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NORTH-HOLLAND

## Relative Perturbation Theory. III. More Bounds on Eigenvalue Variation\*

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### ABSTRACT

Classically, the relative error in  $\tilde{\alpha} = \alpha(1 + \delta)$  as an approximation to  $\alpha$  is measured by  $\delta = (\text{relative error in } \tilde{\alpha}) = (\tilde{\alpha} - \alpha)/\alpha$ . The quantity  $-\log_{10} |\delta|$  is usually used for the number of correct decimal digits in numerical results, although this  $\delta$ -measure is clearly not a metric, since it lacks symmetry between  $\alpha$  and  $\tilde{\alpha}$ . In part I of this series, two other kinds of relative distances which have much better mathematical properties have been introduced and employed to establish theories. It is shown that these different measurements are topologically equivalent. However, the  $\delta$ -measure is more convenient to use in practice. In this part, we established relative perturbation bounds directly using the classical measure. The new bounds for diagonalizable matrices are cleaner than the corresponding ones in part I and yield nice bounds for Hermitian matrices, too. But when applied to nonnegative definite Hermitian matrices, the new bounds are weaker than those in part I. © 1997 Elsevier Science Inc.

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## 1. INTRODUCTION

Classically, the relative error in  $\tilde{\alpha} = \alpha(1 + \delta)$  as an approximation to  $\alpha$  is measured by

$$\delta = (\text{relative error in } \tilde{\alpha}) = \frac{\tilde{\alpha} - \alpha}{\alpha}. \quad (1.1)$$

When  $|\delta| \leq \epsilon$  it is said the relative perturbation to  $\alpha$  is at most  $\epsilon$  (see, e.g., [2]). This classical measure is not a metric, since it lacks symmetry between  $\alpha$  and  $\tilde{\alpha}$ . Nonetheless, it is good enough for measuring correct digits in numerical approximations and is often used in numerical analysis.

In parts I and II [6, 7], two other kinds of relative distances  $\varrho_p$  and  $\chi$ , defined for (complex)  $\alpha$ ,  $\tilde{\alpha}$  by

$$\varrho_p(\alpha, \tilde{\alpha}) = \frac{|\alpha - \tilde{\alpha}|}{\sqrt[p]{|\alpha|^p + |\tilde{\alpha}|^p}} \quad \text{for } 1 \leq p \leq \infty, \quad (1.2)$$

$$\chi(\alpha, \tilde{\alpha}) = \frac{|\alpha - \tilde{\alpha}|}{\sqrt{|\alpha\tilde{\alpha}|}},$$

were proposed and studied (with convention  $0/0 = 0$  for convenience). It was proved [6] that  $\varrho_p$  is a metric on the set of real numbers, while  $\chi$  is not, even on the set of nonnegative real numbers. The invariance of  $\varrho_p$  and  $\chi$  under scaling by a nonzero number and undertaking reciprocals makes them good candidates for measuring relative errors in numerical approximations, rather than the classical way. Topologically all these measurements are equivalent in the sense that each of these relative distances can be bounded by others provided  $\alpha$  and  $\tilde{\alpha}$  are sufficiently close [6].

The relative distances (1.2) have much better mathematical properties than the classical  $\delta$ -measure (1.1), but the  $\delta$ -measure is more convenient to use in practice. For this reason, in this paper we establish relative perturbation bounds directly in terms of the classical  $\delta$ -measure (1.1), in contrast to those in [6] using  $\varrho_p$  and  $\chi$ . The bounds here are generally sharper than would be derived from bounds in [6] by the topological relationships among these measurements.

2. MAIN THEOREMS

In what follows,  $A$  and  $\tilde{A}$  denote two  $n \times n$  matrices with eigenvalues, counted according to their algebraic multiplicities,

$$\lambda(A) = \{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \lambda(\tilde{A}) = \{\tilde{\lambda}_1, \dots, \tilde{\lambda}_n\}. \tag{2.1}$$

Whenever all  $\lambda_i$ 's and  $\tilde{\lambda}_j$ 's are real, we order them descendingly

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_n. \tag{2.2}$$

$\|X\|_2$  is  $X$ 's spectral norm, i.e.,  $X$ 's largest singular value;  $\|X\|_F$  is  $X$ 's Frobenius norm, the square root of the trace of  $X^*X$ , where  $X^*$  is  $X$ 's conjugate transpose; and  $\kappa(X) = \|X\|_2\|X^{-1}\|_2$  is  $X$ 's spectral condition number.

