

# Abstract

## Associative Conformal Algebras and Pseudoalgebras and Their Representations

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Lie conformal algebras axiomatically describe singular parts of vertex algebras. Conversely, a vertex algebra can be reconstructed from a conformal algebra and its highest weight module. The main subject of this thesis, associative conformal algebras, plays an important role in conformal representation theory. In particular, all pseudolinear maps of a finite module of rank  $n$  form a conformal algebra  $\text{Cend}_n$ . Pseudoalgebras generalize conformal algebras and are also related to differential Lie algebras of Ritt and Hamiltonian formalism in the calculus of variations.

This thesis is roughly divided into two parts. We begin by defining a particular class of associative pseudoalgebras called unital. They resemble unital algebras in “ordinary” algebra. Not every pseudoalgebra is unital; however,  $\text{Cend}_n$  are. We describe how unital pseudoalgebras that satisfy a broad technical condition are completely determined by an associative algebra and a family of locally nilpotent operators acting on it. This allows us to classify representations of all semisimple unital associative pseudoalgebras. In particular, we provide an explicit description of finite modules over conformal  $\text{Cend}_n$ .

The second part of this thesis is devoted to classifying pseudoalgebras that are algebraically similar to  $\text{Cend}_n$ . We introduce the concept of Gelfand-Kirillov dimension for pseudoalgebras and, in particular, conformal algebras, and prove that a simple unital associative conformal algebra of Gelfand-Kirillov dimension 1 is necessarily  $\text{Cend}_n$  for some  $n$ . We also generalize this result to the case of semisimple conformal algebras. In the case of pseudoalgebras such a nice classification does not go through. Instead, we provide a number of conditions that describe a family of associative pseudoalgebras similar to  $\text{Cend}_n$ . In the process we obtain new examples of pseudoalgebras.

**Associative Conformal Algebras  
and Pseudoalgebras  
and Their Representations**

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# Introduction

## General remarks

**Vertex algebras.** In the last several decades vertex algebras and related structures played an important role in many diverse mathematical fields.

Historically, they first appeared in the guise of vertex operators in the context of string theory, see e.g. [BPZ]. Another important development was the realization of representations of affine Lie algebras via vertex operators [LW, FrK].

Yet in a different settings, vertex algebras were used in constructing the representation of the Fischer-Griess Monster group [FLM], the so-called Moonshine module  $V^\natural$ . In fact,  $V^\natural$  has the structure of a vertex algebra and the Monster is its full group of automorphisms. The axiomatic definition of vertex algebras first appeared in [Bo] in connection with the work on the Monster. Another approach can be found in [FHL].

It was well-known that the Monster is related to the modular form  $j(q)$ ; however, there exists a deeper and more general connection between vertex algebras and modular forms [Zh]. Among other areas where vertex algebras started to appear recently, one should mention several branches of algebraic geometry (such as  $\mathcal{D}$ -modules [AD] and singular varieties [BL]).

**Conformal algebras and pseudoalgebras.** Despite the progress in the study of vertex algebras, their structure remains mysterious. It has long been understood that from an algebraic standpoint, the structure of a vertex algebra is mostly encoded by its singular part. As separate objects, these were considered, in particular, in [K1, LZ, Pr]. Axiomatically, their theory was developed in [K1] where they were called conformal algebras.

Roughly speaking, many vertex algebras can be presented as highest weight modules over corresponding conformal algebras. These conformal algebras are smaller and more transparent objects, thus one can hope that the study of the structure and representations of these conformal algebras will lead to a greater understanding of vertex algebras. So far, conformal algebras were useful, in particular, in the classification of simple linearly compact Lie superalgebras [K3] and the study of the structure of lattice vertex algebras [Ro3].

A development simultaneous to the beginnings of the study of conformal algebras was the coordinate-independent approach to vertex algebras (see, e.g. [Hu]). Similar ideas led, in part, to the theory of chiral algebras developed in [BD] (see also [Ga] for the physical applications of this theory and [Fr] for relations between chiral and vertex algebras). Chiral algebras arise as naturally defined objects in pseudotensor categories. A more geometric approach to vertex and chiral algebras [KV] also begins with the construction of an analogue of the singular part of a vertex algebra.

Using the language of pseudotensor categories, a natural generalization of conformal algebras was introduced in [BDK]. These objects, called pseudoalgebras, are also related to the differential Lie algebras of Ritt and Hamiltonian formalism in the theory of nonlinear evolution equations (on this, see also [GD], where conformal algebras already appear, albeit in disguise, and [Xu]).

Classification of semisimple finite Lie conformal algebras and Lie pseudoalgebras was achieved, respectively, in [DK] (for the super case, see [FK]) and [BDK]. Representations of semisimple finite Lie conformal algebras were studied and classified in [CK] and [CKW].

**Pseudolinear algebras.** In the theory of conformal algebras, as well as in the theory of pseudoalgebras, a major role is played by pseudolinear algebras  $\text{Cend}_n$  of all pseudolinear endomorphisms of a free module of rank  $n$ , somewhat analogous to  $n \times n$ -matrices in ordinary algebra. However,  $\text{Cend}_n$  are not finite in any sense and this is a major obstacle for their study. It should also be mentioned that apart from playing an important role in the study of vertex algebras,  $\text{Cend}_n$  are closely related to the algebra  $\mathcal{W}_{1+\infty}$  and vertex algebras  $\mathcal{W}(gl_N)$  [FKRW, K1]. The related study of representations of orthogonal and symplectic subalgebras of  $\text{End}_n(\mathbb{C}) \otimes \mathbb{C}[\partial_z]$  is also of interest [Ze2] and has some physical applications as well (e.g., in the study of quantum Hall effect [CTZ]).

The starting point for this research is the study of the structure and representations of  $\text{Cend}_n$ .

We shall describe a class of associative pseudoalgebras that naturally includes  $\text{Cend}_n$ , classify their representations, and describe conformal algebras and pseudoalgebras with similar properties.

Most of original results presented here can be found in [Re1] and [Re2]

### Organization of this manuscript

In Chapter 1 we outline the general theory of conformal algebras and pseudoalgebras and provide several examples that are often used in subsequent chapters. This chapter is mostly expository and relies on material contained in [K1] and [BDK]. We certainly do not review all major facts about conformal algebras and pseudoalgebras (in particular, Lie conformal algebras are mentioned only in

passing); nonetheless, for our purposes the review is self-contained. Some examples, in particular Example 1.38, are new.

In Chapter 2 we define an important subclass of pseudoalgebras, unital algebras, and describe their structure in the semisimple case (Theorem 2.1, for the proof see Subsection 2.1.2). Similar methods are used to obtain a complete description of representations of such algebras (Theorem 2.2, the proof is in Section 2.2). As an immediate corollary, we obtain the classification of representations of  $\text{Cend}_n$  from **[K2]** (Propositions 2.29 and 2.30).

Chapter 3 is technical and is devoted to Gelfand-Kirillov dimension of conformal algebras and pseudoalgebras.

In Chapter 4 we classify conformal algebras and pseudoalgebras similar to  $\text{Cend}_n$ . For conformal algebras, we are able to obtain a complete description of simple and semisimple conformal algebras of linear growth (Theorems 4.1 and 4.19) in Section 4.2. We rely heavily on the theory of growth developed in the previous chapter and the results of **[SSW]** on the structure of associative algebras of linear growth. We also construct new examples of subalgebras of current conformal algebras in Subsection 4.2.4. In the more general case of pseudoalgebras, a similarly nice classification seems impossible. Nonetheless, under certain conditions we classify pseudoalgebras with properties similar to  $\text{Cend}_n$ 's (Theorem 4.2) in Section 4.3. We also state several conjectures on how these conditions might be relaxed.

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## Preliminaries: Basic Facts and Examples

In this chapter we introduce our basic concepts: conformal algebras and pseudoalgebras. Major sources for these subjects are, respectively, [K1, K2] and [BDK]. When appropriate, more precise references are given in the text.

We always work over an algebraically closed field  $\mathbb{k}$  of characteristic 0. By a module, we always understand a left module. Unless specified otherwise, the word “algebra” always stands for an associative unital algebra.

### 1.1. Conformal algebras

Most of the material in this section can be found in [K1, Chapter 2].

#### 1.1.1. Basic definition and examples.

DEFINITION 1.1. A *conformal algebra*  $C$  is a  $\mathbb{k}[\partial]$ -module endowed with bilinear operations  $\circledast_n : C \otimes C \rightarrow C$   $n \in \mathbb{Z}_{\geq 0}$ , called multiplications of order  $n$ , that satisfies the following axioms. For any  $f, g \in C$ ,

$$\text{C1 (Locality): } a \circledast_n b = 0 \text{ for } n \gg 0;$$

$$\text{C2 (Leibniz rule): } \partial(f \circledast_n g) = (\partial f) \circledast_n g + f \circledast_n (\partial g);$$

$$\text{C3: } (\partial f) \circledast_n g = -nf \circledast_{n-1} g.$$

The minimal  $N$ , such that  $f \circledast_n g = 0$  for all  $n \geq N$ , is called the *degree of locality* of  $f$  and  $g$  and is denoted  $N(f, g)$ .

EXAMPLE 1.2. Any ordinary (i.e. non-conformal) algebra  $A$  naturally gives rise to a *current conformal algebra*  $\text{Cur } A = \mathbb{k}[\partial] \otimes A$ . Namely, for  $a, b \in A$  put  $(1 \otimes a) \circledast_n (1 \otimes b) = \delta_{0,n} 1 \otimes ab$  and extend the operations to all of  $\text{Cur } A$  via the axioms C2 and C3.

EXAMPLE 1.3. The following example was the main rationale behind the introduction of conformal algebras in [K1]. Again, let  $A$  be an arbitrary algebra. Consider the algebra  $A[[z, z^{-1}]]$  of *formal distributions* (called in physical literature chiral fields) over  $A$ . For  $n \geq 0$  any two formal

distributions  $f(z), g(z)$  define the  $n$ -th product as

$$f(z) \circledast_n g(z) = \text{Res}_{w=0} f(w)g(z)(w-z)^n$$

(by  $\text{Res}_{w=0} h(w, z)$  we mean a formal distribution in  $z$  that is a coefficient at  $w^{-1}$  in  $h(w, z)$  viewed as a formal distribution over  $\mathbb{k}[[z, z^{-1}]]$ ).

When two formal distributions are local (i.e. when  $f(z)g(w)(z-w)^N = 0$  for some  $N$ ), we arrive at the formula known as the operator product expansion:

$$f(z)g(w) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{f(w) \circledast_n g(w)}{(z-w)^{n+1}}, \quad (1.1)$$

which plays an important role in conformal field theory.

It is easy to see that formal distributions automatically satisfy C2 and C3 for  $\partial = \partial/\partial z$ . Whenever  $f$  and  $g$  are local, so are the pairs  $\partial f, g$ , and  $f, \partial g$ . Namely, the axioms imply  $N(f, \partial g) = N(\partial f, g) = N(f, g) + 1$ .

Therefore, a set of pairwise mutually local formal distributions over  $A$  which is closed with respect to multiplications and differentiation  $\partial/\partial z$  forms a conformal algebra.

In general, a set of mutually local formal distributions does not generate a conformal algebra, as locality property may be lost (for instance, this might happen in the conformal Tits-Kantor-Koecher construction of a Jordan conformal algebra, see [Ze1]). However, when  $A$  is Lie or associative, this is never the case:

LEMMA 1.4 (Dong's lemma [Li, K1]). *Let  $f, g$ , and  $h$  be pairwise mutually local formal distributions over either a Lie or associative algebra. Then for any  $n \geq 0$ ,  $f \circledast_n g$  and  $h$  are again pairwise mutually local.*

REMARK 1.5. Formal distributions over algebras from some other varieties also allow for statements similar to Dong's lemma. For example, if formal distributions  $f_1, f_2, f_3, f_4$  over a Jordan algebra are pairwise mutually local and for any  $i, j, k$ , and  $n$ ,  $f_i \circledast_n f_j$  and  $f_k$  are also pairwise mutually local, then a product of any three of these distributions with any orders of multiplications and the fourth distribution are pairwise mutually local.

The number of distributions in such Dong-like statements generally depends on degrees of identities for a given variety.

One can easily transform basic definitions from the structural theory of (ordinary) algebras into conformal language. Thus, a conformal *ideal*  $I$  of a conformal algebra  $C$  is a subalgebra such that

for any  $f \in I, g \in C$  and any  $n, f \circledast_n g \in I$  and  $g \circledast_n f \in I$ ; a conformal algebra is *simple* if its only ideals are 0 and itself; etc.

**1.1.2. Varieties of conformal algebras.** In fact, every conformal algebra can be represented as an algebra of formal distributions. The following construction (affinization) was introduced in [K1] generalizing the construction of [Bo] for vertex algebras.

Let  $C$  be a conformal algebra. We can consider  $C$  equipped only with the operation  $\circledast$ ; this makes it into an ordinary algebra. Notice that  $\partial C$  is an ideal of  $(C, \circledast)$ . Define another conformal algebra  $\hat{C} = C[t, t^{-1}]$  equipped with the derivative  $\hat{\partial} = \partial + \partial/\partial t$  and the  $n$ -th product

$$ft^l \circledast_n gt^m = \sum_{j \in \mathbb{Z}_+} \binom{l}{j} (f \circledast_{n+j} g) t^{l+m-j}. \quad (1.2)$$

DEFINITION 1.6. The *coefficient algebra*  $\text{Coeff } C$  of  $C$  is the algebra  $(\hat{C}, \circledast)/(\hat{\partial}\hat{C})$ .

Denote the map  $\hat{C} \rightarrow \text{Coeff } C$  by  $\phi$ . The algebra of formal distributions over  $\text{Coeff } C$  consisting of series  $\sum \phi(ft^n)z^{-n-1}, f \in C$ , is isomorphic to  $C$ .

