p'} \left( n^{1/[p'(n-1)]} / 2, 1 \right) \leq \min_{\alpha_j \geq 0} \kappa_p(V) \leq n^{1/p} \frac{\sqrt{2}}{4} S_{n,1}(1/2, 1).$$

THEOREM 6.2'. *Let  $\beta_{\text{opt}}^+ = 2(n/\Lambda_n(1))^{1/(n-1)}$  with  $\Lambda_n(1)$  defined by (4.4). Then*

$$(6.5) \quad S_{n-1,1} \left( n^{1/(n-1)} / 2, 1 \right) \leq \min_{\alpha_j \geq 0} \kappa_\infty(V) \leq \frac{\beta_{\text{opt}}^+}{2\sqrt{1 + \beta_{\text{opt}}^+}} S_{n,1}(\beta_{\text{opt}}^+/2, 1).$$

THEOREM 6.3. *Let  $\delta > 0$ , and let  $\delta' = [2/(1+c)]\delta \geq \delta$ , where  $c = \cos \frac{\pi}{2n}$ . If  $\delta < 1$ , then*

$$(6.6) \quad S_{n-1,p'}(\delta/2, 1) \leq \min_{0 \leq \alpha_j \leq \delta} \kappa_p(V) \leq \min_{0 \leq \alpha_j, \alpha_{\max} = \delta} \kappa_p(V) \leq \frac{n^{1/p} \delta'}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1).$$

*If  $\delta > 1$ , then*

$$(6.7) \quad \frac{\delta^{n-1} S_{n-1,p'}(\delta/2, 1)}{n^{1/p'}} \leq \min_{0 \leq \alpha_j, \alpha_{\max} = \delta} \kappa_p(V) \leq \left( \frac{1+c}{2} \right)^{n-1} \frac{n^{1/p} (\delta')^n}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1),$$

$$(6.8) \quad S_{n-1,p'} \left( \frac{n^{1/[p'(n-1)]}}{2}, 1 \right) \leq \min_{0 \leq \alpha_j \leq \delta} \kappa_p(V) \leq n^{1/p} \frac{\sqrt{2}}{4} S_{n,1}(1/2, 1).$$

THEOREM 6.3'. *Let  $\delta > 1$ , and let  $\delta' = [2/(1+c)]\delta \geq \delta$ , where  $c = \cos \frac{\pi}{2n}$ . Then*

$$(6.9) \quad \frac{\delta^{n-1} S_{n-1,1}(\delta/2, 1)}{n} \leq \min_{0 \leq \alpha_j, \alpha_{\max} = \delta} \kappa_\infty(V) \leq \frac{\max\{n, 2^{-(n-1)} \Lambda_n(1) (\delta')^{n-1}\}}{n} \times \frac{\delta'}{2\sqrt{1 + \delta'}} S_{n,1}(\delta'/2, 1),$$

$$(6.10) \quad S_{n-1,1} \left( n^{1/(n-1)} / 2, 1 \right) \leq \min_{0 \leq \alpha_j \leq \delta} \kappa_\infty(V) \leq \frac{\delta_1}{2\sqrt{1 + \delta_1}} S_{n,1}(\delta_1/2, 1),$$

*where  $\delta_1 = \min\{\delta', 2(n/\Lambda_n(1))^{1/(n-1)}\}$ . It can be seen that  $\delta_1 = 2(n/\Lambda_n(1))^{1/(n-1)}$  for  $n$  sufficiently large.*

Table 6.1 from [17] lists the asymptotically dominant terms for the ratios of the upper bounds over the lower bounds. The conclusion is that these bounds are very

TABLE 6.1  
*Ratios of the upper bounds over the lower bounds for  $p = \infty$  and nonnegative nodes.*