We first establish Hoffman-Wielandt type theorems ([4], 1953).

**THEOREM 2.1.** *Assume that the  $n \times n$  matrix  $A$  is perturbed to  $\tilde{A} = D_1^*AD_2$  and both  $D_1$  and  $D_2$  are nonsingular. Assume also that both  $A$  and  $\tilde{A}$  are diagonalizable and admit the following decompositions:*

$$A = X\Lambda X^{-1} \quad \text{and} \quad \tilde{A} = \tilde{X}\tilde{\Lambda}\tilde{X}^{-1}, \tag{2.3}$$

where  $X$  and  $\tilde{X}$  are nonsingular, and  $\Lambda$  and  $\tilde{\Lambda}$  are diagonal with the eigenvalues of  $A$  and  $\tilde{A}$  on their diagonals, respectively. Then there are permutations  $\mu$  and  $\nu$  of  $\{1, 2, \dots, n\}$  such that

$$\begin{aligned} \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\lambda_i} \right|^2} &\leq \|X^{-1}\|_2\|\tilde{X}\|_2\|D_2\|_2\|\tilde{X}^{-1}(D_1^* - D_2^{-1})X\|_F \\ &\leq \kappa(X)\kappa(\tilde{X})\|D_2\|_2\|D_1^* - D_2^{-1}\|_F, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\nu(i)}}{\tilde{\lambda}_{\nu(i)}} \right|^2} &\leq \|X^{-1}\|_2\|\tilde{X}\|_2\|D_1^{-*}\|_2\|\tilde{X}^{-1}(D_1^* - D_2^{-1})X\|_F \\ &\leq \kappa(X)\kappa(\tilde{X})\|D_1^{-*}\|_2\|D_1^* - D_2^{-1}\|_F. \end{aligned} \tag{2.5}$$

Applying Theorem 2.1 to  $\tilde{A}$  and  $D_1^{-*}\tilde{A}D_2^{-1} = A$  yields a dual theorem.

**THEOREM 2.1'.** *Let all conditions of Theorem 2.1 hold. Then there are permutations  $\mu$  and  $\nu$  of  $\{1, 2, \dots, n\}$  such that*

$$\begin{aligned} \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\nu(i)}}{\tilde{\lambda}_{\nu(i)}} \right|^2} &\leq \|\tilde{X}^{-1}\|_2 \|X\|_2 \|D_2^{-1}\|_2 \|X^{-1}(D_1^{-*} - D_2)\tilde{X}\|_F \\ &\leq \kappa(X) \kappa(\tilde{X}) \|D_2^{-1}\|_2 \|D_1^{-*} - D_2\|_F, \end{aligned}$$

$$\begin{aligned} \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\lambda_i} \right|^2} &\leq \|\tilde{X}^{-1}\|_2 \|X\|_2 \|D_1^*\|_2 \|X^{-1}(D_1^{-*} - D_2)\tilde{X}\|_F \\ &\leq \kappa(X) \kappa(\tilde{X}) \|D_1^*\|_2 \|D_1^{-*} - D_2\|_F. \end{aligned}$$

**REMARK 1.** Theorem 6.1s of Li [6], in a slightly weaker form, reads: *Under the conditions of Theorem 2.1, there is a permutation  $\tau$  such that*

$$\begin{aligned} &\sqrt{\sum_{i=1}^n \left[ \varrho_2(\lambda_i, \tilde{\lambda}_{\tau(i)}) \right]^2} \\ &\leq \kappa(X) \kappa(\tilde{X}) \min_{\text{unitary } U} \sqrt{\|U - D_1\|_F^2 + \|U^* - D_2^{-1}\|_F^2}. \end{aligned} \tag{2.6}$$

The potential improvement in sharpness of Theorems 2.1 and 2.1' over (2.6) is that the inequalities in Theorems 2.1 and 2.1' are perfect equalities when  $D_1^*D_2 = I$ , as they should be, since similarity transformations do not change spectra.