EXAMPLE 1.7. For a current algebra  $\text{Cur } A$ , its coefficient algebra is  $A[t, t^{-1}]$  (its elements are usually called “currents” in the literature, hence the name). Thus  $\text{Cur } A$  is generated over  $\mathbb{k}[\partial]$  by formal distributions  $\sum at^n z^{-n-1}$ .

REMARK 1.8. In Example 1.2,  $\text{Cur } A$  was generated by elements of  $A$ , i.e. it could have been represented by formal distributions over  $A$  of the form  $az^{-1}$ . This shows that a presentation of a conformal algebra  $C$  as an algebra of formal distributions over  $\text{Coeff } C$  is not unique. However, the above construction is universal: if  $C$  embeds into an algebra  $A[[z, z^{-1}]]$  of formal distributions, then there is a unique homomorphism  $\text{Coeff } C \rightarrow A$  such that the following diagram commutes

$$\begin{array}{ccc} \text{Coeff } C[[z, z^{-1}]] & \longrightarrow & A[[z, z^{-1}]] \\ & \searrow & \nearrow \\ & C & \end{array} .$$

We will write  $\phi(ft^n)$  simply as  $f(n)$  and call this element the  $n$ -th coefficient of  $f$ .  $\text{Coeff } C$  contains an important subalgebra  $\phi(C)$  of zeroth coefficients denoted  $(\text{Coeff})_0 C$ .

The axioms of conformal algebras together with (1.2) imply the following identities in  $\text{Coeff } C$ . For arbitrary  $n, m \in \mathbb{Z}$ ,

$$\sum_{j=0}^{N(f,g)} (-1)^j \binom{N(f,g)}{j} f(n-j)g(m+j) = 0, \quad (\text{locality})$$

$$\begin{aligned}
(\partial f)(n) &= -nf(n-1), \\
(f \circledast g)(m) &= \sum_{j=0}^n (-1)^j \binom{n}{j} f(n-j)g(m+j), \\
f(n)g(m) &= \sum_{j \geq 0} \binom{n}{j} (f \circledast g)(n+m-j).
\end{aligned}$$

These formulas will be freely used below, especially in Chapter 3.

We say that a conformal algebra  $C$  is  $\mathcal{X}$  if  $\text{Coeff } C$  belongs to the variety of  $\mathcal{X}$  algebras. In this way, we can define Lie conformal or associative conformal algebras. In general, given (completely linearized) identities for a variety of ordinary algebras one can always write down the corresponding set of identities for conformal algebras.

REMARK 1.9. To avoid confusion, we will follow the conventions of [Ro2] and denote multiplications in Lie conformal algebras as  $\overline{\square}$ .

Thus, Lie conformal algebras are determined by

$$\begin{aligned}
f \overline{\square} g &= - \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{(-1)^{n+j}}{j!} \partial(g \overline{\square}^{n+j} f), \quad (\text{anticommutativity}) \\
f \overline{\square} (g \overline{\square} h) &= \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{m}{j} (f \overline{\square}^j g) \overline{\square}^{m+n-j} h + g \overline{\square} (f \overline{\square} h), \quad (\text{Jacobi identity})
\end{aligned}$$

and associative conformal algebras by either of the two identities for associativity:

$$\begin{aligned}
f \overline{\circ} (g \overline{\circ} h) &= \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{m}{j} (f \overline{\circ}^j g) \overline{\circ}^{m+n-j} h, \\
(f \overline{\circ} g) \overline{\circ} h &= \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j \binom{m}{j} f \overline{\circ}^{m-j} (g \overline{\circ}^{n+j} h).
\end{aligned} \tag{1.3}$$

The above, as well as the identity for conformal commutativity are deduced in [K1, Chapter 2].

EXAMPLE 1.10. As in the ordinary case, given an associative conformal algebra  $C$ , one can endow it with a Lie conformal structure:

$$f \overline{\square} g = f \overline{\circ} g + \sum_{j \geq 0} (-1)^{n+j} \frac{\partial^j}{j!} g \overline{\circ}^{n+j} f.$$

The resulting algebra is denoted  $C^{(-)}$ .

**1.1.3. Modules over conformal algebras and conformal linear maps.** Similarly to Definition 1.1 we can define modules over conformal algebras from particular varieties. In this work we will work only with modules over associative conformal algebras.

DEFINITION 1.11. A module  $M$  over an associative conformal algebra  $C$  is a  $\mathbb{k}[\partial]$ -module equipped with bilinear operations  $\circledast_n : C \otimes M \rightarrow M$ ,  $n \in \mathbb{Z}_{\geq 0}$ , satisfying the following axioms. For any  $f, g \in C, v \in M$ :

M1 (Locality):  $f \circledast_n v = 0$  for  $n \gg 0$ ;

M2 (Leibniz rule):  $\partial(f \circledast_n v) = \partial f \circledast_n v + f \circledast_n \partial v$ ;

M3:  $(\partial f) \circledast_n v = -nf \circledast_{n-1} v$ ;

MA (associativity):  $f \circledast_m (g \circledast_n v) = \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{m}{j} (f \circledast_j g) \circledast_{m+n-j} v$ .

The definition of modules over Lie conformal algebras is similar to the above; one only needs to replace MA with a version of the conformal Jacobi identity.

For more details on conformal modules, the reader is referred to [CK].

Given a  $\mathbb{k}[\partial]$ -module  $M$ , one can define a *conformal linear map*  $f$  as a collection of  $\mathbb{k}$ -linear maps  $f \circledast_n : M \rightarrow M$ ,  $n \geq 0$ , that satisfy M2 and M3. The set of conformal linear maps is denoted  $\text{Cend } M$ . It has the structure of an associative conformal algebra (with an obvious definition of multiplication via the associativity rule on  $\text{Cend } M \otimes \text{Cend } M \otimes M$ ); however, locality is not necessarily satisfied. Yet, when  $M$  is finite over  $\mathbb{k}[\partial]$ ,  $\text{Cend } M$  becomes an associative conformal algebra. For  $M = \mathbb{k}[\partial]^n$ , it is denoted simply  $\text{Cend}_n$ . For a fuller discussion, see [K1, 2.10], we will only provide a description of  $\text{Cend}_n$  slightly different from that in [K1]:

EXAMPLE 1.12. Denote by  $W$  the Weyl algebra  $\mathbb{k}\langle x, t \mid xt - tx = 1 \rangle$  and by  $W_t$  its localization at  $t$ . We define conformal algebra  $\mathfrak{W}_n$  as an algebra of formal distributions over  $W_t$  generated by distributions  $L_A^k = \sum Ax^k t^n z^{-n-1}$ ,  $k \geq 0$ ,  $A \in \text{End}_n(\mathbb{k})$ . In particular, we will call  $\mathfrak{W}_1$  the conformal Weyl algebra.

It is clear that the conformal Weyl algebra is generated by elements  $L^k = \sum x^k t^n z^{-n-1}$  for  $k = 0, 1$ . Their non-zero products are

$$\begin{aligned} L^0 \circledast L^0 &= L^0, & L^0 \circledast L^1 &= L^1 \circledast L^0 = L, \\ L^0 \circledast L^1 &= L^1 \circledast L^0 = -L^0, & & \\ L^1 \circledast L^1 &= L^2, & L^1 \circledast L^1 &= -L^1. \end{aligned} \tag{1.4}$$

Notice that the conformal Weyl algebra is not finite over  $\mathbb{k}[\partial]$ .

Consider now the standard model of a finite  $\mathbb{k}[\partial]$ -module of  $\text{rk } n$ :

$$E_n = \left\{ a(z) = \sum_{a \in \mathbb{k}^n} at^n z^{-n-1} \right\}.$$

Its algebra of conformal linear maps  $\text{Cend}_n$  is spanned by formal distributions of the form  $J_A^m = \sum At^n(-\partial_t)^m z^{-n-1}$ , where  $A \in \text{End}_n(\mathbb{k})$  [K1, 2.10]. (The action is standard.) It is easy to see that these elements provide another basis of  $\mathfrak{W}_n$  (see Example 1.34). Thus,  $\text{Cend}_n \cong \mathfrak{W}_n$ . In particular, this implies that  $\text{Coeff } \text{Cend}_n \cong \text{End}_n(W_t)$ .

To define a representation  $M$  of an associative conformal algebra  $C$  such that  $M$  is finite over  $\mathbb{k}[\partial]$ ,  $\text{rk } M = n$ , is equivalent to providing a map  $C \rightarrow \text{Cend}_n$ . Thus, the knowledge of the structure of  $\text{Cend}_n$  and its modules provides an insight into representation theory of any conformal algebra.

In the Lie case,  $\text{Cend}_n$  should be replaced with the general conformal algebra  $\text{gc}_n = \text{Cend}_n^{(-)}$ .

## 1.2. Pseudoalgebras

**1.2.1. Pseudotensor categories.** The theory of pseudotensor categories was developed in [BD] as a way of expressing such notions as Lie algebras, representations etc. in purely categorical terms. The ultimate goal is to define these notions for categories of modules that have an interesting action on tensor products, e.g.  $\mathcal{D}$ -modules (as in [BD]) or modules over a Hopf algebra (as in [BDK] or this paper).

More details on pseudo-tensor categories can be found in [BD, Chapter 1] and [BDK, Chapter 4], we will get by with a short presentation of main definitions and several examples.

Denote by  $\mathcal{S}$  the category of finite non-empty sets with surjective maps. For a morphism  $\pi : J \rightarrow I$  and  $i \in I$ , we denote  $\pi^{-1}(i) = J_i$ .

**DEFINITION 1.13.** A *pseudotensor category* is a class of objects  $\mathcal{M}$  together with the following data:

- For any  $I \in \mathcal{S}$ , a family of objects  $\{L_i\}_{i \in I}$  and an object  $M$ , one has the set of *polynomial maps*  $\text{Lin}_I(\{L_i\}, M)$ ; the symmetric group  $S_I$  acts on  $\text{Lin}_I(\{L_i\})$ .
- For any morphism  $\pi : J \rightarrow I$  in  $\mathcal{S}$ , the families of objects  $\{L_i\}_{i \in I}$  and  $\{N_j\}_{j \in J}$ , and an object  $M$ , there exists the composition map

$$\text{Lin}_I(\{L_i\}, M) \otimes \bigotimes_{i \in I} \text{Lin}_{J_i}(\{N_j\}, L_i) \rightarrow \text{Lin}_J(\{N_j\}, M),$$

$$\phi \times \{\psi_i\}_{i \in I} \mapsto \phi \circ (\otimes_i \psi_i) = \phi(\{\psi_i\}_{i \in I}).$$

This data satisfies the following properties:

**Associativity:** For a surjective map  $K \rightarrow J$  and a  $K$ -family of objects  $\{P_k\}_{k \in K}$ , one has

$$\phi(\{\psi_i(\{\chi_j\})\}) = (\phi(\{\psi_i\}))(\{\chi_j\}) \in \text{Lin}_K(\{P_k\}, M), \text{ given } \chi_j \in \text{Lin}_{K_j}(\{P_k\}, N_j);$$

**Unit:** For any object  $M$ , there exists an element  $\text{id}_M \in \text{Lin}(\{M\}, M)$  such that for any  $\phi \in \text{Lin}_I(\{L_i\}, M)$ , one has  $\text{id}_M(\phi) = \phi(\{\text{id}_{L_i}\}) = \phi$ ;

**Equivariance:** The compositions of polylinear maps are equivariant with respect to the natural action of the symmetric group.

EXAMPLES 1.14. 1. For the category  $\mathcal{V}ec$  of vector spaces, put  $\text{Lin}_I(\{L_i\}, M) = \text{Hom}(\otimes_i L_i, M)$ . The symmetric group acts on  $\text{Lin}_I(\{L_i\}, M)$  by permuting the factors of  $\otimes_i L_i$ .

2. Let  $H$  be a cocommutative bialgebra with a comultiplication  $\Delta : H \rightarrow H^{\otimes 2}$  and  $\mathcal{M}^l(H)$  its category of left modules. This is a symmetric tensor category; hence, it can be made into a pseudotensor category:  $\text{Lin}_I(\{L_i\}, M) = \text{Hom}_H(\otimes_i L_i, M)$ .

3. We will introduce another pseudotensor structure on  $\mathcal{M}^l(H)$ .

Recall that  $\Delta$  gives rise to a functor  $\mathcal{M}^l(H) \rightarrow \mathcal{M}^l(H^{\otimes 2})$ ,  $M \mapsto H^{\otimes 2} \otimes_H M$ , where  $H$  acts on  $H^{\otimes 2}$  via  $\Delta$ . This may be generalized as follows. For every surjection  $\pi : J \twoheadrightarrow I$ , define a functor  $\Delta^{(\pi)} : \mathcal{M}^l(H^{\otimes I}) \rightarrow \mathcal{M}^l(H^{\otimes J})$ ,  $M \mapsto H^{\otimes J} \otimes_{H^{\otimes I}} M$ , where  $H^{\otimes I}$  acts on  $H^{\otimes J}$  via the iterated comultiplication determined by  $\pi$  (the  $i$ -th copy of  $H$  is mapped into  $H^{\otimes J_i}$ ). This is well-defined because of coassociativity.

Denote the tensor product functor  $\mathcal{M}^l(H)^I \rightarrow \mathcal{M}^l(H^{\otimes I})$  by  $\boxtimes_{i \in I}$ . Then we can define a pseudotensor category  $\mathcal{M}^*(H)$  that has the same objects as  $\mathcal{M}^l(H)$  but with

$$\text{Lin}_I(\{L_i\}, M) = \text{Hom}_{H^{\otimes I}}(\boxtimes_{i \in I} L_i, H^{\otimes I} \otimes_H M). \quad (1.5)$$

For  $\pi : J \twoheadrightarrow I$ , the composition of polylinear maps is defined as follows:

$$\phi(\{\psi_i\}) = \Delta^{(\pi)}(\phi) \circ (\boxtimes_i \psi_i). \quad (1.6)$$

The symmetric group acts on  $\text{Lin}_I(\{L_i\}, M)$  by simultaneously permuting the factors in  $\otimes_i L_i$  and  $H^{\otimes I}$ . This is well-defined because of cocommutativity.