$\min \kappa_\infty(V)$ subject to $\dots$	Ratio (asymptotically dominant term)	Ineq.
$\alpha_j \geq 0$	$\frac{4+3\sqrt{2}}{4} \times n^{1/\sqrt{2}}$	(6.4)
	$\frac{\sqrt{2}}{4} \pi \sqrt{2}/4 \times n^{\sqrt{2}/4}$	(6.5)
$0 \leq \alpha_j, \alpha_{\max} = \delta$ or $\leq \delta$ for $\delta \leq 1$	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}} \times n^0$	(6.6)
$0 \leq \alpha_j, \alpha_{\max} = \delta$ for $\delta > 1$	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}} \times n^1$	(6.7)
	$\frac{(1+\sqrt{1+\delta})^2}{2\sqrt{1+\delta}} \frac{1}{\pi} \times n^{1/2}$	(6.9)
$0 \leq \alpha_j, \alpha_{\max} \leq \delta$ for $\delta > 1$	$\frac{4+3\sqrt{2}}{4} \times n^{1/\sqrt{2}}$	(6.8)
	$\frac{\sqrt{2}}{4} \pi \sqrt{2}/4 \times n^{\sqrt{2}/4}$	(6.10)

tight. These bounds, together with Lemma 5.1, lead to the qualitative behavior of  $\min_{\alpha_j \geq 0} \kappa_\infty(V)$  as  $\alpha_{\max}$  varies, depicted in Figure 1.1. Also, proofs in [17] yield

*For a fixed  $\alpha_{\max}$ , a nearly optimally conditioned  $V$  is the one with the translated Chebyshev nodes on an interval slightly larger than  $[0, \alpha_{\max}]$  (so that  $\alpha_{\max}$  is a node).*

(6.11)

As was pointed out in section 1, optimal nodes with respect to  $\min_{\alpha_j \geq 0} \kappa_1(V)$  subject to  $\alpha_{\max} = \gamma$  were obtained by [1, Theorem 5.9].

COROLLARY 6.4. *Let  $0 < \delta$ . We then have*

$$\begin{aligned} \min_{0 \leq \alpha_j \leq \delta} \kappa_\infty(V), \min_{0 \leq \alpha_j, \alpha_{\max} = \delta} \kappa_\infty(V) &= \mathcal{O} \left( \left[ \delta^{-1/2} + 1 + \sqrt{1 + \delta^{-1}} \right]^{2n} \right) \text{ for } \delta \leq 1, \\ \min_{0 \leq \alpha_j, \alpha_{\max} = \delta} \kappa_\infty(V) &= \mathcal{O}_n \left( (1 + \sqrt{1 + \delta})^{2n} \right) \text{ for } \delta > 1, \\ \min_{0 \leq \alpha_j \leq \delta} \kappa_\infty(V) &= \mathcal{O}_n \left( (3 + 2\sqrt{2})^n \right) \text{ for } \delta > 1, \\ \min_{\alpha_j \geq 0} \kappa_\infty(V) &= \mathcal{O}_n \left( (3 + 2\sqrt{2})^n \right). \end{aligned}$$

**7. Concluding remarks.** We have obtained a series of lower and upper bounds on the optimal condition number  $\min \kappa_p(V)$  of real Vandermonde matrices. These bounds are proved to be asymptotically optimal, except possibly the one in Theorem 3.2 in the case when interval  $[\alpha, \beta]$  is *not* one of the following three kinds: (1) symmetrical ( $-\alpha = \beta$ ), (2) nonnegative ( $\alpha = 0$ ), (3) nonpositive ( $\beta = 0$ ). Asymptotically optimal bounds have been established for the case  $\alpha > 0$  and the case  $\beta < 0$  (too [17]).

Our results led us to deduce the qualitative behaviors of optimally conditioned Vandermonde matrices as the largest absolute value  $\alpha_{\max}$  of all nodes varies, as shown in Figure 1.1 at the beginning of this paper. Our proofs yielded nearly optimally conditioned Vandermonde matrices in various circumstances.

Similar bounds, though unclear about their asymptotical optimality, have been established, too, for confluent Vandermonde matrices [18].

**Acknowledgment.** The author wishes to thank an anonymous referee for his constructive suggestions that improved and shortened the paper considerably, and for bringing the identity  $T_n(2t^2 - 1) = T_{2n}(t)$  to the author's attention.

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