

A Note on Eigenvalues of Perturbed Hermitian Matrices

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Abstract

Let

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$$

be Hermitian matrices with eigenvalues $\lambda_1 \geq \dots \geq \lambda_k$ and $\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_k$, respectively. Denote by $\|E\|$ the spectral norm of the matrix E , and η the spectral gap between the spectra of H_1 and H_2 . It is shown that

$$|\lambda_i - \tilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

which improves all the existing results. Similar bounds are obtained for singular values of matrices under block perturbations.

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1 Introduction

Consider a partitioned Hermitian matrix

$$A = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{matrix}, \quad (1.1)$$

where E^* is E 's complex conjugate transpose. At various situations (typically when E is *small*), one is interested in knowing the impact of removing E and E^* on the eigenvalues

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of A . More specifically, one would like to obtain bounds for the differences between that eigenvalues of A and those of its perturbed matrix

$$\tilde{A} = \begin{matrix} & m & n \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{matrix}. \quad (1.2)$$

Let $\lambda(X)$ be the spectrum of the square matrix X , and let $\|Y\|$ be the spectral norm of a matrix Y , i.e., the largest singular value of Y . There are two kinds of bounds for the eigenvalues $\lambda_1 \geq \dots \geq \lambda_{m+n}$ and $\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_{m+n}$ of A and \tilde{A} , respectively:

1. [1, 7, 8]

$$|\lambda_i - \tilde{\lambda}_i| \leq \|E\|. \quad (1.3)$$

2. [1, 2, 3, 5, 7, 8] If the spectra of H_1 and H_2 are disjoint, then

$$|\lambda_i - \tilde{\lambda}_i| \leq \|E\|^2/\eta, \quad (1.4)$$

where

$$\eta \stackrel{\text{def}}{=} \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|.$$

The bound of the first kind does not use information of the spectral distributions of the H_1 and H_2 , which will give (much) weaker bounds when η is not so small; while the bound of the second kind may blow up whenever H_1 and H_2 have a common eigenvalue. Thus both kinds have their own drawbacks, and it would be advantageous to have bounds that are always no bigger than $\|E\|$, of $\mathcal{O}(\|E\|)$ as $\eta \rightarrow 0$, and at the same time behave like $\mathcal{O}(\|E\|^2/\eta)$ for not so small η . To further motivate our study, let us look at the following 2×2 example.

Example 1 Consider the 2×2 Hermitian matrix

$$A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}. \quad (1.5)$$

Interesting cases are when ϵ is *small*, and thus α and β are *approximate* eigenvalues of A . We shall analyze by how much the eigenvalues of A differ from α and β . Without loss of generality, assume

$$\alpha > \beta, \quad \text{and } \epsilon \text{ is real.}$$

The eigenvalues of A , denoted by λ_{\pm} , satisfy $\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta - \epsilon^2 = 0$; and thus

$$\lambda_{\pm} = \frac{\alpha + \beta \pm \sqrt{(\alpha + \beta)^2 - 4(\alpha\beta - \epsilon^2)}}{2} = \frac{\alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}}{2}.$$

Now

$$\begin{aligned} 0 < \begin{Bmatrix} \lambda_+ - \alpha \\ \beta - \lambda_- \end{Bmatrix} &= \frac{-(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}}{2} \\ &= \frac{2\epsilon^2}{(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}} \end{aligned} \quad (1.6)$$

which provides a difference that enjoys the following properties:

$$\frac{2\epsilon^2}{(\alpha - \beta) + \sqrt{(\alpha - \beta)^2 + 4\epsilon^2}} \begin{cases} \leq \epsilon & \text{always,} \\ \rightarrow \epsilon & \text{as } \alpha \rightarrow \beta^+, \\ \leq \epsilon^2/(\alpha - \beta). \end{cases}$$

The purpose of this note is to extend this 2×2 example and obtain bounds which improve both (1.3) and (1.4). Such results are not only of theoretical interest but also important in the computations of eigenvalues of Hermitian matrices [4, 6, 9].

As an application, similar bounds are presented for the singular value problem.

2 Main Result

Theorem 2 *Let*

$$A = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{matrix} \quad \text{and} \quad \tilde{A} = \begin{matrix} & \begin{matrix} m & n \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{matrix}$$

be Hermitian matrices with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+n} \quad \text{and} \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{m+n}, \quad (2.1)$$

respectively. Define

$$\eta_i \stackrel{\text{def}}{=} \begin{cases} \min_{\mu_2 \in \lambda(H_2)} |\tilde{\lambda}_i - \mu_2|, & \text{if } \tilde{\lambda}_i \in \lambda(H_1), \\ \min_{\mu_1 \in \lambda(H_1)} |\tilde{\lambda}_i - \mu_1|, & \text{if } \tilde{\lambda}_i \in \lambda(H_2), \end{cases} \quad (2.2)$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|. \quad (2.3)$$

Then for $i = 1, 2, \dots, m + n$, we have

$$|\lambda_i - \tilde{\lambda}_i| \leq \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \quad (2.4)$$

$$\leq \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}. \quad (2.5)$$

Proof. Suppose U^*H_1U and V^*H_2V are in the diagonal form with their diagonal entries arranged in the descending order, respectively. We may assume that $U = I_m$ and $V = I_n$. Otherwise, replace A by

$$(U \oplus V)^* A (U \oplus V).$$

We may perturb the diagonal of A so that all entries are distinct, and apply continuity argument for the general case.