Examples of explicit calculations in  $\mathcal{M}^*(H)$  will be provided below.

Given a pseudotensor category, one can define corresponding structures. Namely, a *Lie algebra* in a pseudotensor category  $\mathcal{M}$  is an object  $L$  together with a polylinear map  $\beta \in \text{Lin}(\{L, L\}, L)$  that satisfies analogues of skew-commutativity and the Jacobi identity:  $\beta = -(12)\beta$ , where  $(12) \in S_2$ , and  $\beta(\beta(\cdot, \cdot), \cdot) = \beta(\cdot, \beta(\cdot, \cdot)) - (12)\beta(\cdot, \beta(\cdot, \cdot))$ , where  $(12)$  now lies in  $S_3$ .

An *associative algebra* is an object  $R \in \mathcal{M}$  together with a polylinear map  $\mu \in \text{Lin}(\{R, R\}, R)$  satisfying associativity  $\mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot))$ .

In  $\mathcal{V}ec$  associative (Lie) algebras are just the associative (Lie) algebras in their usual sense. To avoid confusion we will sometimes call these algebras *ordinary*. The same is true of representations (to be defined below), cohomology (see [BDK]), etc.

REMARK 1.15. Consider an associative algebra  $(R, \mu)$  in a pseudotensor category  $\mathcal{M}$ . Then the pair  $(R, \mu - (12)\mu)$  is a Lie algebra in  $\mathcal{M}$  [BDK, Prop. 3.11]. However, not every Lie algebra can be represented in this form, i.e. the PBW theorem does not necessarily hold in a generic pseudotensor category (see [Ro2] for the case of conformal algebras).

A *representation* of an associative algebra  $(R, \mu)$  in  $\mathcal{M}$  is an object  $M$  (a *module*) together with  $\rho \in \text{Lin}(\{R, M\}, R)$  satisfying  $\rho(\mu(\cdot, \cdot), \cdot) = \rho(\cdot, \rho(\cdot, \cdot))$ . Representations of Lie algebras are defined in the similar way.

**1.2.2. Preliminaries on Hopf algebras.** Before proceeding further, we need to recall several facts about Hopf algebras (see [Jo, Chapter 1] or [Sw] for basic definitions and notations).

In this manuscript  $H$  will always stand for a Hopf algebra with a coproduct  $\Delta$ , a counit  $\varepsilon$ , and an antipode  $S$ . As usual, we will use Sweedler's notations:  $\Delta(h) = h_{(1)} \otimes h_{(2)}$  (summation is implied),  $(\Delta \otimes \text{id})\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ ,  $(S \otimes \text{id})\Delta(h) = h_{(-1)} \otimes h_{(2)}$  etc.

The following formulas will be quite useful:

$$\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = h, \quad (1.7)$$

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = h_{(1)}h_{(-2)} \otimes h_{(3)} = 1 \otimes h. \quad (1.8)$$

$H$  also possesses the opposite coproduct  $\Delta^{cop} : h \mapsto h_{(2)} \otimes h_{(1)}$ ; we denote the Hopf algebra  $(H, \Delta^{cop}, \varepsilon, S)$  as  $H^{cop}$ . As usual,  $H^{op}$  will stand for the algebra  $H$  with the opposite multiplication.

An associative algebra  $A$  is called  *$H$ -differential* if it is a left  $H$ -module such that

$$h(xy) = (h_{(1)}x)(h_{(2)}y). \quad (1.9)$$

REMARK 1.16. An  $H$ -differential algebra is an associative algebra in the pseudotensor category  $\mathcal{M}^l(H)$ .

For an  $H$ -differential algebra  $A$  one can define a *smash product*  $A \sharp H$  as a tensor product  $A \otimes H$  of underlying vector spaces with a new multiplication

$$(a \sharp g)(b \sharp h) = a(g_{(1)}b) \sharp g_{(2)}h.$$

As usual we will denote by  $G(H)$  the set of group-like elements of  $H$ , i.e.,  $h \in H$  such that  $\Delta(h) = (h \otimes h)$ . Another distinguished subspace of  $H$  is the set  $P(H)$  of primitive elements, i.e.,  $h \in H$  with  $\Delta(h) = 1 \otimes h + h \otimes 1$ . Group-like elements form a group with multiplication inherited from  $H$ , and  $P(H)$  is a Lie subalgebra of  $H$  with respect to the standard commutator  $[g, h] = gh - hg$ .

$G(H)$  acts on  $P(H)$  by inner automorphisms, namely, for  $g \in G(H)$  and  $h \in P(H)$ ,  $g(h) = gpg^{-1} \in P(H)$ . In this way one obtains a typical example of a smash product:  $U(P(H)) \# \mathbb{k}[G(H)]$ .

REMARK 1.17. A theorem due to Kostant [Sw, Theorem 8.1.5] states that a cocommutative Hopf algebra  $H$  is, in fact, isomorphic to  $U(P(H)) \# \mathbb{k}[G(H)]$ .

We will also require a standard filtration on  $H$ :

$$F^n H = 0 \text{ for } n < 0;$$

$$F^0 H = \mathbb{k}[G(H)];$$

$$F^n H = \left\{ h \in H \mid \Delta(h) \in F^0 H \otimes h + h \otimes F^0 H + \sum_{i=1}^{n-1} F^i H \otimes F^{n-i} H \right\}.$$

When  $H = U(\mathfrak{g})$  is a universal enveloping algebra, we get the canonical filtration. Remark that when  $\mathfrak{g}$  is finite-dimensional,  $\dim F^n H < \infty$  for all  $n$ .

Clearly, operations on  $H$  respect the filtration:

$$(F^m H)(F^n H) \subset F^{m+n} H,$$

$$\Delta(F^n H) \subset \sum_{i=0}^n F^i H \otimes F^{n-i} H,$$

$$S(F^n H) \subset F^n H.$$

When  $H$  is cocommutative, Remark 1.17 implies  $\bigcup_n F^n H = H$ . If so, we say that a non-zero element  $h \in H$  has *degree*  $n$  if  $h \in F^n H \setminus F^{n-1} H$ .

In order to define certain operations on pseudoalgebras (see below), we will need the following:

LEMMA 1.18 ([BDK, Lemma 2.5]). *Every element of  $H \otimes H$  can be uniquely represented in the form  $\sum_i (h_i \otimes 1)\Delta(l_i)$ , where  $\{h_i\}$  is a fixed basis of  $H$  and  $l_i \in H$ . Also, for any  $H$ -module  $V$ ,*

$$(F^n H \otimes \mathbb{k})\Delta(H) = F^n(H \otimes H)\Delta(H) = (\mathbb{k} \otimes F^n H)\Delta(H),$$

where  $F^n(H \otimes H) = \sum_{i+j=n} F^i H \otimes F^j H$ .

**1.2.3. Dual algebra of a Hopf algebra.** Denote the dual algebra of  $H$  by  $X = H^* = \text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ .

It is an  $H$ -differential algebra with the action defined by

$$\langle hx, f \rangle = \langle x, S(h)f \rangle, \text{ for } f, h \in H, x \in X.$$

Moreover, one can similarly define the structure of a right  $H$ -module on  $X$ ; because of associativity of  $H$ , this makes  $X$  into an  $H$ -bimodule.

$X$  possesses a standard filtration  $X = F_{-1}X \supset F_0X \supset \dots$  where  $F_nX = (F^nH)^\perp$ . The fundamental system of neighborhoods at 0,  $\{F_nX\}$ , defines a standard topology on  $X$ .

When  $H$  is cocommutative,  $X$  is commutative and  $\bigcap_n F_nX = 0$ . If this conditions holds,  $X$  is Hausdorff in the standard topology. The multiplication and the action of  $H$  are continuous.

By a basis of  $X$  we will always mean a topological basis  $\{x_i\}$  such that for any  $n$  only a finite number of  $x_i$ 's does not lie in  $F_nX$  (i.e.  $x_i \rightarrow 0$  in the standard topology). Let  $\{h_i\}$  be a basis of  $H$  compatible with the standard filtration. If  $\dim F^nH < \infty$  for all  $n$ , the dual basis of  $X$  satisfies the above condition, i.e. tends to 0. For  $h \in H$  and  $x \in X$ , we have

$$h = \sum_i \langle h, x_i \rangle h_i, \quad x = \sum_i \langle x, h_i \rangle x_i,$$

where the first sum is finite and the second converges in the standard topology.

$X$  possesses some additional algebraic structure. Namely, define the antipode  $S$  as a dual of that of  $H$ :  $\langle S(x), h \rangle = \langle x, S(h) \rangle$ . Also, we introduce a coproduct  $\Delta : X \rightarrow X \hat{\otimes} X$  where  $X \hat{\otimes} X = (H \otimes H)^*$  is the completed tensor product. By definition, for  $x, y \in X$  and  $f, g \in H$

$$\langle xy, f \rangle = \langle x \otimes y, \Delta(f) \rangle = \langle x, f_{(1)} \rangle \langle y, f_{(2)} \rangle, \quad (1.10)$$

$$\langle x, fg \rangle = \langle \Delta(x), f \otimes g \rangle = \langle x_{(1)}, f \rangle \langle x_{(2)}, g \rangle. \quad (1.11)$$

REMARK 1.19. For  $X$  such that  $\dim F_nX < \infty$  for all  $n$ , one can endow  $H$  with the structure of an  $X$ -differential algebra. As in the case of  $H$ -action on  $X$ , the action is defined by  $\langle y, xh \rangle = \langle S(x)y, h \rangle$ . The proof is also similar to that for the  $H$ -action on  $X$  and uses (1.11) instead of (1.10) and (1.8). Formula (1.9) makes sense as the right-hand side will be finite for every pair of elements of  $H$ .

**1.2.4. Notations for universal enveloping algebras.** In this paper we mostly restrict our attention to universal enveloping algebras of finite-dimensional Lie algebras. Some notations are in order:

Put  $H = U(\mathfrak{g})$  where  $\mathfrak{g}$  is a  $n$ -dimensional Lie algebra spanned over  $\mathbb{k}$  by  $\partial_1, \partial_2, \dots, \partial_n$ .

We fix the canonical (but not PBW) basis of  $H$  indexed by elements of  $\mathbb{Z}_{\geq 0}^n$ :

$$\partial^I = \frac{\partial_1^{i_1} \cdots \partial_n^{i_n}}{i_1! \cdots i_n!}, \quad \text{for } I = (i_1, \dots, i_n). \quad (1.12)$$

It is easy to see that  $\Delta(\partial^I) = \sum_{J+K=I} \partial^J \otimes \partial^K$ .

REMARK 1.20. For future reference, we need to describe our notations for the multiindex set  $\mathbb{Z}_{\geq 0}^n$ . The addition is pointwise. There is the standard partial ordering, i.e.,  $(i_1, \dots, i_n) > (j_1, \dots, j_n)$  iff  $i_m > j_m$  for all  $m$ ; if neither  $I \geq J$  nor  $J \geq I$ , we call  $I$  and  $J$  *incompatible*. The index  $(0, \dots, 0)$  is denoted simply by 0. Also, for a multiindex  $I = (i_1, \dots, i_n)$  we put  $|I| = i_1 + \cdots + i_n$  and  $(-1)^I = (-1)^{|I|}$ .

The dual Hopf algebra of  $H$  is  $X = \mathbb{k}[[t_1, \dots, t_n]]$  with the canonical dual basis  $t^I = t_1^{i_1} \cdots t_n^{i_n}$ . As usual,  $t^0 = 1$ . The action of  $H$  on  $X$  is given by differential operators:  $\partial_i = -\partial/\partial t_i$ . The right action is the same:  $xh = hx$  for  $x \in X$ ,  $h \in H$ . Similarly, the action of  $X$  on  $H$  is defined by  $t_i = -\partial/\partial \partial_i$ .

REMARK 1.21. The standard filtration on  $X$  regarded as the dual algebra of  $H$  is not the standard decreasing filtration on the polynomial algebra  $\mathbb{k}[t_1, \dots, t_n]$  (it is shifted by 1). In particular, the standard total degree function on  $X$  does not respect multiplication.

It is easy to see that  $\Delta(t_i) = 1 \otimes t_i + t_i \otimes 1 + \text{summands with both entries in the tensor product of degree higher than 0}$ . This can be generalized for any  $t^I$ :

$$\Delta(t^I) = \sum_{J \leq I} t^J \otimes t^{I-J} + \sum_j c_j t^{K_j} \otimes t^{L_j}, \quad |K_j| + |L_j| \geq |I| + 1, \quad c_j \in \mathbb{k}. \quad (1.13)$$

In particular, one can deduce from (1.13) that

$$\Delta(\mathbb{F}_{n-1} X) \subset \sum_{i=0}^n \mathbb{F}_{i-1} X \hat{\otimes} \mathbb{F}_{n-i-1} X. \quad (1.14)$$

**1.2.5. Pseudoalgebras and their representations.** Recall the description of the pseudotensor category  $\mathcal{M}^*(H)$  (see Example 1.14(3)).

DEFINITION 1.22. An associative (Lie) *pseudoalgebra* over a cocommutative Hopf algebra  $H$  is an associative (Lie) algebra in  $\mathcal{M}^*(H)$ . A representation of a pseudoalgebra  $\mathfrak{a}$  (*pseudomodule*) is its representation in  $\mathcal{M}^*(H)$ .

We denote multiplication in a pseudoalgebra  $R$  by  $*$ :  $R \otimes R \rightarrow (H \otimes H) \otimes_H R$  and call  $a * b$  the *pseudoproduct* of  $a$  and  $b$ . The same notation will be used for the action of  $R$  on its modules.

This operation satisfies  $H$ -bilinearity: for  $f, g \in H$ ,  $(fa) * (gb) = ((f \otimes g) \otimes_H 1)(a * b)$ , and associativity  $(a * b) * c = a * (b * c)$ . The explicit expressions for the latter equality are calculated below (this calculation is from [BDK, Chapter 3]).

