

## Solutions to PRACTICE EXAM I

1. (a)  $S$  is linearly independent if none of the  $A_i$  can be expressed as a linear combination of other elements of  $S$ .

(b) No:  $x^2 + 1 = (x^2 - x) + (x + 1)$ .

(c) Yes: if  $\cos x$  depends on  $\sin x$ , then  $\cos x = c \sin x$  for some number  $c$ . But then for  $x = 0$ ,  $1 = \cos x = c \sin x = c \cdot 0 = 0$ . On the other hand, if  $\sin x$  depends on  $\cos x$ ,  $\sin x = c \cos x$ . Then for  $x = \pi/2$ ,  $1 = \sin x = c \cos x = c \cdot 0 = 0$ . Thus, no dependence can be found between  $\cos x$  and  $\sin x$ .

2. Vectors in the span:  $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

Vector not in the span:  $Z = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ . If  $Z$  lied in the span of  $X$  and  $Y$ , then  $Z = aX + bY$ . Comparing second entries of  $X, Y$ , and  $Z$ , we see that  $a = 1$ ; comparing third entries, we see that  $b = 1$ . But then the first entry of  $Z$  must be  $a - b = 0$  which it is not.

3. Let  $V = aX + bY$ ,  $W = cX + dY$ . Then any linear combination of  $V$  and  $W$ ,  $eV + fW$  equals  $e(aX + bY) + f(cX + dY) = (ea + fc)X + (eb + fd)Y$ , a linear combination of  $X$  and  $Y$ .

4. The augmented matrix is  $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 & 3 \\ 2 & 6 & 3 & 2 & 1 \\ 5 & -3 & 3 & 5 & 7 \end{bmatrix}$ . After operations  $R_2 -$

$2R_1 \rightarrow R_2$ ,  $R_3 - 2R_1 \rightarrow R_3$ , and  $R_4 - 5R_1 \rightarrow R_4$ , we obtain the matrix

$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 2 \end{bmatrix}$ . After operations  $R_3 + R_2 \rightarrow R_3$  and  $R_4 - 2R_2$ , we

obtain  $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ , a matrix in echelon form. The non-pivot variables

are  $z$  and  $w$ . Set  $z = s$ ,  $w = t$ . The equation corresponding to the second row is  $-4y - z = 1$ , hence  $y = -1/4 - s/4$ . The equation corresponding to the first row is  $x + y + z + w = 1$ , hence  $x = 5/4 - (3/4)s - t$ . In parametric form,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5/4 \\ -1/4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The translation vector is  $\begin{bmatrix} 5/4 \\ -1/4 \\ 0 \\ 0 \end{bmatrix}$ , the spanning vectors are  $\begin{bmatrix} -3/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

$$(b) A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 2 & 6 & 3 & 2 \\ 5 & -3 & 3 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}.$$

5. RREF: A, C, E, G.

Echelon form: A, B, C, E, G, I.

Neither: D, F, H.

$$6. AX = \begin{bmatrix} 1 \\ -8 \\ 8 \end{bmatrix}.$$

7. The nullspace is the set of solutions of  $AX = 0$ .

Reducing the augmented matrix  $\begin{bmatrix} 1 & 3 & 2 & 0 & 0 \\ -1 & 3 & 1 & -2 & 0 \\ 2 & 0 & 1 & 2 & 0 \end{bmatrix}$  of this system to echelon form: after operations  $R_2 + R_1 \rightarrow R_1$  and  $R_3 - 2R_1 \rightarrow R_1$ , we obtain the matrix  $\begin{bmatrix} 1 & 3 & 2 & 0 & 0 \\ 0 & 6 & 3 & -2 & 0 \\ 0 & -6 & -3 & 2 & 0 \end{bmatrix}$ . After operation  $R_3 + R_2 \rightarrow R_3$ , we obtain the matrix  $\begin{bmatrix} 1 & 3 & 2 & 0 & 0 \\ 0 & 6 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . Hence the nullspace is the solution of the system

$$x_1 + 3x_2 + 2x_3 = 0$$

$$x_2 + 3x_3 - 2x_4 = 0$$

(for which the last matrix above is the augmented matrix). The non-pivot variables are  $x_3$  and  $x_4$ . Thus,  $x_2 = -3x_3 + 2x_4$  and  $x_1 = 7x_3 - 6x_4$ . The

general solutions has the form  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ . The nullspace is

the span of the spanning vectors  $U = \begin{bmatrix} 7 \\ -3 \\ 1 \\ 0 \end{bmatrix}$  and  $V = \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ .

8. The coefficient matrix of this system is the matrix  $A$  from Questions 6 and 7. In particular, we know the nullspace of  $A$ . Thus, the set of solutions is  $T + sU + tV$  ( $U$  and  $V$  as in the answer to Question 7).

9. Reducing  $A$  to the echelon form:  $R_3 - 2R_2 \rightarrow R_3$ , we get the matrix  $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The pivot entries are in the first and third columns. Therefore

the column space of  $A$  is spanned by its first and third columns:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ .

10. We use test for linear independence, i.e. look for non-zero solutions of the system  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x_3 = 0$ . The augmented matrix is  $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ .

We switch the last and the first rows (not necessary but makes for easier calculations down the road):  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$  After row operations,  $R_2 - 2R_1 \rightarrow$

$R_2$ ,  $R_3 - 3R_1 \rightarrow R_3$ ,  $R_4 - R_1 \rightarrow R_4$ , we obtain  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$ . Switch

the second and fourth rows:  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ . After  $R_3 - 1/2R_2 \rightarrow R_3$ , get

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ . After  $R_4 - 1/2R_3$ , get  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The only solution is the zero vector. Therefore, the vectors  $V$ ,  $U$ , and  $W$  are linearly independent.