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## A note on eigenvalues of perturbed Hermitian matrices

Chi-Kwong Li <sup>a,1</sup>, Ren-Cang Li <sup>b,\*,2</sup>

<sup>a</sup>*Department of Mathematics, The College of William and Mary, Williamsburg, VA 23185, USA*

<sup>b</sup>*Department of Mathematics, University of Kentucky, 715 Patterson Office Tower,  
Lexington, KY 40506, USA*

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### Abstract

Let

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$$

be Hermitian matrices with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_k$  and  $\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_k$ , respectively. Denote by  $\|E\|$  the spectral norm of the matrix  $E$ , and  $\eta$  the spectral gap between the spectra of  $H_1$  and  $H_2$ . It is shown that

$$|\lambda_i - \tilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

which improves all the existing results. Similar bounds are obtained for singular values of matrices under block perturbations.

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\* Corresponding author.

*E-mail addresses:* [ckli@math.wm.edu](mailto:ckli@math.wm.edu) (C.-K. Li), [rccli@ms.uky.edu](mailto:rccli@ms.uky.edu) (R.-C. Li).

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## 1. Introduction

Consider a partitioned Hermitian matrix

$$A = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{matrix}, \quad (1.1)$$

where  $E^*$  is  $E$ 's complex conjugate transpose. At various situations (typically when  $E$  is *small*), one is interested in knowing the impact of removing  $E$  and  $E^*$  on the eigenvalues of  $A$ . More specifically, one would like to obtain bounds for the differences between that eigenvalues of  $A$  and those of its perturbed matrix

$$\tilde{A} = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{matrix}. \quad (1.2)$$

Let  $\lambda(X)$  be the spectrum of the square matrix  $X$ , and let  $\|Y\|$  be the spectral norm of a matrix  $Y$ , i.e., the largest singular value of  $Y$ . There are two kinds of bounds for the eigenvalues  $\lambda_1 \geq \dots \geq \lambda_{m+n}$  and  $\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{m+n}$  of  $A$  and  $\tilde{A}$ , respectively:

(1) [1,7,8]

$$|\lambda_i - \tilde{\lambda}_i| \leq \|E\|. \quad (1.3)$$

(2) [1–4,7,8] If the spectra of  $H_1$  and  $H_2$  are disjoint, then

$$|\lambda_i - \tilde{\lambda}_i| \leq \|E\|^2/\eta, \quad (1.4)$$

where

$$\eta \stackrel{\text{def}}{=} \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|,$$

and  $\lambda(H_i)$  is the spectrum of  $H_i$ .

The bounds of the first kind do not use information of the spectral distribution of the  $H_1$  and  $H_2$ , which will give (much) weaker bounds when  $\eta$  is not so small; while the bounds of the second kind may blow up whenever  $H_1$  and  $H_2$  have a common eigenvalue. Thus both kinds have their own drawbacks, and it would be advantageous to have bounds that are always no bigger than  $\|E\|$ , of  $\mathcal{O}(\|E\|)$  as  $\eta \rightarrow 0$ , and at the same time behave like  $\mathcal{O}(\|E\|^2/\eta)$  for not so small  $\eta$ . To further motivate our study, let us look at the following  $2 \times 2$  example.

**Example 1.** Consider the  $2 \times 2$  Hermitian matrix

$$A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}. \quad (1.5)$$

Interesting cases are when  $\epsilon$  is *small*, and thus  $\alpha$  and  $\beta$  are *approximate* eigenvalues of  $A$ . We shall analyze by how much the eigenvalues of  $A$  differ from  $\alpha$  and  $\beta$ . Without loss of generality, assume

$$\alpha > \beta.$$

The eigenvalues of  $A$ , denoted by  $\lambda_{\pm}$ , satisfy  $\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta - \epsilon^2 = 0$ ; and thus

$$\lambda_{\pm} = \frac{\alpha + \beta \pm \sqrt{(\alpha + \beta)^2 - 4(\alpha\beta - \epsilon^2)}}{2} = \frac{\alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}}{2}.$$

Now

$$\begin{aligned} 0 < \left\{ \begin{array}{l} \lambda_+ - \alpha \\ \beta - \lambda_- \end{array} \right\} &= \frac{-(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}}{2} \\ &= \frac{2\epsilon^2}{(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}} \end{aligned} \tag{1.6}$$

which provides a difference that enjoys the following properties:

$$\frac{2\epsilon^2}{(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}} \begin{cases} \leq \epsilon & \text{always,} \\ \rightarrow \epsilon & \text{as } \alpha \rightarrow \beta^+, \\ \leq \epsilon^2 / (\alpha - \beta). \end{cases}$$

The purpose of this note is to extend this  $2 \times 2$  example and obtain bounds which improve both (1.3) and (1.4). Such results are not only of theoretical interest but also important in the computations of eigenvalues of Hermitian matrices [5,6,9].