Let  $a, b, c \in R$ . To calculate  $(a * b) * c \in H^{\otimes 3} \otimes_H R$  in accordance with (1.6), notice that here  $J = \{1, 2, 3\}, I = \{1, 2\}, \psi_1 = \phi = *, \psi_2 = \text{id}$ , and the map  $\pi : J \rightarrow I$  is given by  $\pi(1) = \pi(2) = 1, \pi(3) = 2$ . Put

$$\begin{aligned} a * b &= \sum_i (f_i \otimes g_i) \otimes_H d_i, \\ d_i * c &= \sum_j (f_{ij} \otimes g_{ij}) \otimes_H d_{ij}. \end{aligned}$$

Then, as  $\Delta^{(\pi)} = \Delta \otimes \text{id}$ ,

$$(a * b) * c = \sum_{i,j} (f_i f_{ij(1)} \otimes g_i f_{ij(2)} \otimes g_{ij}) \otimes_H d_{ij}. \quad (1.15)$$

Similarly, for the product  $a * (b * c)$ ,  $\Delta^{(\pi)} = \text{id} \otimes \Delta$ , and we obtain

$$a * (b * c) = \sum_{ij} (h_{ij} \otimes h_j k_{ij(1)} \otimes k_j k_{ij(2)}) \otimes_H e_{ij}, \quad (1.16)$$

$$\text{where } b * c = \sum_i (h_i \otimes k_i) \otimes_H e_i, \quad a * e_i = \sum_j (h_{ij} \otimes k_{ij}) \otimes_h e_{ij}.$$

For a representation  $V$  of  $R$ , we will also denote the action by  $*$ :  $a * v \in (H \otimes H) \otimes_H V$ . It also satisfies  $H$ -bilinearity and associativity.

REMARK 1.23. Recall that a cocommutative Hopf is a smash product of  $G(H)$  and  $U(P(H))$  (Remark 1.17). For brevity denote  $\Gamma = G(H)$ ,  $H' = U(P(H))$ . The action of  $\Gamma$  on  $H'$  can be extended to the action on  $(H')^{\otimes I}$  via  $\Delta^{(I)}(g) = \otimes_i g$ , and in an obvious way to the action on  $(H')^{\otimes I} \otimes_{H'} M$  for an  $H'$ -module  $M$ . It can be shown that the category  $\mathcal{M}^*(H)$  is equivalent to a subcategory of  $\mathcal{M}^*(H')$  that consists of  $H$ -modules and polylinear maps that commute with the action of  $\Gamma$ . Thus, it follows [BDK, Corollary 5.3] that an  $H$ -pseudoalgebra is an  $H'$ -pseudoalgebra with an action of  $\Gamma$  such that  $ga * gb = g(a * b)$ . Moreover, the pseudoproduct over  $H$  is defined as

$$a * b = \sum_{g \in \Gamma} ((g^{-1} \otimes 1) \otimes_H 1)(ga * b), \quad (1.17)$$

where the products on the RHS are taken over  $H'$  and the sum is finite.

This shows that the case of pseudoalgebras over general cocommutative algebras can be in most cases reduced to the study of pseudoalgebras over universal enveloping algebras.

**1.2.6. Annihilation algebra and  $x$ -products.** As before,  $X = H^*$ . Let  $R$  be a left module of a cocommutative Hopf algebra  $H$ . Define another  $H$ -module  $\mathcal{A}(R) = X \otimes_H R$  with an obvious left action  $h(x \otimes_H a) = hx \otimes_H a$ . If  $R$  is also an associative  $H$ -pseudoalgebra,  $\mathcal{A}(R)$  is an associative algebra with multiplication defined by

$$(x \otimes_H a)(y \otimes_H b) = \sum_i (xf_i)(yg_i) \otimes_H e_i, \quad (1.18)$$

$$\text{where } a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i.$$

It is not difficult to see that  $\mathcal{A}(R)$  is also an  $H$ -differential algebra. Remark also that an  $R$ -pseudomodule  $V$  gives rise to an  $\mathcal{A}(R)$ -module  $\mathcal{A}(V) = X \otimes_H V$ .

The algebra  $\mathcal{A}(R)$  is called the *annihilation algebra* of the pseudoalgebra  $R$ . Its elements  $x \otimes_H a$  are denoted  $a_x$  and are called *Fourier coefficients* of  $a$ .

The annihilation algebra closely mirrors the properties of the corresponding pseudoalgebra. In particular, when  $R$  is torsion-free,  $\mathcal{A}(R)$  “distinguishes” its elements:

LEMMA 1.24 (cf. [BDK, Proposition 11.5]). *Let  $M$  be a left  $H$ -module over a universal enveloping algebra  $H$ . All Fourier coefficients of  $a \in M$  are zero if and only if  $a$  is torsion.*

Moreover, sometimes it is possible to get back from  $\mathcal{A}(M)$  to  $M$ . Given a topological left  $H$ -module  $L$ , one can construct another module  $\mathcal{C}(L) = \text{Hom}_H^{\text{cont}}(X, L)$  (“cont” stands for continuous in the standard topologies of  $X$  and  $H$ ). Define the map  $\Phi : M \rightarrow \mathcal{C}(\mathcal{A}(M))$  as  $\Phi(a)(x) = x \otimes_H a$ . In most interesting cases,  $M$  imbeds into  $\Phi(M)$  and, if  $M$  possesses a pseudoalgebra structure, so does  $\Phi(M)$  (see Lemma 1.25).

Let  $R$  be an associative pseudoalgebra with the pseudoproduct  $a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i$ . The choice of  $f_i, g_i$ , and  $e_i$  is certainly not unique. By Lemma 1.18 we can assume that  $g_i = 1$ . This defines the new operation  $R \otimes R \rightarrow H \otimes R$ :  $a \cdot b = \sum_i f_i \otimes e_i$ . For any  $x \in X$  we introduce the  *$x$ -product*:

$$a_x b = (\langle S(x), \cdot \rangle \otimes \text{id}) a \cdot b = \sum_i \langle S(x), f_i \rangle e_i. \quad (1.19)$$

Given the  $x$ -products of  $a$  and  $b$ , one can also pass back to their pseudoproduct:

$$a * b = \sum_i (S(h_i) \otimes 1) \otimes_H (a_{x_i} b), \quad \text{for dual bases } \{h_i\}, \{x_i\} \text{ of } H, H^*. \quad (1.20)$$

Notice that the sum in (1.20) is finite, i.e. for almost all  $x_i$ ,  $a_{x_i} b = 0$ .

Thus, one can define an associative  $H$ -pseudoalgebra as a left  $H$ -module  $R$  equipped with the  $x$ -products satisfying:

**Locality:**

$$\text{codim}\{x \in X \mid a_x b = 0\} < \infty \quad \text{for any } a, b \in R; \quad (1.21)$$

 **$H$ -sesquilinearity:**

$$\begin{aligned} (ha)_x b &= a_x h b, \\ a_x (hb) &= h_{(2)}(a_{h_{(-1)}x} b) \quad \text{for any } a, b \in R, h \in H; \end{aligned} \quad (1.22)$$

**Associativity:**

$$a_x (b_y c) = (a_{x_{(2)}} b)_{x_{(1)}y} c. \quad (1.23)$$

Locality suggests the following definition: we will call  $x \in X$  *maximal with respect to  $a$  and  $b$*  if  $a_x b \neq 0$  but for any  $y \in F_0 X$ ,  $a_{xy} b = 0$ .

Associativity (1.23) can be equivalently stated as

$$(a_x b)_y c = a_{x_{(2)}} (b_{x_{(-1)}y} c). \quad (1.24)$$

Most of the above properties survive the passing to  $\mathcal{C}(\mathcal{A}(R))$ :

LEMMA 1.25 (cf. [BDK, Proposition 11.2]). *Let  $R$  be an associative pseudoalgebra. Then  $\Phi(R) = \mathcal{C}(\mathcal{A}(R))$  satisfies (1.22-1.24).*

The above formulas and statements, of course, remain true for the action of  $R$  on a pseudomodule  $M$  and of  $\mathcal{A}(R)$  on  $\mathcal{A}(M)$ .

Using (1.20) we can obtain formulas similar to (1.23) and (1.24) for the multiplication in  $\mathcal{A}(R)$ :

$$\begin{aligned} a_x \cdot b_y &= (a_{x_{(2)}} b)_{x_{(1)}y}, \\ (a_x b)_y &= (a_{x_{(2)}}) \cdot (b_{x_{(-1)}y}). \end{aligned} \quad (1.25)$$

We can now define structural concepts for associative pseudoalgebras. Denote by  $A_x B$  the set  $\{a_x b \mid a \in A, b \in B\}$ . An *ideal*  $I$  of  $R$  is a pseudoalgebra such that for all  $x \in X$ ,  $I_x R \subset I$ ,  $R_x I \subset I$ . A pseudoalgebra  $R$  whose only ideals are 0 and  $R$  is called *simple*. A pseudoalgebra  $R$  such that for a fixed  $n$   $R_{x_1} R_{x_2} \cdots R_{x_n} R = 0$  for any collection of  $\{x_1, \dots, x_n\} \subset X$  is called *nilpotent* (as  $x_i$ 's are arbitrary, we can omit the brackets). A pseudoalgebra that contains no nilpotent ideals is called *semisimple*. Similar definitions for pseudomodules are given in Section 2.2.

### 1.3. Examples of Associative Pseudoalgebras

In this section we provide several important examples of associative pseudoalgebras. In general, we do not assume that  $H$  is a universal enveloping algebra of a Lie algebra.

### 1.3.1. General examples.

EXAMPLE 1.26. Let  $H'$  be a Hopf subalgebra of  $H$ , and  $R$  an  $H'$ -pseudoalgebra.

DEFINITION 1.27. The *current extension* of  $R$  is the  $H$ -pseudoalgebra  $\text{Cur}_{H'}^H R$  which is the  $H$ -module  $H \otimes_{H'} R$  with the pseudoproduct  $*$  extending the pseudoproduct of  $R$  by  $H$ -bilinearity.

More explicitly, for  $a, b \in R$  with  $a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} c_i$  we define

$$\begin{aligned} (f \otimes_{H'} a) * (g \otimes_{H'} b) &= ((f \otimes g) \otimes_H 1)(a * b) \\ &= \sum_i ((f f_i) \otimes (g g_i)) \otimes_H (1 \otimes_{H'} c_i). \end{aligned} \tag{1.26}$$

The same construction, of course, is possible for any variety of (pseudo)algebras defined over  $H$ .

REMARK 1.28. This terminology is different from that of [BDK] where current extensions were called current pseudoalgebras. However, here this term is restricted to a smaller class (see immediately below). In this author's view this makes some statements below less cluttered.

In particular, when  $H' = \mathbb{k}$ , an associative  $H'$ -pseudoalgebra  $R$  is an associative  $\mathbb{k}$ -algebra with the ordinary product. Then  $\text{Cur}_{H'}^H R$ , which we will denote simply  $\text{Cur } R$ , has the pseudoproduct

$$(f \otimes a) * (g \otimes b) = (f \otimes g) \otimes_H (1 \otimes ab).$$

We will call such a pseudoalgebra  $\text{Cur } R$  a *current pseudoalgebra*.

Current pseudoalgebras have a simple characterization: if in a pseudoalgebra  $R$   $a_x b = 0$  for all  $a, b$  from some  $H$ -basis of  $R$  and  $x \in F_0 X$ , then  $R$  is current.

EXAMPLE 1.29. Let  $H = U(\mathfrak{g})$  be a universal enveloping algebra and  $R$  an associative pseudoalgebra over  $H$  that is free and of rk 1 as an  $H$ -module. We can easily classify all such pseudoalgebras:

LEMMA 1.30. *Let  $R$  be as above. Then either the multiplication in  $R$  is trivial (i.e.  $a * b = 0$  for any  $a, b \in R$ ) or  $R \cong \text{Cur } \mathbb{k}$ .*

PROOF. Let  $e$  be the generator of  $R$  over  $H$ , i.e.  $R = He$ . By  $H$ -bilinearity, multiplication in  $R$  is completely determined by the values of the coefficients in the product  $e * e$ . Namely, put

$$e * e = \alpha \otimes_H e, \quad \text{where } \alpha = \sum_{(I,J)} c_{IJ} \partial^I \otimes \partial^J \in H \otimes H.$$

Then, to classify pseudoalgebras of rk 1, it suffices to classify all appropriate  $\alpha$ 's.

Associativity implies

$$(\alpha \otimes 1)(\Delta \otimes \text{id})(\alpha) = (1 \otimes \alpha)(\text{id} \otimes \Delta)(\alpha),$$

which can be rewritten as

$$\sum_{(I,J),K+L=I} c_{IJ} \partial^I \partial^K \otimes \partial^J \partial^L \otimes \partial^J = \sum_{(I,J),M+N=J} c_{IJ} \partial^I \otimes \partial^I \partial^M \otimes \partial^J \partial^N. \quad (1.27)$$

Pick  $I$  with a maximal degree among all such that  $c_{IJ} \neq 0$ . Then by comparing the degrees of the first terms in (1.27), we see that  $|K| = 0$ , thus  $|I| = 0$ . Similarly,  $|J| = 0$ , i.e.  $\alpha = c \otimes 1$  for some  $c \in \mathbb{k}$ . For a non-zero  $c$  we can normalize  $e$ , so that  $e * e = (1 \otimes 1) \otimes_H e$ . This makes  $R$  isomorphic to  $\text{Cur } \mathbb{k}$ .  $\square$

Lie pseudoalgebras of rk 1 were classified in [BDK, 4.3] by essentially similar methods.

### 1.3.2. Conformal algebras.

EXAMPLE 1.31. Below we shall partly follow the introduction to [BDK].

Let  $H = \mathbb{k}[\partial]$ . Then  $H$ -pseudoalgebras (of any variety) are conformal algebras.

REMARK 1.32. The annihilation algebra of a conformal algebra is a proper subalgebra of its coefficient algebra; namely the subalgebra of all non-negative coefficients. In other words,  $\text{Coeff } R = X[t^{-1}] \otimes_H R$  for a conformal algebra  $R$ .

