Math 3330

Solutions to PRACTICE EXAM I

1. (a) A reduces to an echelon form $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, a basis of the row space is $\begin{bmatrix} 1 & -2 & 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -0 & -1 & 0 \end{bmatrix}$.

(b) rank $A = 2$ (dimension of the row space).

(c) A basis of the column space consists of columns corresponding to the pivots in the echelon form above, i.e. the first and the third columns: $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix}$.

(d) The number of basis vectors of both spaces is the rank of A.

2. (a) $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. $\dim \mathbb{R}^n = n$.

(b) Consider the dependence equation: $x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = 0$. To solve it, reduce the augmented matrix $\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -1 & 5 & 0 \end{bmatrix}$. We obtain $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & \frac{33}{7} & 0 \end{bmatrix}$ which has the unique solution 0. Thus these vectors are linearly independent. There are 3 of them, thus they form the basis of $\mathbb{R}^3$, thus span it.

(c) The maximum number of linearly independent vectors in a 2-dimensional space is 2 and we are given 3 vectors. The answer is, no.

3. (a) For an $m \times n$ matrix A, rank $A +$ null $A = n$.

(b) null $A = 2$.

4. (a) Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. $T(X_1 + X_2) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 - y_1 - y_2 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix} = T(X_1) + T(X_2)$. Also, $T(aX_1) = T\left(\begin{bmatrix} ax_1 \\ ay_1 \end{bmatrix}\right) = \begin{bmatrix} 2ax_1 - ay_1 \\ ax_1 + ay_1 \end{bmatrix} = aT(X_1)$. Thus T is linear.

(b) The images of the corners of the unit square are: $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$, $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$, $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$. Thus the image is the rectanlge with corners $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$.

(c) The first column of A is $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$, the second column is $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$, thus $A = \begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}$.

5. $T(\sin x) = \cos x$, $T(\cos x) = -\sin x$, thus in the basis $\sin x, \cos x$, the matrix of T is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

6. First of all, $T(0) = 0$, therefore 0 lies in the image of T and the image is non-empty.

Then, let X, Y be elements in the image of T: for some $X', Y' \in \mathcal{V}$, $T(X') = X$ and $T(Y') = Y$. Then $T(aX' + bY') = aT(X') + bT(Y') = aX + bY$ by linearity. We see that for any scalars a, b, $aX + bY$ also lies in the image of T, hence it is a subspace of $\mathcal{W}$.

7. (a) This matrix is $AB = \begin{bmatrix}5 & -1\\5 & 4\\5 & -3\\-11 & 8\end{bmatrix}$.

(b) Not defined because the target space of S has dimension 4 and the domain of T has dimension 2.

8. (a) An echelon form of A is $\begin{bmatrix}1 & -1 & 2\\0 & 1 & -5\\0 & 0 & 1\end{bmatrix}$. Thus its nullspace is zero and T is one-to-one.

(b) From (a) we see that rank $A = 3$, thus the image of the domain is *all* the target space. Hence T is onto.

(c) A is 3×3 and has rank 3, thus it is nonsingular, thus invertible. Computing the inverse: $\left[\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\3 & -2 & 1 & 0 & 1 & 0\\2 & -1 & 0 & 0 & 0 & 1\end{array}\right]$ reduces to $\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -2 & 3\\0 & 1 & 0 & 2 & -4 & 5\\0 & 0 & 1 & 1 & -1 & 1\end{array}\right]$. Therefore, $A^{-1} = \begin{bmatrix}1 & -2 & 3\\2 & -4 & 5\\1 & -1 & 1\end{bmatrix}$.