As an application, similar bounds are presented for the singular value problem.

## 2. Main result

**Theorem 2.** *Let*

$$A = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{matrix} \quad \text{and} \quad \tilde{A} = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{matrix}$$

*be Hermitian matrices with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{m+n} \quad \text{and} \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_{m+n}, \tag{2.1}$$

*respectively. Define*

$$\eta_i \stackrel{\text{def}}{=} \begin{cases} \min_{\mu_2 \in \lambda(H_2)} |\tilde{\lambda}_i - \mu_2|, & \text{if } \tilde{\lambda}_i \in \lambda(H_1), \\ \min_{\mu_1 \in \lambda(H_1)} |\tilde{\lambda}_i - \mu_1|, & \text{if } \tilde{\lambda}_i \in \lambda(H_2), \end{cases} \tag{2.2}$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|. \tag{2.3}$$

Then for  $i = 1, 2, \dots, m + n$ , we have

$$|\lambda_i - \tilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \quad (2.4)$$

$$\leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}. \quad (2.5)$$

**Proof.** Suppose  $U^*H_1U$  and  $V^*H_2V$  are in diagonal form with diagonal entries arranged in descending order. We may assume that  $U = I_m$  and  $V = I_n$ . Otherwise, replace  $A$  by

$$(U \oplus V)^*A(U \oplus V).$$

We may perturb the diagonal of  $A$  so that all entries are distinct, and apply continuity argument for the general case.

We prove the result by induction on  $m + n$ . If  $m + n = 2$ , the result is clear (from our example). Assume that  $m + n > 2$ , and the result is true for Hermitian matrices of size  $m + n - 1$ .

First, refining an argument of Mathias [4], we show that (2.4) holds for  $i = 1$ . Assume that the  $(1, 1)$  entry of  $H_1$  equals  $\tilde{\lambda}_1$ . By the min-max principle [1,7,8], we have

$$\lambda_1 \geq e_1^* A e_1 = \tilde{\lambda}_1,$$

where  $e_1$  is the first column of the identity matrix. Let

$$X = \begin{pmatrix} I_m & 0 \\ -(H_2 - \mu_1 I_n)^{-1} E & I_n \end{pmatrix}.$$

Then

$$X^*(A - \lambda_1 I)X = \begin{pmatrix} H_1(\lambda_1) & 0 \\ 0 & H_2 - \lambda_1 I_n \end{pmatrix},$$

where

$$H_1(\lambda_1) = H_1 - \lambda_1 I_m - E^*(H_2 - \lambda_1 I_n)^{-1} E.$$

Since  $A$  and  $X^*AX$  have the same inertia, we see that  $H_1(\lambda_1)$  has zero as the largest eigenvalue. Notice that the largest eigenvalue of  $H_1 - \lambda_1 I$  is  $\tilde{\lambda}_1 - \lambda_1 \leq 0$ . Thus, for  $\delta_1 = |\lambda_1 - \tilde{\lambda}_1| = \lambda_1 - \tilde{\lambda}_1$ , we have (see [7–(10.9)])

$$\lambda_1 \leq \tilde{\lambda}_1 + \|E\|_2^2 / (\delta_1 + \eta_1),$$

and hence

$$\delta_1 \leq \|E\|^2 / (\delta_1 + \eta_1).$$

Consequently,

$$\delta_1 \leq \frac{2\|E\|}{\eta_1 + \sqrt{\eta_1^2 + 4\|E\|^2}}$$

as asserted. Similarly, we can prove the result if the (1, 1) entry of  $H_2$  equals  $\tilde{\lambda}_1$ . In this case, we will apply the inertia arguments to  $A$  and  $YAY^*$  with

$$Y = \begin{pmatrix} I_m & 0 \\ -E(H_1 - \lambda_1 I_m)^{-1} & I_n \end{pmatrix}.$$

Applying the result of the last paragraph to  $-A$ , we see that (2.2) holds for  $i = m + n$ .