The concise way to present the operations on the conformal algebra is the so-called  $\lambda$ -product:  $f \lambda g = \sum_n \frac{\lambda^n}{n!} f \circledast_n g$ .

Let  $C$  be a conformal algebra generated over  $H = \mathbb{k}[\partial]$  by  $f_i$ 's. Then for each  $i, j$ , there exist a collection of polynomials  $\{p_{ij}^k\}$  such that

$$f_i \lambda f_j = \sum_k p_{ij}^k(\lambda, \partial) f_k.$$

It is not difficult to see that the operator product expansion (1.1) for  $f_i(w)f_j(z)$  can be written as

$$f_i(w)f_j(z) = \sum_{k,n} p_{ij}^k(-\partial_w, \partial_w + \partial_z)(f_k(z)w^n z^{-n-1}),$$

hence the  $p_{ij}^k$ 's induce the  $H$ -bilinear map  $R \otimes R \rightarrow (H \otimes H) \otimes_H R$ . Explicitly,

$$f_i \otimes f_j \mapsto \sum_k p_{ij}^k(-\partial \otimes 1, \Delta(\partial)) \otimes_H f_k.$$

This shows that  $R$  is an  $H$ -pseudoalgebra.

Moreover, let  $X$  be the dual of  $H = \mathbb{k}[\partial]$ , i.e.  $X = \mathbb{k}[[t]]$ . Then for  $f, g \in R$ ,

$$f \circledast g = f_{t^n} g, \quad \text{and} \quad f \cdot g = \sum S\left(\frac{\partial^n}{n!}\right) \otimes (f \circledast g).$$

Thus,  $f \cdot g = f_\lambda g$  for  $\lambda = -\partial \otimes 1$ .

**1.3.3. Pseudolinear algebras.** Let  $V, W$  be  $H$ -modules. An  $H$ -pseudolinear map from  $V$  to  $W$  is a linear map  $\phi : V \rightarrow (H \otimes H) \otimes_H W$  such that  $\phi(h) = ((1 \otimes h) \otimes 1)\phi(v)$  for  $h \in H, v \in V$ . The space of all such maps, denoted  $\text{Chom}(V, W)$ , is a left  $H$ -module: put  $(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\phi(v)$ . When  $V = W$ , we denote the set of all pseudolinear maps as  $\text{Cend}(V)$ . Though it is possible to define the action of the product  $\phi * \psi$  on  $V$  for  $\phi, \psi \in \text{Cend} V$ , it might not be represented by a finite sum, i.e.,  $\text{Cend} V$  is not necessarily a pseudoalgebra. However, when  $V$  is finite over  $H$ ,  $\text{Cend} V$  becomes an associative  $H$ -pseudoalgebra with a naturally defined multiplication.

EXAMPLE 1.33. If  $V$  is a finite free  $H$ -module, i.e.,  $V = H \otimes V_0$  for some finite dimensional vector space  $V_0$  with a trivial action of  $H$ , then  $\text{Cend} V = H \otimes H \otimes \text{End} V_0$  with the pseudoproduct defined as

$$(f \otimes a \otimes A) * (g \otimes b \otimes B) = (f \otimes ga_{(1)}) \otimes_H (1 \otimes ba_{(2)} \otimes AB). \quad (1.28)$$

(see [BDK, Propositions 10.5, 10.11]).

Clearly, in the above case  $\text{Cend} V$  depends only on  $\text{rk} V$ . To emphasize this, for a module of rank  $n$ , its pseudoalgebra of endomorphisms will be denoted simply  $\text{Cend}_n$ .

It is not difficult to see that  $\text{Cend}_n$  is simple [BDK, Proposition 13.34]; however, unlike the case of ordinary algebras of linear endomorphisms, it is not finite as an  $H$ -module.

When  $H = \mathbb{k}[\partial]$  the conformal algebra  $\text{Cend}_n$  is isomorphic to  $\mathfrak{W}_n$  (cf. Example 1.12).

#### 1.3.4. Differential algebras.

EXAMPLE 1.34. Recall that the bialgebra  $X^{cop}$  is isomorphic to  $X$  as an algebra and has the comultiplication  $\Delta^{op} : x \mapsto x_{(2)} \otimes x_{(1)}$ . Consider an  $X^{cop}$ -differential algebra  $A$ , i.e., a topological associative algebra with a left  $X^{cop}$ -action such that for  $x \in X^{cop}, a, b \in A$

$$x(ab) = (x_{(2)}a)(x_{(1)}b). \quad (1.29)$$

Recall that  $\Delta(x) \in X \hat{\otimes} X$  is not, in general, a finite sum. Thus, in order for (1.29) to make sense, we must require that for any  $a \in A$ ,  $\text{codim Ann } a < \infty$ .

REMARK 1.35. For brevity, the above property will never be stated in the further exposition but will always be assumed when we discuss  $X^{cop}$ -differential algebras. We will simplify terminology even further and simply call such algebras  $X^{cop}$ -algebras, always implying the structure from (1.29).

REMARK 1.36. For  $X$  such that  $\dim F_n X < \infty$  for all  $n$ , a typical example of an  $X^{cop}$ -algebra is  $H^{op}$ . This statement is “dual” to Remark 1.19 and can be deduced in the same way.

We introduce the pseudoalgebra structure on  $\text{Diff } A = H \otimes A$ . Notice that by  $H$ -sesquilinearity it is enough to define the products between elements of the type  $1 \otimes a$ :

$$(1 \otimes a)_x(1 \otimes b) = 1 \otimes (ax(b)), \quad \text{for any } x \in X. \quad (1.30)$$

Associativity of these products follows from (1.23) and (1.29). Finite codimension of the annihilator of every  $a \in A$  implies locality.

Notice that  $\text{Diff } A$  is generated over  $H$  by  $1 \otimes A$ . For brevity we will denote such elements  $1 \otimes a$  by  $\tilde{a}$ .

For a free finite  $H$ -module  $V$ ,  $\text{Cend } V$  is a differential pseudoalgebra  $\text{Diff}(H^{op} \otimes \text{End}_{\text{rk } V}(\mathbb{k}))$ . This may be shown directly but in the case of arbitrary  $H$  the calculations are cumbersome; this result will follow from a more general statement in the next chapter (see, in particular, Theorem 2.12 and Corollary 2.14).

However, the case of  $H = \mathbb{k}[\partial]$  is much simpler. Indeed, put  $A = \mathbb{k}[\partial_t] \otimes \text{End}_n(\mathbb{k})$  and let the generator  $t$  of  $X = \mathbb{k}[[t]]$  act as a derivation in  $\partial_t$ . For  $a \in A$ , put  $\tilde{a} = \sum at^n z^{-n-1}$ . Then  $J_A^m = (-1)^m \sum_j (-1)^j \partial^j (\tilde{a}^{(j)})$  where  $a^{(j)}$  is the  $j$ -th derivative of  $a$  with respect to  $\partial_t$ . Similarly,  $\tilde{a}$  can be expressed via  $J_A^m$ 's. This shows that  $\text{Cend}_n \cong \text{Diff}(A)$ .

In notations of Example 1.12,  $L^k = \tilde{x}^k$ . Notice that  $L^0 = \tilde{1}$  behaves, in some sense, as a left identity. In general, so does  $\tilde{1}$  in any differential pseudoalgebra. This will play out in Definition 2.5.

REMARK 1.37. A current algebra over a differential pseudoalgebra is itself a differential pseudoalgebra. Namely, let  $H'$  be a subalgebra of  $H$ . Choose a topological basis of  $X = \mathbb{k}[[t_1, \dots, t_n]]$  such that  $X' = \mathbb{k}[[t_1, \dots, t_r]] = (H')^*$  for some  $r < n$ . Let  $A$  be a differential  $(X')^{cop}$ -algebra. One can consider the induced action of  $X^{cop}$  on  $A$ , namely let  $t_i, i > r$ , act on  $A$  trivially. Then  $\text{Cur}_{H'}^H \text{Diff}_{H'}(A) = \text{Diff}_H A$ .

A particular case of the above setting is an arbitrary algebra  $A$  with a trivial  $X^{cop}$  action. Then  $\text{Diff}_H A = \text{Cur } A$ .

EXAMPLE 1.38. Here we present a differential pseudoalgebra that is neither current, nor isomorphic to  $\text{Cend}_n$ .

Recall that the  $n$ -th Weyl algebra  $A_n$  is generated by  $\{x_i, y_i\}_{i=1}^n$  such that

$$\begin{aligned} x_i x_j &= x_j x_i, & y_i y_j &= y_j y_i, \\ x_i y_j - y_j x_i &= \delta_{ij}. \end{aligned}$$

Let  $H = \mathbb{k}[\partial_1, \dots, \partial_{2n}]$ . Then  $X = \mathbb{k}[[t_1, \dots, t_{2n}]]$ . Since  $X$  is cocommutative, every  $X$ -differential algebra gives rise to an  $H$ -pseudoalgebra.

To define the action of  $X$  on  $A_n$  it is enough to describe the action of each  $t_i$  and check that it conforms to the Leibniz rule (i.e. that  $t_i$  is a derivation of  $A_n$ ). For  $1 \leq i \leq n$  put  $t_i = \partial/\partial x_i$  and for  $n+1 \leq i \leq 2n$ ,  $t_i = \partial/\partial y_i$ . Less formally, we put for  $i \leq n$ ,  $t_i = -\text{ad } y_i$ , and for  $i > n$ ,  $t_i = \text{ad } x_i$ ; this immediately implies the Leibniz rule.

Remark that as  $A_n$  is simple,  $\text{Diff}_H A_n$  is also simple (see Lemma 4.3).

This example generalizes to the case of  $H = U(\mathfrak{g})$ . Let  $\hat{\mathfrak{g}}$  be a one-dimensional central extension of  $\mathfrak{g}$ :  $0 \rightarrow \mathbb{k}\mathfrak{c} \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$ . Let  $\phi$  be the corresponding cocycle. The action of  $X$  (or  $X^{cop}$ ) on  $\mathfrak{g}$  extends trivially to  $\hat{\mathfrak{g}}$ . Put  $A = U(\hat{\mathfrak{g}})/(1 - \mathfrak{c})$ , then  $\text{Diff } A$  is a simple  $H$ -pseudoalgebra. We will denote it  $\text{Cend}_n^\phi$ .

REMARK 1.39. When  $\phi$  is the trivial cocycle,  $\text{Cend}_n^\phi = \text{Cend}_n$ .

For an abelian  $\mathfrak{g}$ ,  $\hat{\mathfrak{g}}$  is the Heisenberg algebra, and  $A$  as described above is the Weyl algebra  $A_{\dim \mathfrak{g}}$ .

## Unital Pseudoalgebras and Their Representations

In this chapter we introduce the concept of unitality for associative pseudoalgebras and describe the structure of associative pseudoalgebras and their representations. In particular, we prove

**THEOREM 2.1.** *A semisimple unital associative pseudoalgebra is differential.*

**THEOREM 2.2.** *Let  $V$  be a representation of unital differential pseudoalgebra  $R = \text{Diff } A$ . Then  $V = V^0 \oplus V^1$ , where  $R * V^0 = 0$  and  $V^1$  is constructed from a unitary  $A$ -module. Moreover,  $V^1$  is irreducible (indecomposable) if and only if  $A$  is irreducible (indecomposable).*

### 2.1. Unital pseudoalgebras

Unless otherwise specified, in this section we will only consider the case of  $H = U(\mathfrak{g})$  where  $\mathfrak{g}$  is a finite-dimensional Lie algebra. We are interested in some sort of classification of associative  $H$ -pseudoalgebras and their representations.

**2.1.1. Definition of unital algebras.** Any study of pseudoalgebras is ultimately a study of corresponding annihilation algebras. The standard trick in the study of ordinary associative algebras is to adjoin the identity; however, it is unclear if such an operation can be performed on the pseudoalgebra level, i.e. if we will still remain in the class of annihilation algebras. Thus, it is necessary to introduce some concept of “identity” for pseudoalgebras themselves.

The trivial observation is that an ordinary algebra  $A$  is unital (i.e. possesses an identity) if there is an embedding  $\mathbb{k} \rightarrow A$  that agrees with the faithful  $\mathbb{k}$ -action on  $A$ . We shall introduce a similar concept for  $H$ -pseudoalgebras. The role of  $\mathbb{k}$  will be played by the “smallest” pseudoalgebra  $\text{Cur } \mathbb{k}$  (cf. Lemma 1.30).

In order to define unital pseudoalgebras, we shall study in greater detail the representations of  $\text{Cur } \mathbb{k}$ . From now on, we will always denote the generator of  $\text{Cur } \mathbb{k}$  as an  $H$ -module by  $e$ .

**LEMMA 2.3.** (i) *Let  $V$  be a  $\text{Cur } \mathbb{k}$ -module. Then  $V = V^0 \oplus V^1$ , where  $V^0$  and  $V^1$  are submodules of  $V$  such that  $e * V^0 = 0$  and for every  $v \in V^1$ ,  $e_1 v = v$ .*

(ii)  *$V^1$  is a torsion-free  $H$ -module.*

PROOF. (i)  $\text{Cur } \mathbb{k}$  is an ordinary associative algebra with respect to the product  $\cdot$ , hence  $V$  splits into the direct sum of ordinary submodules  $V^0 \oplus V^1$  such that  $e$  acts as a multiplication by  $i$  on  $V^i$ ,  $i = 0, 1$ .

For any  $x \in X$ , if  $e_1 v = 0$ , we have  $0 = e_x(e_1 v) = (e_{x(2)} e)_{x(1)} v = (e_1 e)_x v = e_x v$ , thus  $e * V^0 = 0$ .

A direct calculation shows that  $V^0$  and  $V^1$  are  $H$ -stable.