Theorem 2.1 is applicable when both  $A$  and  $\tilde{A} = D^*AD$  are Hermitian.

**THEOREM 2.2.** *Let  $A$  and  $\tilde{A} = D^*AD$  be two  $n \times n$  Hermitian matrices with eigenvalues (2.1). Then there are permutations  $\mu$  and  $\nu$  of  $\{1, 2, \dots, n\}$  such that*

$$\sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\lambda_i} \right|^2} \leq \|D\|_2 \|D^* - D^{-1}\|_F, \tag{2.7}$$

$$\sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\nu(i)}}{\tilde{\lambda}_{\nu(i)}} \right|^2} \leq \|D^{-*}\|_2 \|D^* - D^{-1}\|_F. \tag{2.8}$$

REMARK 2. It is hard to compare the sharpness of Theorem 2.2 with that of Theorem 6.3 in Li [6]: *Under the conditions of Theorem 2.2, there is a permutation  $\tau$  such that*

$$\sqrt{\sum_{i=1}^n [\varrho_2(\lambda_i, \tilde{\lambda}_{\tau(i)})]^2} \leq \sqrt{\|I - \Sigma\|_F^2 + \|I - \Sigma^{-1}\|_F^2}, \tag{2.9}$$

where  $\Sigma$  is diagonal with  $D$ 's singular values on its diagonal. But for nonnegative definite matrices, both (2.7) and (2.8) of Theorem 2.2 are corollaries of an inequality proved in [6]: *In addition to the conditions of Theorem 2.2, if  $A$  is nonnegative definite, then*

$$\sqrt{\sum_{i=1}^n [\chi(\lambda_i, \tilde{\lambda}_i)]^2} \leq \|D^* - D^{-1}\|_F. \tag{2.10}$$

Here the ordering (2.2) is assumed. To see that (2.10) implies (2.7) and (2.8), we notice

$$\left| \frac{\lambda_i - \tilde{\lambda}_i}{\lambda_i} \right|^2 = [\chi(\lambda_i, \tilde{\lambda}_i)]^2 \frac{\tilde{\lambda}_i}{\lambda_i} \leq \|D\|_2^2 [\chi(\lambda_i, \tilde{\lambda}_i)]^2,$$

$$\left| \frac{\lambda_i - \tilde{\lambda}_i}{\tilde{\lambda}_i} \right|^2 = [\chi(\lambda_i, \tilde{\lambda}_i)]^2 \frac{\lambda_i}{\tilde{\lambda}_i} \leq \|D^{-1}\|_2^2 [\chi(\lambda_i, \tilde{\lambda}_i)]^2,$$

where we have used a theorem due to Ostrowski ([9], 1959; see also [5, pp. 224–225]) to get

$$\tilde{\lambda}_i/\lambda_i \leq \|D\|_2^2 \quad \text{and} \quad \lambda_i/\tilde{\lambda}_i \leq \|D^{-1}\|_2^2.$$

We also see that in the nonnegative definite case,  $\mu$  and  $\nu$  can be expressed explicitly: both can be the identity permutation under the ordering (2.2).

REMARK 3. Generally, it is natural to order the  $\lambda_i$ 's and  $\tilde{\lambda}_j$ 's in Theorem 2.2 descendingly as in (2.2), and to expect that  $\mu$  and  $\nu$  will be the identity permutation. But this may not be true without further assumptions. Consider

$\alpha \leq \beta$  and  $\tilde{\alpha} \leq \tilde{\beta}$ . We ask if

$$\left| \frac{\alpha - \tilde{\alpha}}{\alpha} \right|^2 + \left| \frac{\beta - \tilde{\beta}}{\beta} \right|^2 \leq \left| \frac{\alpha - \tilde{\beta}}{\alpha} \right|^2 + \left| \frac{\beta - \tilde{\alpha}}{\beta} \right|^2. \tag{2.11}$$

Unfortunately, (2.11) may fail, since

$$\begin{aligned} & \alpha^2 \beta^2 \left( \left| \frac{\alpha - \tilde{\beta}}{\alpha} \right|^2 + \left| \frac{\beta - \tilde{\alpha}}{\beta} \right|^2 - \left| \frac{\alpha - \tilde{\alpha}}{\alpha} \right|^2 - \left| \frac{\beta - \tilde{\beta}}{\beta} \right|^2 \right) \\ &= (\alpha - \beta)(\tilde{\alpha} - \tilde{\beta})[(\alpha + \beta)(\tilde{\alpha} + \tilde{\beta}) - 2\alpha\beta], \end{aligned}$$

which could be negative.