We prove the result by induction on $m + n$. If $m + n = 2$, the result is clear (from our Example). Assume that $m + n > 2$, and the result is true for Hermitian matrices of size $m + n - 1$.

First, refining an argument of Mathias [5], we show that (2.4) holds for $i = 1$. Assume that the $(1, 1)$ th entry of H_1 equals $\tilde{\lambda}_1$. By the min-max principle [1, 7, 8], we have

$$\lambda_1 \geq e_1^* A e_1 = \tilde{\lambda}_1,$$

where e_1 is the first column of the identity matrix. No proof is necessary if $\lambda_1 = \tilde{\lambda}_1$. Assume $\lambda_1 > \tilde{\lambda}_1$ and let

$$X = \begin{pmatrix} I_m & 0 \\ -(H_2 - \lambda_1 I_n)^{-1} E & I_n \end{pmatrix}.$$

Then

$$X^*(A - \lambda_1 I)X = \begin{pmatrix} M(\lambda_1) & 0 \\ 0 & H_2 - \lambda_1 I_n \end{pmatrix},$$

where

$$M(\lambda) = H_1 - \lambda I_m - E^*(H_2 - \lambda I_n)^{-1} E.$$

Since A and X^*AX have the same inertia, we see that $M(\lambda_1)$ has zero as the largest eigenvalue. Notice that the largest eigenvalue of $H_1 - \lambda_1 I$ is $\tilde{\lambda}_1 - \lambda_1 \leq 0$. Thus, for $\delta_1 = |\lambda_1 - \tilde{\lambda}_1| = \lambda_1 - \tilde{\lambda}_1$, we have (see [7, (10.9)])

$$\lambda_1 \leq \tilde{\lambda}_1 + \|E\|_2^2 / (\delta_1 + \eta_1),$$

and hence

$$\delta_1 \leq \|E\|^2 / (\delta_1 + \eta_1).$$

Consequently,

$$\delta_1 \leq \frac{2\|E\|^2}{\eta_1 + \sqrt{\eta_1^2 + 4\|E\|^2}}$$

as asserted. Similarly, we can prove the result if the $(1, 1)$ th entry of H_2 equals $\tilde{\lambda}_1$. In this case, we will apply the inertia arguments to A and YAY^* with

$$Y = \begin{pmatrix} I_m & 0 \\ -E(H_1 - \lambda_1 I_m)^{-1} & I_n \end{pmatrix}.$$

Applying the result of the last paragraph to $-A$, we see that (2.4) holds for $i = m + n$.

Now, suppose $1 < i < m + n$. The result trivially holds if $\lambda_i = \tilde{\lambda}_i$. Suppose $\lambda_i \neq \tilde{\lambda}_i$. We may assume that $\tilde{\lambda}_i > \lambda_i$. Otherwise, replace (A, \tilde{A}, i) by $(-A, -\tilde{A}, m + n - i + 1)$. Delete the row and column of A that contain the diagonal entry $\tilde{\lambda}_{m+n}$. Suppose the resulted matrix \hat{A} has eigenvalues $\nu_1 \geq \dots \geq \nu_{m+n-1}$. By the interlacing inequalities [7, Section 10.1], we have

$$\lambda_i \geq \nu_i \quad \text{and hence} \quad \tilde{\lambda}_i - \lambda_i \leq \tilde{\lambda}_i - \nu_i. \quad (2.6)$$

Note that $\tilde{\lambda}_i$ is the i th largest diagonal entries in \hat{A} . Let $\hat{\eta}_i$ be the minimum distance between $\tilde{\lambda}_i$ and the diagonal entries in the diagonal block \hat{H}_j in \hat{A} not containing $\tilde{\lambda}_i$, where $j \in \{1, 2\}$. Then

$$\hat{\eta}_i \geq \eta_i$$

because \hat{H}_j may have one fewer diagonal entries than H_j . Let \hat{E} be the off-diagonal block of \hat{A} . Then $\|\hat{E}\| \leq \|E\|$. Thus,

$$\begin{aligned} |\lambda_i - \tilde{\lambda}_i| &= \tilde{\lambda}_i - \lambda_i && \text{because } \tilde{\lambda}_i > \lambda_i \\ &\leq \tilde{\lambda}_i - \nu_i && \text{by (2.6)} \\ &\leq \frac{2\|\hat{E}\|^2}{\hat{\eta}_i + \sqrt{\hat{\eta}_i^2 + 4\|\hat{E}\|^2}} && \text{by induction assumption} \\ &\leq \frac{2\|\hat{E}\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|\hat{E}\|^2}} && \text{because } \hat{\eta}_i \geq \eta_i \\ &= \frac{1}{2}\sqrt{\eta_i^2 + 4\|\hat{E}\|^2} - \eta_i \\ &\leq \frac{1}{2}\sqrt{\eta_i^2 + 4\|E\|^2} - \eta_i && \text{because } \|\hat{E}\| \leq \|E\| \\ &= \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \end{aligned}$$

