Math 3330

SOLUTIONS FOR PRACTICE FINAL EXAM

1. Augmented matrix: $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 2 & 3 & -1 & 1 & 4 \\ 0 & -1 & -1 & 3 & 0 \end{bmatrix}$.

Gaussian elimination: $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 2 & 3 & -1 & 1 & 4 \\ 0 & -1 & -1 & 3 & 0 \end{bmatrix} \underset{R_2 \to R_2 - 2R_1}{\leadsto} \begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & -1 & -1 & 3 & 0 \\ 0 & -1 & -1 & 3 & 0 \end{bmatrix} \underset{R_3 \to R_3 - R_2}{\leadsto}$
$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Pivot variables: x and y, non-pivot variables: z and w.

Let $z = s$, $w = t$. Then $y = -s + 3t$, $x = 2 + 2s - 5t$.

In paramtertic form, $\begin{bmatrix} x \\ y \\ w \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

The spanning vectors are $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ and the translation vector, $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

2. W contains the 0 function.

Let f and g be two functions from W. $(f+g)(2) = f(2) + g(2) = 0 + 0 = 0$, thus $f + g$ lies in W as well.

If f is a function from W and c any constant, then $(cf)(2) = cf(2) = 0$, thus cf lies in W as well.

3. The dependence equation is $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \\ -2 \\ -5 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix} z + \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix} w = 0$.

I.e. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & -3 & 1 & 0 \\ -1 & -2 & 2 & 2 \\ 1 & -5 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$.

Gaussian elimination:
$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -1 & -2 & 2 & 2 & 0 \\ 1 & -5 & 5 & 3 & 0 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & -7 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & -7 & 5 & 4 & 0 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & -7 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix} \leadsto$
$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & -7 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Not every variable is pivot, hence there are non-zero solutions of the dependence equations, hence the vectors are not linearly independent.

4. A is the coefficient matrix from Problem 1. After reducing (as in Problem 1), we get a matrix in echelon form: $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) The basis of the row space: $\begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 3 \end{bmatrix}$.

(b) The basis of the column space (pivot columns of A): $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

(c) Rank=dimension of the row space= 2.

(d) $\operatorname{null} A + \operatorname{rank} A = 4$, hence $\operatorname{null} A = 2$.

5. A transformation is linear if it satisfies $T(X_1 + X_2) = T(X_1) + T(X_2)$ and $T(cX) = cT(X)$.

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. $T(X_1 + X_2) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) + (y_1 + y_2) - 1 \\ x_1 + x_2 - (y_1 + y_2) \\ 3(y_1 + y_2) \end{bmatrix}$,

whereas $T(X_1) + T(X_2) = \begin{bmatrix} x_1 + y_1 - 1 \\ x_1 - y_1 \\ 3y_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 - 1 \\ x_2 - y_2 \\ 3y_2 \end{bmatrix}$.

Since $T(X_1 + X_2) \neq T(X_1) + T(X_2)$, T is not linear.

6. $AB = \begin{bmatrix} 1(1)+2(2)+0(0) & 1(2)+2(-2)+0(-1) & 1(0)+2(0)+0(-1) & 1(-1)+2(1)+0(3) \\ 0(1)-1(2)-1(0) & 0(2)-1(-2)-1(-1) & 0(0)-1(0)-1(-1) & 0(-1)-1(1)-1(3) \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 & 1 \\ -2 & 3 & 1 & -4 \end{bmatrix}$.

7. $P_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$. $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$.

Computing $C_{\mathcal{B}}$: $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 5 & -3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \rightsquigarrow$
$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$. Then $C_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ -1 & 2 & -1 \end{bmatrix}$.

$X' = C_{\mathcal{B}} X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

8. a) $\det A = 2\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = 16$.

b) $\det A = 16$, $\det A_{21} = \begin{vmatrix} 5 & -3 \\ -2 & 1 \end{vmatrix} = 11$. Answer: $(-1)^{1+2}\dfrac{\det A_{21}}{\det A} = -\dfrac{11}{16}$.

9. Coefficient matrix: same as in problem 8, $\det A = 16$.

For x: $\begin{vmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 3$, $x = \dfrac{3}{16}$.

For y: $\begin{vmatrix} 2 & 1 & -3 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -1$, $y = -\dfrac{1}{16}$.

For z: $\begin{vmatrix} 2 & 5 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{vmatrix} = -5$, $z = -\dfrac{5}{16}$.

10. a) $\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & -2 & 0 \\ 2 & -2-\lambda & 0 \\ 0 & -1 & -1-\lambda \end{bmatrix} = (-1-\lambda)\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} =$ $(-1-\lambda)\left((3-\lambda)(-2-\lambda)+4\right) = -(1+\lambda)(\lambda^2-\lambda-2)$. Eigenvalues: $\lambda = -1, 2$.

Eigenvectors: for -1, solve $\begin{bmatrix} 4 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} X = 0$. Solution (in parametric form): $X = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, solve $\begin{bmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \\ 0 & -1 & -3 \end{bmatrix} X = 0$. Solution: $X = t \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$.

Eigenvectors: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$.

b) If a matrix is diagonalizable, its eigenvectors form a basis of the vector space. Here the space is 3-dimensional but we can obtain only two linearly independet eigenvectors. Hence, A is not diagonalizable.

11. a) $P_1 \cdot P_2 = 1(-1) + 0(1) + (1)1 = 0$, $P_1 \cdot P_3 = 1(2) + 0(4) + 1(-2) = 0$, $P_2 \cdot P_3 = (-1)2 + 1(4) + 1(-2) = 0$.

b) $x_1' = \dfrac{X \cdot P_1}{P_1 \cdot P_1} = \dfrac{4}{2}$, $x_2' = \dfrac{X \cdot P_2}{P_2 \cdot P_2} = \dfrac{5}{3}$, $x_3' = \dfrac{X \cdot P_3}{P_3 \cdot P_3} = \dfrac{4}{24} = \dfrac{1}{6}$.

12. $P_1 = A_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$P_2 = A_2 - \dfrac{A_2 \cdot A_1}{A_1 \cdot A_1} A_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \dfrac{2(1) + 1(-1) + 1(1)}{1(1) - 1(-1) + 1(1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 5/3 \\ 1/3 \end{bmatrix}$.