Could  $\mu$  and  $\nu$  in Theorem 2.2 be the same? We do not know either. But we suspect they may not be without additional assumptions.

**COROLLARY 2.1.** *If, in Theorem 2.2,  $\mu = \nu$ , then*

$$\sqrt{\sum_{i=1}^n \left[ \chi(\lambda_i, \tilde{\lambda}_{\mu(i)}) \right]^2} \leq \sqrt{\kappa(D)} \|D^* - D^{-1}\|_F.$$

When applied to nonnegative definite matrices, this inequality is slightly weaker than the corresponding (2.10) due to [6]. This is not surprising, since we have remarked that Theorem 2.2 is a corollary in the nonnegative definite case.

*Proof of Corollary 2.1.* Multiplying the two sides of (2.7) and (2.8) yields

$$\begin{aligned} \|D\|_2 \|D^{-1}\|_2 \|D^* - D^{-1}\|_F^2 &\geq \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\lambda_i} \right|^2} \sqrt{\sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\tilde{\lambda}_{\mu(i)}} \right|^2} \\ &\geq \sum_{i=1}^n \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\lambda_i} \right| \cdot \left| \frac{\lambda_i - \tilde{\lambda}_{\mu(i)}}{\tilde{\lambda}_{\mu(i)}} \right| \\ &= \sum_{i=1}^n \left[ \chi(\lambda_i, \tilde{\lambda}_{\mu(i)}) \right]^2, \end{aligned}$$

as required. ■

Now we state a Weyl-Lidskii type theorem for diagonalizable matrices.

**THEOREM 2.3.** *To the hypotheses of Theorem 2.1 add: all  $\lambda_i \geq 0$  and  $\tilde{\lambda}_j \geq 0$  and ordered as in (2.2). Then*

$$\begin{aligned} \max_{1 \leq i \leq n} \left| \frac{\lambda_i - \tilde{\lambda}_i}{\lambda_i} \right| &\leq \|X^{-1}\|_2 \|\tilde{X}\|_2 \|D_2\|_2 \|\tilde{X}^{-1}(D_1^* - D_2^{-1})X\|_2 \\ &\leq \kappa(X) \kappa(\tilde{X}) \|D_2\|_2 \|D_1^* - D_2^{-1}\|_2, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \max_{1 \leq i \leq n} \left| \frac{\lambda_i - \tilde{\lambda}_i}{\tilde{\lambda}_i} \right| &\leq \|X^{-1}\|_2 \|\tilde{X}\|_2 \|D_1^{-*}\|_2 \|\tilde{X}^{-1}(D_1^* - D_2^{-1})X\|_2 \\ &\leq \kappa(X) \kappa(\tilde{X}) \|D_1^{-*}\|_2 \|D_1^* - D_2^{-1}\|_2. \end{aligned} \tag{2.13}$$

Similarly to Theorem 2.1', a dual theorem can be obtained. We leave it to the reader.

**REMARK 4.** The inequalities in Theorem 2.3 become equalities when  $D_1^* D_2 = I$ , as they should; the corresponding inequalities in Theorem 6.4 of Li [6] do not. Theorem 2.3, applied to Hermitian matrices ( $D_1 = D_2 = D$ , then), yields a weaker inequality than Ostrowski's theorem, which implies

$$\left| \frac{\lambda_i - \tilde{\lambda}_i}{\lambda_i} \right| \leq \|I - D^* D\|_2 = \|(D^{-1} - D^*) D\|_2 \leq \|D\|_2 \|D^{-1} - D^*\|_2.$$

**REMARK 5.** Lu [8] and more recently Bhatia, Kittaneh, and Li [1] significantly improved previous spectral perturbation bounds for diagonalizable matrices by a factor of  $\sqrt{\kappa(X) \kappa(\tilde{X})}$ . Could the factor  $\kappa(X) \kappa(\tilde{X})$  in Theorems 2.1, 2.1', and 2.3 be replaced by  $\sqrt{\kappa(X) \kappa(\tilde{X})}$  as well?

