Math 4322

SOLUTIONS TO PRACTICE FINAL EXAM

1. $\dfrac{2+4i}{2-i} = \dfrac{(2+4i)(2+i)}{(2-i)(2+i)} = \dfrac{10i}{5} = 2i.$
$2i = 2\exp(i\pi/2).$
$\sqrt[3]{2\exp(i\pi/2)} = \sqrt[3]{2}\exp\left(i\left[\dfrac{\pi}{6}+\dfrac{2\pi n}{3}\right]\right) = \sqrt[3]{2}\exp(i\pi/6);\ \sqrt[3]{2}\exp(5i\pi/6);\ \sqrt[3]{2}\exp(2i\pi/3).$

2. $(1+3i)(2-i) = 5+5i.$ $|5+5i| = 5\sqrt{2}.$
$5+5i = 5\sqrt{2}\left(\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i\right).$ $\operatorname{Arg}(5+5i) = \cos^{-1}\dfrac{1}{\sqrt{2}} = \pi/4.$
(Alternative solution: find arguments of $(1+3i)$ and $(2-i)$, add them up.)

3. $\dfrac{z+2i}{z} = \dfrac{x+i(y+2)}{x+iy} = \dfrac{(x+i(y+2))(x-iy)}{x^2+y^2} = \dfrac{x^2+y(y+2)+i(x(y+2)-xy)}{x^2+y^2} =$
$\dfrac{x^2+y(y+2)+2ix}{x^2+y^2}$. I.e. $\operatorname{Im}\dfrac{z+2i}{z} = \dfrac{2x}{x^2+y^2} > 1.$
$2x > x^2+y^2$, i.e. $x^2-2x+y^2<0$ or $x^2-2x+1+y^2<1$. We get that the set is $(x-1)^2+y^2<1$, an <u>open</u> disk of radius 1 with the center at $(1,0)$.

4. $u(x,y) = x^2-y^2, v(x,y) = 2xy$. Partial derivatives: $u_x = 2x, u_y = -2y, v_x = 2y, v_y = 2x$. Cauchy–Riemann equations, $u_x = v_y, u_y = -v_x$, hold; the partial derivatives are continuous. Therefore, $f(x,y) = u(x,y)+iv(x,y)$ is differentiable at every point.
$f'(x,y) = u_x + iv_x = 2x+2yi.$

5. $u(x,y) = \dfrac{y}{x^2+y^2}$, $v(x,y) = \dfrac{x}{x^2+y^2}$. The partial derivatives are: $u_x = \dfrac{-2xy}{(x^2+y^2)^2}, u_y = \dfrac{x^2-y^2}{(x^2+y^2)^2}, v_x = \dfrac{x^2-y^2}{(x^2+y^2)^2}, v_y = \dfrac{-2xy}{(x^2+y^2)^2}$. Thus $f(x+iy)$ satisfies Cauchy–Riemann equations if and only if $u_x = v_y$ and $u_y = -v_x$. The first condition holds always and the second, only when $(x^2-y^2) = -(x^2-y^2)$. This implies that $x^2 = y^2$, i.e. either $x = y$ or $x = -y$. Thus, $f(x+iy)$ can be differentiable only at points with either $x = y$ or $x = -y$. However, in order for the function to be analytic at a point, it must be differentiable in a neighborhood of this point. Since every neighborhood of a point on the line $x = y$ (or $x = -y$) will have points outside of the line, the function is not analytic.

6. $|2-2i| = 2^{\frac{3}{2}}$, $\operatorname{Arg}(2-2i) = \dfrac{7\pi}{4}.$
$\operatorname{Log}(2-2i) = \ln 2^{\frac{3}{2}} + i\operatorname{Arg}(2-2i) = \dfrac{3}{2}\ln 2 + \dfrac{7\pi i}{4}.$

7. z^3 is analytic and the endpoints of C are 0 and i, therefore $\int_C z^3 dz = \left.\dfrac{z^4}{4}\right|_0^i = \dfrac{i^4}{4} = \dfrac{1}{4}.$

8. This is an entire function and the contour is closed, thus the answer is 0.

9. By Cauchy integral formula, $\int_C \frac{\cos z}{(z-\pi)^3} dz = \frac{2\pi i}{2!} \cos^{(2)}(\pi)$. $(\cos z)' = -\sin z$, $(-\sin z)' = -\cos z$, $-\cos\pi = -1$. The answer is $\frac{2\pi i}{2!}(-1) = -\pi i$.

10. For $z \neq 0$, $f(z) = 2\frac{\cos z - 1}{z^2} = 2\frac{-\frac{z^2}{2} + \frac{z^4}{4!} - \dots}{z^2} = -1 + \frac{z^2}{12} - \dots$. At $z = 0$ this series also equals $f(0)$, thus it equals $f(z)$ everywhere. Since every power series is entire, so is $f(z)$.