Now, suppose  $1 < i < m + n$ . The result trivially holds if  $\lambda_i = \tilde{\lambda}_i$ . Suppose  $\lambda_i \neq \tilde{\lambda}_i$ . We may assume that  $\tilde{\lambda}_i > \lambda_i$ . Otherwise, replace  $(A, \tilde{A}, i)$  by  $(-A, -\tilde{A}, m + n - i + 1)$ . Delete the row and column of  $A$  that contain the diagonal entry  $\tilde{\lambda}_n$ . Suppose the resulting matrix  $\hat{A}$  has eigenvalues  $\nu_1 \geq \dots \geq \nu_{m+n-1}$ . By the interlacing inequalities [7–Section 10.1], we have

$$\lambda_i \geq \nu_i \quad \text{and hence} \quad \tilde{\lambda}_i - \lambda_i \leq \tilde{\lambda}_i - \nu_i. \tag{2.6}$$

Note that  $\tilde{\lambda}_i$  is the  $i$ th largest diagonal entries in  $\hat{A}$ . Let  $\hat{\eta}_i$  be the minimum distance between  $\tilde{\lambda}_i$  and the diagonal entries in the diagonal block  $\hat{H}_j$  in  $\hat{A}$  not containing  $\tilde{\lambda}_i$ ; here  $j \in \{1, 2\}$ . Then

$$\hat{\eta}_i \geq \eta_i$$

because  $\hat{H}_j$  may have one fewer diagonal entries than  $H_j$ . Let  $\hat{E}$  be the off-diagonal block of  $\hat{A}$ . Then  $\|\hat{E}\| \leq \|E\|$ . Thus,

$$\begin{aligned} |\lambda_i - \tilde{\lambda}_i| &= \tilde{\lambda}_i - \lambda_i && \text{because } \tilde{\lambda}_i > \lambda_i \\ &\leq \tilde{\lambda}_i - \nu_i && \text{by (2.6)} \\ &\leq \frac{2\|\hat{E}\|^2}{\hat{\eta}_i + \sqrt{\hat{\eta}_i^2 + 4\|\hat{E}\|^2}} && \text{by induction assumption} \\ &\leq \frac{2\|\hat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\hat{E}\|^2}} && \text{because } \hat{\eta}_i \geq \eta_i \\ &= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\hat{E}\|^2} - \eta_i \\ &\leq \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\hat{E}\| \leq \|E\| \\ &= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \end{aligned}$$

as asserted.  $\square$

### 3. Application to singular value problem

In this section, we apply the result in Section 2 to study singular values of matrices. For notational convenience in connection to our discussion, we define the sequence of singular values of a complex  $p \times q$  matrix  $X$  by

$$\sigma(X) = (\sigma_1(X), \dots, \sigma_k(X)),$$

where  $k = \max\{p, q\}$  and  $\sigma_1(X) \geq \dots \geq \sigma_k(X)$  are the nonnegative square roots of the eigenvalues of the matrix  $XX^*$  or  $X^*X$  depending on which one has a larger size. Note that the nonzero eigenvalues of  $XX^*$  and  $X^*X$  are the same, and they give rise to the nonzero singular values of  $X$  which are of importance. We have the following result concerning the nonzero singular values of perturbed matrices.

**Theorem 3.** *Let*

$$B = \begin{matrix} & k & \ell \\ m & \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix} \\ n & \end{matrix} \quad \text{and} \quad \tilde{B} = \begin{matrix} & k & \ell \\ m & \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix} \\ n & \end{matrix}$$

be complex matrices with singular values

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\max\{m+n, k+\ell\}} \quad \text{and} \quad \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \dots \geq \tilde{\sigma}_{\max\{m+n, k+\ell\}}, \quad (3.1)$$

respectively, so that  $G_1$  and  $G_2$  are nontrivial. Define  $\epsilon = \max\{\|E_1\|, \|E_2\|\}$ , and

$$\eta_i \stackrel{\text{def}}{=} \begin{cases} \min_{\mu_2 \in \sigma(G_2)} |\tilde{\sigma}_i - \mu_2|, & \text{if } \tilde{\sigma}_i \in \sigma(G_1), \\ \min_{\mu_1 \in \sigma(G_1)} |\tilde{\sigma}_i - \mu_1|, & \text{if } \tilde{\sigma}_i \in \sigma(G_2), \end{cases} \quad (3.2)$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \sigma(G_1), \mu_2 \in \sigma(G_2)} |\mu_1 - \mu_2|. \quad (3.3)$$