(ii) Assume that  $v \in V^1$  is torsion, i.e., there exists  $h \in H$  such that  $hv = 0$ . Suppose we can choose  $x$  such that  $S(h)x$  is maximal with respect to  $e$  and  $v$ . But since

$$0 = e_x(hv) = h_{(2)}(e_{h(-1)x} v) = e_{S(h)x} v \neq 0,$$

this is impossible and for all  $x \in X$ ,  $e_x v = 0$ . Hence,  $v \in V^0 \cap V^1$  and  $v = 0$ .  $\square$

REMARK 2.4. In fact, we will show in the proof of Lemma 2.17 that  $V^1$  is free as an  $H$ -module, but for now torsion-freeness will suffice.

DEFINITION 2.5. An  $H$ -pseudoalgebra  $R$  is called *unital* if

- 1) there exists an embedding  $\text{Cur } \mathbb{k} \rightarrow R$ ;
- 2) for  $R$  regarded as a  $\text{Cur } \mathbb{k}$ -module,  $R^0 = \{0\}$ .

We shall denote the image of the generator of  $\text{Cur } \mathbb{k}$  in  $R$  by  $e$  as well and call it the *pseudoidentity* of  $R$ .

Differential pseudoalgebras (see Example 1.34) over unital algebras are unital and, since identity can be adjoined to any ordinary algebra, any differential pseudoalgebra can be embedded into a unital one. Thus, speaking of differential pseudoalgebras, we will always assume them to be unital.

REMARK 2.6. It is unknown, in general, what pseudoalgebras can be embedded into unital ones. Torsion-freeness over  $H$  is a necessary condition (Lemma 2.8), and one can provide a number of sufficient conditions as well, e.g. having a faithful finite representation (then there is an embedding into  $\text{Cend}_n$ ).

REMARK 2.7. Unlike the case of ordinary algebras with an ordinary identity, pseudoidentity is not unique. Consider, for instance, the conformal algebra (i.e. a pseudoalgebra over  $\mathbb{k}[\partial]$ )  $\text{Cur End}_n(\mathbb{k})$ . Clearly,  $\tilde{1}$  is a pseudoidentity, but so is  $\tilde{1} + \partial \tilde{r}$  for any  $r$  of nilpotency degree 2.

Nonetheless, unital algebras possess a number of good properties.

LEMMA 2.8. *Let  $R$  be a unital  $H$ -pseudoalgebra. Then  $R$  is a torsion-free  $H$ -module.*

REMARK 2.9. In Lie conformal algebras torsion elements necessarily lie in its center [DK, Proposition 3.1].

PROOF. Follows from Lemma 2.3 (ii).  $\square$

**2.1.2. Classification.** Remark that  $e_1$  acts on  $\mathcal{A}(R)$  as a left identity. On the other hand, it is easy to construct an example of a pseudoalgebra such that  $\mathcal{A}(R)$  possesses no right identities. E.g., consider  $R = \text{Cur } A$  where  $A$  has no right identities, then  $\mathcal{A}(R)$  is a tensor product of algebras  $X \otimes A$ . In order to provide a good classification of unital algebras, we need to exclude such degenerate examples.

DEFINITION 2.10. The *left annihilator* of a pseudoalgebra  $R$  is the set of elements  $a$  such that  $a * b = 0$  for all  $b \in R$ .

It is clear that  $L(R)$  is an ideal of  $R$ .

LEMMA 2.11. For a unital  $R$ ,  $L(R) = \{a \mid a * e = 0\}$ .

PROOF. Let  $a$  be such that  $a * e = 0$ . For any  $x \in X, b \in R$ ,  $a_x b = a_x(e_1 b) = (a_{x(2)} e)_{x(1)} b = 0$ . Hence,  $a * b = 0$ .  $\square$

THEOREM 2.12. A unital pseudoalgebra  $R$  with a zero left annihilator is differential:  $R = \text{Diff } A$  for some associative  $A$ . Moreover, if  $R$  is finitely generated as a pseudoalgebra,  $A$  is a finitely generated algebra.

PROOF. Consider the subset  $A = 1 \otimes_H R$  of the annihilator algebra  $\mathcal{A}(R)$ . Clearly, it is a subalgebra of  $\mathcal{A}(R)$  with a left identity  $1 \otimes e$ . We will show that for a unital pseudoalgebra,  $R = \text{Diff } A$ .

We shall describe, at first, the annihilator subalgebra of  $\text{Cur } \mathbb{k}$ . Recall that  $e * e = (1 \otimes 1) \otimes_H e$ . Since  $(x \otimes e)(y \otimes e) = xy \otimes e$ ,  $e_1$  is the identity in  $\mathcal{A}(\text{Cur } \mathbb{k})$  and  $\mathcal{A}(\text{Cur } \mathbb{k})$  is generated by  $\{e_{t_i}\}$ .

Assume for now that  $e_1$  is the left and right identity in  $\mathcal{A}(R)$ . Since  $e_x = e_1 e_x$ ,  $e_x$  is not a zero divisor.

Suppose  $R$  is generated over  $H$  by the set  $R_0$  of elements  $a$  such that

$$a * e = (1 \otimes 1) \otimes_H a \tag{2.1}$$

(i.e.  $a_1 e = a$  and  $1 \in X$  is maximal with respect to  $a$  and  $e$ ). Notice that if  $a$  is such a generator, then so is  $e_x a$  for every  $x \in X$  by (1.24).

Clearly  $a_x = (a_1e)_x = a_1e_x$ . As the collection  $\{a_x\}$  is unique for each element of  $R$  (see Lemma 1.24), we conclude that the above set of generators of  $R$  is in 1-1 correspondence with  $A$ . Moreover,  $a_1b_1 = (a_1b)_1$ , hence  $A$  is an algebra with multiplication determined by the  $_1$ -product in  $R$ .

Define the action of  $X$  on  $A$ :  $x(a_1) = (e_x a)_1$ . The following calculation implies that  $x(a_1b_1) = x_{(2)}(a_1)x_{(1)}(b_1)$ :

$$\begin{aligned} e_x(a_1b) &= (e_{x(2)}a)_{x(1)}b = (e_{x(2)}(a_1e))_{x(1)}b \\ &= ((e_{x(2)}a)_1e)_{x(1)}b = (e_{x(2)}a)_1(e_{x(1)}b). \end{aligned}$$

Therefore,  $A$  is an  $X^{cop}$ -differential algebra. Hence, as  $R$  is torsion free over  $H$ , it follows from Lemma 1.24 that the  $x$ -products of elements from  $R_0$  satisfy (1.30).

To show that with the  $X^{cop}$  action defined as above,  $R \cong \text{Diff } A$ , it remains to prove that  $R = H \otimes R_0 \cong H \otimes A$  as  $H$ -modules, i.e. that  $R$  is a free  $H$ -module generated by  $R_0$ . Assume the contrary, namely, that there exist non-zero elements  $b_I \in R_0$  such that  $\sum_I \partial^I b_I = 0$  for some finite collection of  $I$ 's. Among these  $I$ 's, choose a maximal  $J$  (with respect to the natural ordering of  $n$ -tuples). Then  $0 = (\sum_I \partial^I b_I)_{t^J} e = (-1)^J b_J$ , a contradiction. Hence,  $R = H \otimes R_0$ , where  $R_0$  is a generating set of  $R$  satisfying (2.1). By construction of  $A$ ,  $R = \text{Diff } A$ .

Now it remains to construct such a generating set  $R_0$ .

Fix an arbitrary element  $a \in R$ . For  $I$  such that  $t^I$  is maximal with respect to  $a$  and  $e$ , put  $a_I = (-1)^I a_{t^I} e$ . A direct calculation utilizing (1.13) shows that for any  $J$ ,  $(a_I)_{t^J} e = \delta_{0,J} a_I$ , i.e.  $a_I$  satisfies (2.1). Consider now the element  $a - \partial^I a_I$ . For  $J$  such that  $J \not\geq I$  and  $a_{t^J} e = 0$ ,  $\partial^I t^J = 0$ , thus clearly,  $(a - \partial^I a_I)_{t^J} e = 0$ . Also, for  $J > I$ ,  $(a - \partial^I a_I)_{t^J} e = -(a_I)_{t^{J-I}} e = 0$ . Finally,  $(a - \partial^I a_I)_{t^I} e = a_I - (a_I)_1 e = 0$  as well.

We conclude that by subtracting from  $a$  elements of the type  $hb$  where  $h \in H$  and  $b$  satisfies (2.1), we can lower the number of  $I$ 's such that  $a_{t^I} e \neq 0$ . Since in the process we also lower the degree of such  $t^I$ 's, we will at some point obtain an element  $c$  such that  $c_{t^I} e = 0$  for  $I > 0$ . Then either  $c_1 e = 0$  as well or  $c_1 e \neq 0$ . In the former case,  $c * e = 0$ , hence  $c \in L(R)$  and  $c = 0$ . In the second case, for arbitrary  $d$  and  $x$ ,  $c_x d = c_x (e_1 d) = (c_1 e)_x d$ , hence  $c - c_1 e \in L(R)$  and we see that  $c$  satisfies (2.1) as well.

Therefore, given a set of  $H$ -generators of  $R$ , we can construct a set of generators satisfying (2.1) which shows that  $R$  is a differential pseudoalgebra.

Moreover, given a set that generates  $R$  as a pseudoalgebra, the elements obtained from it by the above procedure will also generate  $R$  and their tensor products with 1 will generate  $A$ . Due to our

construction of  $R_0$  and, ultimately, locality, if  $R$  is finally generated, this procedure will result in a finite number of generators for  $A$ .

We turn now to the general case. Let  $\mathcal{B}(R) = \mathcal{A}(R)e_1$ . Clearly  $\mathcal{B}(R)$  is a topological associative algebra. Since for any  $1 \neq h \in H$ ,  $h(1 \otimes e) = 0$ , for any  $a \in R$  and  $x \in X$ ,  $h(a_x e_1) = h(a_x)1(e_1) = h(a_x)e_1$ . We see that  $\mathcal{B}(R)$  is an  $H$ -differential algebra as well. Hence,  $\overline{R} = \mathcal{C}(\mathcal{B}(R)) = \text{Hom}_H^{\text{cont}}(X, \mathcal{B}(R))$  satisfies (1.22-1.24) (cf. Lemma 1.25).

Define a map  $\phi : R \rightarrow \overline{R}$ ,  $\phi(a) = a'$ , where  $a'(x) = a_x e_1$ . By definition of multiplications in  $\overline{R}$ , for  $a, b \in R$  and  $x, y \in X$ ,

$$\begin{aligned} (a'_x b')(y) &= a'(x_{(2)})b'(x_{(-1)}y) = a_{x_{(2)}} e_1 b_{x_{(-1)}y} e_1 = \\ &= (a'_x b')(y) = a_{x_{(2)}} b_{x_{(-1)}y} e_1 = (a_x b)_y e_1, \end{aligned} \tag{2.2}$$

as  $e_1$  is the left identity in  $\mathcal{A}(R)$ . Thus we obtain that  $\phi(a_x b) = a'_x b'$ . Denote  $\text{Im } \phi$  by  $R'$ . The calculation in (2.2) shows that multiplications in  $R'$  are also local, hence  $\phi$  is a pseudoalgebra map. We also conclude that  $R'$  is a unital pseudoalgebra with pseudoidentity  $e'$ .

The above construction of a generating set satisfying (2.1) could be repeated for  $R'$  with  $\mathcal{A}(R')$  replaced with  $\mathcal{B}(R)$  with the conclusion  $R' = \text{Diff } \mathcal{B}(R)$ . Therefore, if  $R \cong R'$ , the proof will be finished.

Assume  $\phi$  is not injective. Thus, there exists  $a \in R$  such that  $a_x e_1 = 0$  for all  $x \in X$ . As  $e_x = e_1 e_x$ ,  $a_x e_y = 0$  for all  $y \in X$  too. This implies that  $(a_x e)_y = 0$  for all  $y$ , hence  $a_x e = 0$ . Therefore, by Lemma 2.11,  $a = 0$ .  $\square$

REMARK 2.13. Given a choice of a pseudoidentity in a unital pseudoalgebra  $R$ , the above theorem provides a direct algorithm for establishing the differential structure of  $R$  (including the canonical basis  $\tilde{a}$ ). In the future, we will simply say that a choice of pseudoidentity endows  $R$  with a particular structure.

The above theorem is especially useful for the cases described below:

COROLLARY 2.14.  $\text{Cend}_n$  is a differential algebra over  $\text{End}_n(\mathbb{k}) \otimes H^{op}$ .

PROOF. Definition (1.28) of the pseudoproduct in  $\text{Cend}_n$  implies that it is unital with a zero left annihilator (the latter also follows from the simplicity of  $\text{Cend}_n$ , see Corollary 2.15 immediately below), hence it is differential. Repeating the construction of  $A$  in the proof of Theorem 2.12 in this particular case gives the description of the underlying  $X^{cop}$ -algebra. Namely, we have  $A =$

$\{1 \otimes_H (1 \otimes b \otimes B) \mid b \in H, B \in \text{End}_n(\mathbb{k})\}$  with the multiplication

$$(1 \otimes_H (1 \otimes b \otimes B))(1 \otimes_H (1 \otimes c \otimes C)) = 1 \otimes_H (1 \otimes cb \otimes BC),$$

which shows that  $A$  is isomorphic to  $\text{End}_n(\mathbb{k}) \otimes H^{op}$ .  $\square$

**COROLLARY 2.15** (Theorem 2.1). *A unital semisimple pseudoalgebra is differential.*

**PROOF.** By associativity the left annihilator is a nilpotent ideal.  $\square$

**2.1.3. Unital pseudoalgebras over cocommutative Hopf algebras.** We briefly discuss here what happens in the case of a more general cocommutative  $H$ .

Let  $R$  be a semisimple unital pseudoalgebra over a cocommutative Hopf algebra  $H$ . Recall (Corollary 1.23) that here  $R$  is a pseudoalgebra over  $H' = U(P(H))$ . Interpretation (1.17) of pseudoproduct over  $H$  in terms of that over  $H'$  together with (1.19) imply that  $R$  is unital over  $H'$ . In particular, if  $e$  is a pseudoidentity over  $H$ , it remains such over  $H'$  (the calculation is direct and is, therefore, omitted).