11. $\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$.
In the annulus $2 < |z-1| < 3$:
$$\frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{1}{z-1}\frac{1}{1-\frac{2}{z-1}} = \frac{1}{z-1}\sum_{k=0}^{\infty}\left(\frac{2}{z-1}\right)^k = \sum_{k=0}^{\infty}\frac{2^k}{(z-1)^{k+1}} = \sum_{m=1}^{\infty}\frac{2^{m-1}}{(z-1)^m}.$$
$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n.$$
Answer: $-\sum_{n=0}^{\infty}(z-1)^n + \sum_{m=1}^{\infty}\frac{2^{m-1}}{(z-1)^m}$.

12. The denominator is $(z-5)(z-1)(z^2-1) = (z-5)(z-1)^2(z+1)$. The poles *inside* C are $z = -1, 1$. (The radius of C is 4, so the $z = 5$ is outside of C.)
$$\operatorname*{Res}_{z=1}\frac{z^3+1}{(z-5)(z-1)^2(z+1)} = \frac{d}{dz}\frac{z^3+1}{(z-5)(z+1)}\bigg|_{z=1} = \frac{3z^2(z^2-4z-5)-(z^3+1)(2z-4)}{(z^2-4z-5)^2}\bigg|_{z=1} = \frac{-20}{64} = -\frac{5}{16}.$$
$$\operatorname*{Res}_{z=-1}\frac{z^3+1}{(z-5)(z-1)^2(z+1)} = \frac{z^3+1}{(z-5)(z-1)^2}\bigg|_{z=-1} = 0.$$
$$\int_C \frac{z^3+1}{(z-5)(z-1)(z^2-1)}dz = 2\pi i\left(\operatorname*{Res}_{z=1}\frac{z^3+1}{(z-5)(z-1)^2(z+1)} + \operatorname*{Res}_{z=-1}\frac{z^3+1}{(z-5)(z-1)^2(z+1)}\right) = 2\pi i\left(-\frac{5}{16}+0\right) = -\frac{5\pi i}{8}.$$

13. The function $\frac{1}{(x^2+1)(x^2+4)}$ is even, so
$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(x^2+1)(x^2+4)}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(z^2+1)(z^2+4)}dz.$$
To compute the latter integral, consider the positively oriented contour C made up of the interval $[-R, R]$ and C_R, the upper half of the circle of radius R centered at the origin.
$$\int_C \frac{1}{(z^2+1)(z^2+4)}dz = 2\pi i\left(\operatorname*{Res}_{z=i}\frac{1}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i}\frac{1}{(z^2+1)(z^2+4)}\right) = 2\pi i\left(\frac{1}{(z+i)(z^2+4)}\bigg|_{z=i} + \frac{1}{(z^2+1)(z+2i)}\bigg|_{z=2i}\right) = 2\pi i\left(\frac{1}{6i}+\frac{1}{-12i}\right) = \frac{\pi}{6}.$$
On C_R, $\left|(z^2+1)(z^2+4)\right| \leq (R^2-1)(R^2-4)$, so $\left|\int_{C_R}\frac{1}{(z^2+1)(z^2+4)}dz\right| \leq \frac{\pi R}{(R^2-1)(R^2-4)} \to 0$ as $R \to \infty$.

It follows that
$\lim_{R\to\infty}\int_{-R}^{R}\frac{dz}{(z^2+1)(z^2+4)} = \int_C \frac{dz}{(z^2+1)(z^2+4)} - \lim_{R\to\infty}\int_{C_R}\frac{dz}{(z^2+1)(z^2+4)} = \frac{\pi}{6}$.

14. The function $\frac{x\sin x}{x^2+1}$ is even, hence $\int_0^\infty \frac{x\sin x}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+1}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin x}{x^2+1}dx = \frac{1}{2}\mathrm{Im}\lim_{R\to\infty}\int_{-R}^{R}\frac{z\exp(iz)}{z^2+1}dz$.

To compute the latter integral, consider the positively oriented contour C made up of the interval $[-R,R]$ and C_R, the upper half of the circle of radius R centered at the origin.

$\int_C \frac{z\exp(iz)}{z^2+1}dz = 2\pi i \operatorname*{Res}_{z=i}\frac{z\exp(iz)}{z^2+1} = 2\pi i\left.\frac{z\exp(iz)}{z+i}\right|_{z=i} = \pi i e^{-1}$.

On C_R, $\left|\frac{z}{z^2+1}\right| \le \frac{R}{R^2-1} \to 0$ as $R\to\infty$. By Jordan's lemma, $\int_{C_R}\frac{z\exp(iz)}{z^2+1}dz \to 0$ as $R\to\infty$.

Thus, $\frac{1}{2}\mathrm{Im}\lim_{R\to\infty}\int_{-R}^{R}\frac{z\exp(iz)}{z^2+1}dz = \frac{1}{2}\mathrm{Im}\pi i e^{-1} = \frac{\pi}{2e}$.

15. Let $f(z)=z^4$. On the circle $|z|=2$, $|g(z)-f(z)| = |7z+1| \le 7|z|+1 = 15$ but $|f(z)| = 2^4 = 16$. Hence, $g(z)$ and $f(z)$ have the same number of roots inside the circle $|z|<2$. Since all four roots of $f(z)$ lie inside the circle, the same holds for $g(z)$.