as asserted. ■

3 Application to Singular Value Problem

In this section, we apply the result in Section 2 to study singular values of matrices. For notational convenience in connection to our discussion, we define the sequence of singular values of a complex $p \times q$ matrix X by

$$\sigma(X) = (\sigma_1(X), \dots, \sigma_k(X)),$$

where $k = \max\{p, q\}$ and $\sigma_1(X) \geq \dots \geq \sigma_k(X)$ are the nonnegative square roots of the eigenvalues of the matrix XX^* or X^*X depending on which one has a larger size. Note that the nonzero eigenvalues of XX^* and X^*X are the same, and they give rise to the nonzero singular values of X which are of importance. We have the following result concerning the nonzero singular values of perturbed matrices.

Theorem 3 *Let*

$$B = \begin{matrix} & \begin{matrix} k & \ell \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} G_1 & E_1 \\ E_2 & G_2 \end{pmatrix} \end{matrix} \quad \text{and} \quad \tilde{B} = \begin{matrix} & \begin{matrix} k & \ell \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix} \end{matrix}$$

be complex matrices with singular values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\max\{m+n, k+\ell\}} \quad \text{and} \quad \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_{\max\{m+n, k+\ell\}}, \quad (3.1)$$

respectively, so that G_1 and G_2 are non-trivial. Define $\epsilon = \max\{\|E_1\|, \|E_2\|\}$, and

$$\eta_i \stackrel{\text{def}}{=} \begin{cases} \min_{\mu_2 \in \sigma(G_2)} |\tilde{\sigma}_i - \mu_2|, & \text{if } \tilde{\sigma}_i \in \sigma(G_1), \\ \min_{\mu_1 \in \sigma(G_1)} |\tilde{\sigma}_i - \mu_1|, & \text{if } \tilde{\sigma}_i \in \sigma(G_2), \end{cases} \quad (3.2)$$

$$\eta \stackrel{\text{def}}{=} \min_{1 \leq i \leq m+n} \eta_i = \min_{\mu_1 \in \sigma(G_1), \mu_2 \in \sigma(G_2)} |\mu_1 - \mu_2|. \quad (3.3)$$

Then for $i = 1, 2, \dots, \min\{m+n, k+\ell\}$, we have

$$|\sigma_i - \tilde{\sigma}_i| \leq \frac{2\epsilon^2}{\eta_i + \sqrt{\eta_i^2 + 4\epsilon^2}} \quad (3.4)$$

$$\leq \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}}, \quad (3.5)$$

and $\sigma_i = \tilde{\sigma}_i = 0$ for $i > \min\{m+n, k+\ell\}$.

PROOF: By Jordan-Wielandt Theorem [8, Theorem I.4.2], the eigenvalues of

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix}$$

are $\pm\sigma_i$ and possibly some zeros adding up to $m+n+k+\ell$ eigenvalues. A similar statement holds for \tilde{B} . Permuting the rows and columns appropriately, we see that

$$\begin{pmatrix} O & B \\ B^* & O \end{pmatrix} \text{ is similar to } \left(\begin{array}{cc|cc} O & G_1 & O & E_1 \\ G_1^* & O & E_2^* & O \\ \hline O & E_2 & O & G_2 \\ E_1^* & O & G_2^* & O \end{array} \right),$$

and

$$\begin{pmatrix} O & \tilde{B} \\ \tilde{B}^* & O \end{pmatrix} \text{ is similar to } \left(\begin{array}{cc|cc} O & G_1 & & \\ G_1^* & O & & \\ \hline & & O & G_2 \\ & & G_2^* & O \end{array} \right).$$

Applying Theorem 2 with

$$H_i = \begin{pmatrix} O & G_i \\ G_i^* & O \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} O & E_2 \\ E_1^* & O \end{pmatrix},$$

we get the result. ■

One can also apply the above proof to the degenerate cases when G_1 or G_2 in the matrix B is trivial, i.e., one of the parameters m, n, k, ℓ is zero. These cases are useful in applications. We state one of them, and one can easily extend it to other cases.

Theorem 4 *Suppose $B = (G \ E)$ and $\tilde{B} = (G \ O)$ are $p \times q$ matrices with singular values*

$$\sigma_1 \geq \cdots \geq \sigma_{\max\{p,q\}} \quad \text{and} \quad \tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_{\max\{p,q\}},$$

respectively. Then for $i = 1, \dots, \min\{p, q\}$,

$$|\sigma_i - \tilde{\sigma}_i| \leq \frac{2\|E\|^2}{2\tilde{\sigma}_i + \sqrt{\tilde{\sigma}_i^2 + 4\|E\|^2}}.$$

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