Consider now the pseudoproduct  $(ge) * e = (g \otimes 1) \otimes_H e$  over  $H$ . It does not survive passing to  $H'$  (cf. the construction in [BDK, Chapter 5]), hence  $L(R) \neq 0$  by Lemma 2.11. Remark that according to Definition 2.10, it is impossible to find another pseudoidentity in  $R$  regarded as an  $H'$ -pseudoalgebra such that  $R$  would satisfy the conditions of Theorem 2.12. Hence,  $R$  is not an  $H'$ -differential pseudoalgebra over a unital algebra.

On a more general note, unitality as defined above does not seem to be the right concept for the study of pseudoalgebras over a generic cocommutative algebra: for instance, there is no analogue of Lemma 1.30.

## 2.2. Representations of unital pseudoalgebras

We turn now to the description of representations of unital pseudoalgebras. Although we will work only with unital differential pseudoalgebras, in light of Theorem 2.12, this simply means that we impose a technical condition  $L(R) = 0$ . Since most of the interesting pseudoalgebras are semisimple, this holds automatically (Corollary 2.15).

The goal is to provide a statement similar to Theorem 2.12, i.e. to establish a correspondence between the categories of modules of  $A$  and  $\text{Diff } A$ . As in the previous section,  $H = U(\mathfrak{g})$  for a finite-dimensional  $\mathfrak{g}$ .

**2.2.1. Structure of representations of unital algebras.** Consider a representation  $V$  of a differential pseudoalgebra  $R = \text{Diff } A$ . Recall that by Lemma 2.3,  $V = V^0 \oplus V^1$  as a Curk-module, where  $e * V^0 = 0$ . Thus for any  $\tilde{a}, a \in A$  and  $v \in V^0$ ,  $\tilde{a}_x v = (\tilde{a}_1 e)_x v = 0$ , and  $R * V^0 = 0$ . Also, since  $e_1(a_x v) = (e_1 a)_x v = a_x v$ ,  $V^1$  is  $R$ -stable. Therefore, the decomposition of  $V$  is valid over  $R$  as well.

DEFINITION 2.16. A module  $V$  of a unital differential pseudoalgebra  $R$  is *unitary* if its zero component  $V^0$  is trivial.

Let now  $R = \text{Diff } A$  be a unital differential pseudoalgebra and  $V$  its unitary module.

As in the proof of Theorem 2.12, for any  $v \in V$ , we can consider elements  $v_I = e_{t^I} v$ . If  $t^I$  is maximal with respect to  $e$  and  $v$ ,  $e_{t^J} v_I = \delta_{0,J} v_I$  for any  $n$ -tuple  $J$ . Now, consider the difference  $w = v - \partial^I v_I$ . Direct calculations show that  $e_{t^J} w = 0$  for  $J$  such that either  $J \geq I$  or  $J$  is incompatible with  $I$  and  $e_{t^J} v = 0$ . By taking such differences repeatedly we will arrive at  $w$  such that  $e_{t^I} w = \delta_{0,I} w$ . Hence,  $V$  is generated over  $H$  by elements  $v$  such that  $e * v = (1 \otimes 1) \otimes_H v$ .

For such an element  $v$ ,  $\tilde{a}_{t^I} v = (\tilde{a}_1 e)_{t^I} v = \tilde{a}_1 (e_{t^I} v)$ , hence

$$\tilde{a} * v = (1 \otimes 1) \otimes_H (\tilde{a}_1 v). \quad (2.3)$$

LEMMA 2.17. *A unitary module of a unital differential pseudoalgebra is free as an  $H$ -module.*

PROOF. Assume the contrary, i.e. the existence of a finite collection of non-zero  $v_I \in V_0$  such that  $\sum_I \partial^I v_I = 0$ . Pick  $J$  to be a maximal  $n$ -tuple such that  $v_J \neq 0$ . Then, by calculating  $e_{t^J}(\sum_I \partial^I v_I)$  via (2.3), we see that  $v_J = 0$ .  $\square$

COROLLARY 2.18. *A unital differential pseudoalgebra is free as an  $H$ -module.*

Now, since  $V$  is free over  $H$ , we see that  $V = H \otimes V_0$ , and  $\tilde{a}$  acts on  $V_0$  in accordance with (2.3).

Remark that  $\mathcal{A}(V) = X \otimes_H V_0 = X \otimes V_0$ . In particular, if we write out  $\tilde{a}_1 v = v_0 + \sum_i h_i v_i$ , where  $v_i \in V_0$ , then  $1 \otimes_H (\tilde{a}_1 v) = 1 \otimes v_0$ . Recall that  $A = 1 \otimes_H R \subset \mathcal{A}(R) = X \otimes_H R$  and  $a = 1 \otimes_H \tilde{a}$ . We can introduce the action of  $A$  on  $V_0$  viewed as the subspace  $1 \otimes V_0$  of  $\mathcal{A}(V)$ . Thus,

$$av = a(1 \otimes_H v) = 1 \otimes_H (\tilde{a}_1 v), \quad v \in V_0 \quad (2.4)$$

We sum up the above discussion in the following lemma:

LEMMA 2.19. *Let  $V$  be a module of a unital differential pseudoalgebra  $R = \text{Diff } A$ . Then  $V = V^0 \oplus V^1$  where  $R * V^0 = 0$  and  $V^1 = H \otimes V_0$  with the action of  $R$  on elements of  $V_0$  described by (2.3). Moreover, there is a structure of an  $A$ -module on  $V_0$  described by (2.4).*

**2.2.2. Constructing representations of unital pseudoalgebras.** Conversely, let  $M$  be a unitary left module for an  $X^{\text{cop}}$ -differential algebra  $A$ . Our goal is to construct a related representation of  $R = \text{Diff } A$  on a left  $H$ -module  $\widetilde{M} = H \otimes M$ . Before this, we shall endow  $X \otimes M$  with the structure of an  $\mathcal{A}(R)$ -module. (Even though  $\mathcal{A}(R) = X \otimes A$ , we will write its basis elements as  $x \otimes_H \tilde{a}$  to emphasize the relation with  $R$ ).

Naturally, we put  $(1 \otimes_H \tilde{a})(1 \otimes m) = (1 \otimes am)$  and  $(x \otimes_H e)(1 \otimes m) = (x \otimes m)$ . In general,  $(x \otimes_H \tilde{a})(y \otimes m) = (1 \otimes_H \tilde{a})(xy \otimes_H e)(1 \otimes m) = (xy \otimes_H \tilde{a})(1 \otimes m)$ . Hence, to describe explicitly the action of  $\mathcal{A}(R)$  on  $X \otimes M$ , it suffices to write out the expression for  $(x \otimes_H \tilde{a})(1 \otimes m)$ . Recall that  $x \otimes_H \tilde{a}$  is  $\tilde{a}_x = \tilde{a}_1 e_x = (e_1 \tilde{a})_x$ . Using (1.23) and (1.24), it is not difficult to see that

$$\begin{aligned} \tilde{a}_1 e_x &= \sum_I (e_{(\partial^I x_{(2)})t^I} \tilde{a})_{x_{(1)}} = \\ &= \sum_I (\partial^I e)_x (e_{t^I} \tilde{a})_1, \end{aligned}$$

where the first equality is valid because  $\sum_I (\partial^I t^J) t^I = 0$  whenever  $J \neq 0$ .

Therefore,  $x \otimes_H \tilde{a} = \sum_I (\partial^I x \otimes_H e)(1 \otimes_H t^I a)$ , and we obtain

$$(x \otimes_H \tilde{a})(1 \otimes m) = \sum_I (\partial^I x) \otimes (t^I(a)m). \quad (2.5)$$

We now turn to the description of  $\widetilde{M}$ . First of all, remark that  $\mathcal{A}(\widetilde{M}) = X \otimes M$ . Thus, for  $x, y \in X$ ,  $(e_x \tilde{m})_y = e_{x_{(2)}} \tilde{m}_{x_{(-1)}y} = (x_{(2)}x_{(-1)}y) \otimes m = \varepsilon(x)y \otimes m$  by (1.7). Hence, if  $\varepsilon(x) = 0$ ,  $e_x m = 0$ , and  $e * \tilde{m} = (1 \otimes 1) \otimes_H \tilde{m}$ . That is,  $\widetilde{M}$  is a unitary module and, according to (2.3),  $\tilde{a} * \tilde{m} = (1 \otimes 1) \otimes_H \tilde{a}_1 \tilde{m}$ . It remains only to determine  $\tilde{a}_1 \tilde{m}$  for arbitrary  $a \in A, m \in M$ . Checking the coefficients, we obtain from (2.5):

$$\tilde{a}_1 \tilde{m} = \sum_I \partial^I (\widetilde{t^I(a)m}). \quad (2.6)$$

We summarize the above discussion as

**LEMMA 2.20.** *Let  $A$  be a unital  $X^{\text{cop}}$  differential algebra, and  $M$  an  $A$ -module. Then  $\widetilde{M} = H \otimes M$  is a representation of  $\text{Diff } A$  with the action described by*

$$\tilde{a} * \tilde{m} = (1 \otimes 1) \otimes_H \left( \sum_I \partial^I (\widetilde{t^I(a)m}) \right). \quad (2.7)$$

**2.2.3. Classification and structural theory.** Summing up, we obtain the full description of representations of unital differential pseudoalgebras.

**THEOREM 2.21.** *Let  $V$  be a module of a unital differential pseudoalgebra  $R = \text{Diff } A$ . Then  $V = V^0 \oplus V^1$ , where  $R * V^0 = 0$  and  $V^1 = \widetilde{M}$  for some  $A$ -module  $M$ . In particular,  $V^1$  is free over  $H$ .*

**PROOF.** The decomposition of  $V$  as well as the 0 action of  $R$  on  $V^0$  follow from Lemma 2.19. Freeness of  $V^1$  is explained in Lemma 2.17, in particular, we know (again, from Lemma 2.19) that  $V^1 = H \otimes V^0$ , where  $V^0$  is an  $A$ -module.

We can construct another representation  $\widetilde{V}_0$  of  $R$ . By comparing (2.3) with (2.7), we see that the  $R$ -action on both  $V^1$  and  $\widetilde{V}_0$  is determined by the  ${}_1$ -action only. Now, define the degree of  $a \in A$  as the maximal value of  $|I|$  such that  $t^I(a) \neq 0$ . Inducting on the degree and comparing (2.4) with (2.6), we conclude that  $\widetilde{V}_0 \cong V^1$ .  $\square$

**REMARK 2.22.** The proofs of both Theorem 2.12 and Theorem 2.21 required constructions of particular  $H$ -generating sets of, respectively, a given pseudoalgebra and a given module. However, if one considers a unital algebra as a  $\text{Cur } \mathbb{k}$ -module, these bases are clearly different (e.g., compare (2.1) and (2.3)). For conformal algebras  $\text{Cend}_n$  both were written out explicitly in Example 1.12 as  $L_A^k$  and  $J_A^m$ .

For the general case of pseudoalgebras  $\text{Cend}_n$ , the basis from Example 1.33 is the one corresponding to its structure as a  $\text{Cur } \mathbb{k}$ -module:  $\text{Cend}_n = \widetilde{M}_n$  where  $M_n = H \otimes \text{End}_n(\mathbb{k})$ .

**REMARK 2.23.** Clearly, a non-unitary  $A$ -module  $M$  gives rise to a non-unitary  $\text{Diff } A$ -module  $\widetilde{M} = H \otimes M$ . However, in this case the converse is not true. For example, a non-unitary  $\text{Diff } A$ -module does not have to be free over  $H$ .

Nonetheless, for consistency we will sometimes use the notation  $\widetilde{M}$  for non-unitary modules. In particular, we will denote the zero-dimensional  $\text{Diff } A$ -module as  $\widetilde{0}$ .

We now turn to the structural theory of representations of unital algebras; obviously, Theorem 2.21 will be our main tool.

The definitions are the same as in the ordinary case. A module  $V$  over a pseudoalgebra is called *irreducible* if it contains no submodules except for 0 and  $V$ , *indecomposable* if it can not be presented as a sum of two non-zero submodules, and *completely reducible* if it decomposes into a direct sum of irreducible ideals.

**COROLLARY 2.24.** *Let  $\widetilde{M}$  be a unitary module of a unital differential pseudoalgebra  $R = \text{Diff } A$  and  $W$  its submodule. Then  $W = \widetilde{N}$  for an  $A$ -module  $N \subset M$ .*

PROOF. The argument is the same as in the proof of Lemma 2.17.

Put  $N = \{m | \widetilde{m} \in W\}$ . Obviously,  $\widetilde{N} \subset W$ . Theorem 2.21 implies that  $W$  is unital as well, so we can apply (2.3). For an arbitrary element  $w = \sum_I \partial^I \widetilde{m}_I \in W$ ,  $m_I \in M$ , let  $J$  be a maximal  $n$ -tuple among  $I$ 's such that  $m_I \neq 0$ . Then  $e_{t^J} w = (-1)^J m_J \in W$ . By induction we obtain that all  $m_I$  lie in  $W$ , and  $w \in \widetilde{N}$ .  $\square$

COROLLARY 2.25. *Let  $V$  be a module over a unital differential pseudoalgebra  $R = \text{Diff } A$ . Then  $V$  is irreducible if and only if  $V = \widetilde{M}$  for an irreducible  $A$ -module  $M$  (not necessarily non-zero).*

PROOF. Since  $V = V^0 \oplus V^1$ , either of the components must be 0. If  $V = V^1$ , then by Theorem 2.21  $V = \widetilde{M}$  and, clearly,  $M$  must be irreducible as well. If  $V = V^0$ , then every element of  $V$  gives rise to an  $R$ -submodule, i.e.  $V = \widetilde{0}$ .