### 3. PROOFS OF THEOREMS 2.1 AND 2.3

$\tilde{A} = D_1^* A D_2$  implies  $\tilde{A} D_2^{-1} - D_1^* A = 0$ . So we have

$$\tilde{A} D_2^{-1} - D_2^{-1} A = (D_1^* - D_2^{-1}) A \quad \text{and} \quad \tilde{A} D_1^* - D_1^* A = \tilde{A} (D_1^* - D_2^{-1}).$$

Substitute  $A = X\Lambda X^{-1}$  and  $\tilde{A} = \tilde{X}\tilde{\Lambda}\tilde{X}^{-1}$  and pre- and postmultiply by  $\tilde{X}^{-1}$  and  $X$  respectively to get

$$\tilde{\Lambda}\tilde{X}^{-1}D_2^{-1}X - \tilde{X}^{-1}D_2^{-1}X\Lambda = \tilde{X}^{-1}(D_1^* - D_2^{-1})X\Lambda, \tag{3.1}$$

$$\tilde{\Lambda}\tilde{X}^{-1}D_1^*X - \tilde{X}^{-1}D_1^*X\Lambda = \tilde{\Lambda}\tilde{X}^{-1}(D_1^* - D_2^{-1})X. \tag{3.2}$$

Now we are ready for the proofs.

*Proof of Theorem 2.1.* We give a proof of (2.4) with the help of (3.1). Similarly one can prove (2.5) using (3.2).

Write  $Y \equiv (y_{ij}) = \tilde{X}^{-1}D_2^{-1}X$  and  $E \equiv (e_{ij}) = \tilde{X}^{-1}(D_1^* - D_2^{-1})X$ . Looking at the  $(i, j)$ th entry of (3.1) gives<sup>1</sup>

$$(\tilde{\lambda}_i - \lambda_j)y_{ij} = e_{ij}\lambda_j \quad \Rightarrow \quad |e_{ij}|^2 \geq \left| \frac{\lambda_i - \lambda_j}{\lambda_j} \right|^2 |y_{ij}|^2.$$

Thus summing over all possible  $i$  and  $j$  gives

$$\|\tilde{X}^{-1}(D_1^* - D_2^{-1})X\|_F^2 = \sum_{i,j} |e_{ij}|^2 \geq \sum_{i,j} \left| \frac{\tilde{\lambda}_i - \lambda_j}{\lambda_j} \right|^2 |y_{ij}|^2.$$

Notice that the smallest singular value of  $Y$ ,

$$\sigma_{\min}(Y) = \|Y^{-1}\|_2^{-1} = \|\tilde{X}D_2X^{-1}\|_2^{-1} \geq \|\tilde{X}\|_2^{-1}\|D_2\|_2^{-1}\|X^{-1}\|_2^{-1}.$$

The rest of the proof, exactly the same as the proof of Theorem 6.1 in [6], is to use the technique in Hoffman and Wielandt [4] together with the main result of Elsner and Friedland [3]. ■

*Proof of Theorem 2.3.* With Equations (3.1) and (3.2), a proof can be given similar to the proof of Theorem 6.4 in [6].

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<sup>1</sup> The inequality becomes an equality if  $\lambda_j \neq 0$ . If  $\lambda_j = 0$ , then one of  $\tilde{\lambda}_i - \lambda_j$  and  $y_{ij}$  must be 0. There are two cases: (1)  $\tilde{\lambda}_i - \lambda_j = 0$ ; then the inequality is true because  $0/0 = 0$  by our convention. (2)  $y_{ij} = 0$ ; then  $(\tilde{\lambda}_i - \lambda_j)/\lambda_j^2 |y_{ij}|^2$  is either  $(0/0) \cdot 0 = 0$  or  $\infty \cdot 0 = 0/0 = 0$ , and the inequality still holds.

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