Math 3330

PRACTICE FINAL EXAM

1. Consider the system of linear equations

$$\begin{aligned} x + 2y - w &= 2 \\ 2x + 3y - z + w &= 4 \\ -y - z + 3w &= 0 \end{aligned}$$

Use Gaussian elimination to find all solutions (if any), keeping track of your elementary row operations. Indicate pivot and non-pivot variables. Express your answer in parametric form and give the translation and spanning vectors.

2. In the space $\mathcal{F}(\mathbb{R})$ of real-valued functions, consider the subset $W = \{f \mid f(2) = 0\}$ (that is, the subset of functions such that $f(2) = 0$). Show that W is a subspace of the space $\mathcal{F}(\mathbb{R})$.

3. Is the following set of vectors linearly independent: $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$?

4. Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 3 & -1 & 1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$.

(a) Find the basis of the row space of A.
(b) Find the basis of the column space of A.
(c) What is the rank of A?
(d) What is the dimension of the null space of A?

5. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T : \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y - 1 \\ x - y \\ 3y \end{bmatrix}$.
Prove or disprove that T is a linear transformation.

6. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^1$ be matrix transformations given by matrices $B = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & -2 & 0 & 1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix}$, respectively. Find the matrix which represents the composition $S \circ T$.

7. Consider the ordered basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Let $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
Find the coordinate vector of X in the basis $\mathcal{B}$.

8. Let $A = \begin{bmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$.

(a) Compute the determinant of A.
(b) Determine if A is invertible and if so, compute the $(1, 2)$-entry of its inverse.

9. Use Cramer's rule to solve the system

$$\begin{aligned} 2x + 5y - 3z &= 1 \\ 2x + y + z &= 0 \\ x - 2y + z &= 0 \end{aligned}$$

10. Let $A = \begin{bmatrix} 3 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix}$.

(a) Find all eigenvalues for the matrix A and a system of linearly independent eigenvectors of A.

(b) Determine if A is diagonalizable. Explain why of why not.

11. (a) Show that the vectors $P_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, P_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, P_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ form an orthogonal basis of $\mathbb{R}^3$.

(b) Find the coordinates for the vector $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis from part (a).

12. Use the Gram-Schmidt orthogonalization procedure to find an orthogonal basis for the subspace spanned by the vectors $A_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.