Then for  $i = 1, 2, \dots, \min\{m+n, k+\ell\}$ , we have

$$|\sigma_i - \tilde{\sigma}_i| \leq \frac{2\epsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\epsilon^2}} \quad (3.4)$$

$$\leq \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}}, \quad (3.5)$$

and  $\sigma_i = \tilde{\sigma}_i = 0$  for  $i > \min\{m+n, k+\ell\}$ .

**Proof.** By Jordan–Wielandt theorem [8–Theorem I.4.2], the eigenvalues of

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix}$$

are  $\pm\sigma_i$  and possibly some zeros adding up to  $m + n + k + \ell$  eigenvalues. A similar statement holds for  $\tilde{B}$ . Permuting the rows and columns appropriately, we see that

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix} \text{ is similar to } \left( \begin{array}{cc|cc} O & G_1 & O & E_1 \\ G_1^* & O & E_2^* & O \\ \hline O & E_2 & O & G_2 \\ E_1^* & O & G_2^* & O \end{array} \right),$$

and

$$\begin{pmatrix} O & \tilde{B} \\ \tilde{B}^* & O \end{pmatrix} \text{ is similar to } \left( \begin{array}{cc|cc} O & G_1 & & \\ G_1^* & O & & \\ \hline & & O & G_2 \\ & & G_2^* & O \end{array} \right).$$

Applying Theorem 2 with

$$H_i = \begin{pmatrix} O & G_i \\ G_i^* & O \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} O & E_2 \\ E_1^* & O \end{pmatrix},$$

we get the result.  $\square$

One can also apply the above proof to the degenerate cases when  $G_1$  or  $G_2$  in the matrix  $B$  is trivial, i.e., one of the parameters  $m, n, k, \ell$  is zero. These cases are useful in applications. We state one of them, and one can easily extend it to other cases.

**Theorem 4.** Suppose  $B = (G \ E)$  and  $\tilde{B} = (G \ O)$  are  $p \times q$  matrices with singular values

$$\sigma_1 \geq \dots \geq \sigma_{\max\{p,q\}} \quad \text{and} \quad \tilde{\sigma}_1 \geq \dots \geq \tilde{\sigma}_{\max\{p,q\}},$$

respectively. Then for  $i = 1, \dots, \min\{p, q\}$ ,

$$|\sigma_i - \tilde{\sigma}_i| \leq \frac{2\|E\|}{2\tilde{\sigma}_i + \sqrt{\tilde{\sigma}_i^2 + 4\|E\|^2}}.$$

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**References**

[1] R. Bhatia, Matrix Analysis, Graduate Texts in Mathematics, vol. 169, Springer, New York, 1996.  
 [2] J. Demmel, Applied Numerical Linear Algebra, SIAM, Philadelphia, 1997.

- [3] G.H. Golub, C.F. van Loan, *Matrix Computations*, second ed., Johns Hopkins University Press, Baltimore, MD, 1989.
- [4] R. Mathias, Quadratic residual bounds for the Hermitian eigenvalue problem, *SIAM J. Matrix Anal. Appl.* 19 (1998) 541–550.
- [5] W. Kahan, When to neglect off-diagonal elements of symmetric tri-diagonal matrices, Technical Report CS42, Computer Science Department, Stanford University, 1966.
- [6] C.C. Paige, Eigenvalues of perturbed Hermitian matrices, *Linear Algebra Appl.* 8 (1974) 1–10.
- [7] B.N. Parlett, *The Symmetric Eigenvalue Problem*, SIAM, Philadelphia, 1998, this SIAM edition is an unabridged, corrected reproduction of the work first published by Prentice-Hall, Inc., Englewood Cliffs, NJ, 1980.
- [8] G.W. Stewart, J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.
- [9] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, England, 1965.