Conversely, by Corollary 2.24, a non-zero irreducible  $A$ -module  $M$  gives rise to an irreducible  $R$ -module  $\widetilde{M}$ .  $\square$

Similarly, we can prove:

COROLLARY 2.26. *Let  $V$  be a module over a unital differential pseudoalgebra  $R$ . Then  $V$  is indecomposable if and only if  $V = \widetilde{M}$  for an indecomposable  $A$ -module  $M$  (not necessarily non-zero).*

COROLLARY 2.27. *Let  $V$  be a module over a unital differential pseudoalgebra  $R$ . Then  $V$  is completely reducible if and only if  $V = \widetilde{M}$  for a completely reducible  $A$ -module  $M$ .*

PROOF. Clearly,  $V^0$  is completely reducible if and only if  $V^0 = \widetilde{0}$ . Complete reducibility of  $V^1$  again follows from Theorem 2.21 and Corollary 2.24.  $\square$

REMARK 2.28. The only major notion that does not immediately carry over from  $A$ -modules to Diff  $A$ -modules is faithfulness. Indeed, if  $\widetilde{M}$  is faithful,  $M$  need not be. The right concept here is to require that the annihilator of  $M$  does not contain any  $X^{\text{cop}}$ -stable ideals.

**2.2.4. Representations of pseudolinear conformal algebras.** The above corollaries allow us, for example, to classify the representations of the pseudoalgebra  $\text{Cend}_n$ . Below we shall do so in the case of conformal algebras.

We will completely describe irreducible and indecomposable modules over the conformal algebra  $\mathfrak{W}_n = \text{Cend}_n$ . By the above corollaries, this comes down to explaining how  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[\partial]$ -modules look like.

In Example 1.12 we described the standard module over  $\mathfrak{W}_n$ :  $E_n = \{a(z) = \sum_{a \in \mathbb{k}^n} at^n z^{-n-1}\}_{a \in \mathbb{k}^n}$ . By Corollary 2.14,  $\text{Cend}_n = \text{Diff } A$  where  $A$  is the algebra of  $n \times n$  matrices over  $\mathbb{k}[\partial]$ . Thus  $E_n = \widetilde{M}_n$  where the  $A$ -module  $M_n$  is an  $n$ -dimensional vector space on which  $\partial$  acts as the identity operator.

This can be generalized to the case of the module  $M_n^\alpha$  which is again an  $n$ -dimensional space on which  $\partial$  acts as  $\alpha \in \text{End}_n(\mathbb{k})$ . Thus we obtain a family of modules  $E_n^\alpha = \widetilde{M}_n^\alpha$  which can be explicitly written as

$$E_n^\alpha = \left\{ a(z) = \sum (at^n e^{-\alpha t}) z^{-n-1}, a \in \mathbb{k}^n \right\}, \quad \alpha \in \text{End}_n(\mathbb{k}).$$

Every irreducible  $\mathbb{k}[\partial]$ -module is one-dimensional ( $M_1^\alpha$  in the above notations) and every irreducible  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[\partial]$ -module is of the form  $M_n^\alpha$ . Thus, Corollary 2.25 implies

**PROPOSITION 2.29.** *Finite irreducible modules over  $\mathfrak{W}_n$  are of the form  $E_n^\alpha$  with the standard action.*

The case of indecomposable modules is similar. Let  $U$  be a finite-dimensional space with an indecomposable endomorphism  $\alpha$ . Then  $U$  is an indecomposable  $\mathbb{k}[\partial]$ -module ( $\partial \mapsto \alpha$ ) and every indecomposable  $\mathbb{k}[\partial]$ -module has this form. For  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[\partial]$  the module  $M_n^\alpha(U) = \mathbb{k}^n \otimes U$  with the obviously defined action is indecomposable. Remark that the irreducible modules  $M_n^\alpha$  defined above are simply  $M_n^\alpha(\mathbb{k})$ .

These describe all indecomposable modules over  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[\partial]$  (consider the decomposition of such as a  $\mathbb{k}[\partial]$ -module; all components will be of the same height). We thus obtain indecomposable  $\mathfrak{W}_n$  modules  $E_n^\alpha(U) = \widetilde{M}_n^\alpha(U)$ . In [K2] they were denoted  $\sigma_\alpha^{as}$  (“as” stands for associative); the following result was stated there as well:

**PROPOSITION 2.30.** *Finite indecomposable modules over  $\mathfrak{W}_n$  are exactly  $E_n^\alpha(U)$ .*

**REMARK 2.31.** In the case of an arbitrary  $\mathfrak{g}$  ( $H = U(\mathfrak{g})$ ), indecomposable and irreducible  $\text{Cend}_n$  modules arise from  $\mathfrak{g}$ -modules.

## Gelfand-Kirillov Dimension

This chapter is mostly dedicated to conformal algebras. The concept of growth has played an important role in the study of ordinary algebras (see [KL] and references therein); here we extend it to conformal algebras. The same approach is possible to the more general class of pseudoalgebras as well; we discuss it briefly at the conclusion of this chapter. However, as far as classification results are concerned, growth does not seem to be the right invariant for pseudoalgebras (see Example 3.13 and Chapter 4).

Unless explicitly stated otherwise, all conformal algebras in this chapter are assumed to be finitely generated.

### 3.1. Gelfand-Kirillov dimension for conformal algebras

The Gelfand-Kirillov dimension of a finitely generated algebra (of any variety)  $A$  is defined as

$$\text{GKdim } A = \limsup_{r \rightarrow \infty} \frac{\log \dim(V^1 + V^2 + \cdots + V^r)}{\log r},$$

where  $V$  is a generating subspace of  $A$ . This definition easily carries over to the conformal case.

Let  $C$  be a finitely generated conformal algebra (over any variety) with generators  $f_1, \dots, f_n$ . Define  $C_r$  to be the  $\mathbb{k}[\partial]$ -span of products of less than  $r$  generators with any positioning of brackets and any multiplications of any order.

Since the powers of  $\partial$  can be gathered in the beginning of conformal monomials (with a probable change in the orders of multiplications), it is clear that  $\bigcup_r C_r = C$ . For a given ordered collection of generators and a given positioning of brackets, the number of non-zero monomials is finite because of the locality axiom C1. Therefore,  $\text{rk } C_r$  is finite.

**DEFINITION 3.1.** Let  $C$  be a finitely generated conformal algebra. Let  $C_r$  be the  $\mathbb{k}[\partial]$ -span of the products of  $r$  generators of  $C$  with any positioning of brackets and any orders of multiplication. Then

$$\text{GKdim } C = \limsup_{r \rightarrow \infty} \frac{\log \text{rk}_{\mathbb{k}[\partial]}(C_1 + C_2 + \cdots + C_r)}{\log r}. \quad (3.1)$$

**EXAMPLE 3.2.** It follows from (1.4) and similar formulas for  $\mathfrak{W}_n$  that  $\text{GKdim } \mathfrak{W}_n = 1$ .

REMARK 3.3. Just as in the case of non-finitely generated associative algebras, it is possible to define the Gelfand-Kirillov dimension of a non-finitely generated conformal algebra:

$$\text{GKdim } C = \sup_{C' \subset C, C' \text{ finitely generated}} \text{GKdim } C'.$$

When  $C$  is an associative conformal algebra, its every element can be written as a sum of monomials  $(\dots (f_{j_1} \overset{(n_1)}{\circlearrowleft} f_{j_2}) \overset{(n_2)}{\circlearrowleft} f_{j_3} \dots) \overset{(n_{r-1})}{\circlearrowleft} f_{j_r}$  over  $\mathbb{k}[\partial]$  and the rewriting process prescribed by associativity identities (1.3) will not increase the number of generators involved in the original presentation of the given element. Therefore,

$$\begin{aligned} C_1 &= \text{Span}_{\mathbb{k}[\partial]}(f_1, \dots, f_n), \\ C_r &= \text{Span}_{\mathbb{k}[\partial]}(g \overset{(m)}{\circlearrowleft} f_j \mid g \in C_{r-1}, m \geq 0, 1 \leq j \leq n) + C_{r-1}. \end{aligned} \tag{3.2}$$

Hereafter this description of  $C_r$ 's will be used.

We will now show that in associative conformal algebras the orders of multiplications in the monomials used in the presentation (3.2) are uniformly bounded. The following lemma is a well-known fact (actually, it can be deduced directly from the standard proof of Dong's lemma 1.4).

LEMMA 3.4. *Let  $N = \max_{i,k} N(f_{j_i}, f_{j_k})$ . If  $(\dots (f_{j_1} \overset{(n_1)}{\circlearrowleft} f_{j_2}) \overset{(n_2)}{\circlearrowleft} f_{j_3} \dots) \overset{(n_{r-1})}{\circlearrowleft} f_{j_r} \neq 0$ , then  $n_j \leq N$  for all  $j$ .*

PROOF. Denote  $(\dots (f_{j_1} \overset{(n_1)}{\circlearrowleft} f_{j_2}) \overset{(n_2)}{\circlearrowleft} f_{j_3} \dots) \overset{(n_{r-3})}{\circlearrowleft} f_{j_{r-2}}$  by  $g$ . If  $n_{r-1} > N$ , then

$$\begin{aligned} (g \overset{(n_{r-2})}{\circlearrowleft} f_{j_{r-1}}) \overset{(n_{r-1})}{\circlearrowleft} f_{j_r} &= \sum_{s \geq n_{r-1}} \binom{n_{r-2}}{s - n_{r-1}} g \overset{(n_{r-2} + n_{r-1} - s)}{\circlearrowleft} (f_{j_{r-1}} \overset{(s)}{\circlearrowleft} f_{j_r}) \\ &= \sum_s \binom{n_{r-2}}{s - n_{r-1}} g \overset{(n_{r-2} + n_{r-1} - s)}{\circlearrowleft} 0 = 0. \end{aligned}$$

The statement follows by induction. □

One can also speak of the growth of  $C$  meaning the growth of function  $\gamma_C(r) = \text{rk}_{\mathbb{k}[\partial]} C_r$ . The Gelfand-Kirillov dimension of  $C$  is finite if and only if  $\gamma_C(r)$  is polynomial. Lemma 3.4 implies that  $\text{rk } C_r \leq \sum_{j=1}^n j^r N^{r-1}$ . Hence, just as in the non-conformal case,  $\gamma_C(r)$  can not be superexponential while exponential growth is possible (e.g. in free conformal algebras, see [Ro1] for the definition and explicit construction of its basis).

REMARK 3.5. Similar results can be proven for Lie conformal algebras since for them the presentation (3.2) holds as well because of the Jacobi identity. However the orders of multiplications in the Lie version of Lemma 3.4 depend linearly on  $r$  [Ro1, 1.17].

The Gelfand-Kirillov dimension is invariant to the change of the generating set: the new generators are contained in some  $C_k$  which in turn is contained in some  $C'_l$  (here  $C'_1, C'_2, \dots$  are the  $\mathbb{k}[\partial]$ -submodules defined by the new set of generators), thus for the new  $C'_r$ 's,  $C'_r \subseteq C_{kr} \subseteq C'_{lr}$  and the Gelfand-Kirillov dimension which measures only the growth of  $\gamma_C(r)$  remains the same. Also, the Gelfand-Kirillov dimensions of a subalgebra or a quotient algebra do not exceed the Gelfand-Kirillov dimension of the full algebra.

REMARK 3.6. If  $C$  is finite as a  $\mathbb{k}[\partial]$ -module,  $\text{GKdim } C = 0$ . Conversely, if  $C$  is finitely generated as a conformal algebra and is an infinite  $\mathbb{k}[\partial]$ -module, it has a non-zero Gelfand-Kirillov dimension.

For Lie and associative conformal algebras, it is also possible to describe what  $\text{GKdim}$  can be attained. Indeed, as  $\text{GKdim } \text{Cur } A = A$  (see below), the set of possible values of  $\text{GKdim}$  of  $\mathcal{X}$  conformal algebras includes the set of possible values of  $\text{GKdim}$  of ordinary  $\mathcal{X}$  algebras. Thus, for any  $r > 2$  (respectively,  $r > 1$ ), there exist an associative (respectively, Lie) conformal algebra of growth  $r$  (see [KL, Pe]).

It is clear from (3.2) and its Lie version that Lie or associative conformal algebras can not attain growth between 0 and 1. For finitely generated ordinary associative algebras, the celebrated Bergman's lemma [KL, Chapter 2] asserts that no growth between 1 and 2 is attainable. The same result can be proven for associative conformal algebras. The proof is similar to the original proof of Bergman's and will be provided elsewhere.

### 3.2. Relation between growths of a conformal algebra and its coefficient algebra

We will now relate the Gelfand-Kirillov dimension of  $C$  to that of  $\text{Coeff } C$ . Recall that  $\hat{C} = C \otimes_{\mathbb{k}} \mathbb{k}[t, t^{-1}]$  and  $\text{Coeff } C = \hat{C}/\hat{\partial}\hat{C}$  where  $\hat{\partial} = \partial + \partial/\partial t$ . The map  $(\hat{C}, \hat{\partial}) \rightarrow \text{Coeff } C$  is again denoted by  $\phi$ .

Notice that by definition,  $\hat{\partial}(ft^k) = \partial ft^k + kft^{k-1}$ , where  $\partial ft^k$  is a shorthand for  $(\partial f)t^k$ . Therefore,  $\phi(\partial ft^k) = -k\phi(ft^{k-1})$ . In general, if an element of  $\text{Coeff } C$  is an image of  $\partial^i ft^k$ , one can choose another preimage for it,  $f't^{k'}$ ,  $f' \notin \partial C$ . Remark also that  $\dim \phi(C') \leq \text{rk}_{\mathbb{k}[\partial]} C'$  for  $C' \subseteq C$ .

THEOREM 3.7. *For an associative conformal algebra  $C$ ,  $\text{GKdim } \text{Coeff } C \leq \text{GKdim } C + 1$ .*

PROOF. Notice that even when  $C$  is finitely generated,  $\text{Coeff } C$  does not have to be. Nonetheless, since we need to prove an upper bound on  $\text{GKdim } \text{Coeff } C$ , it suffices to demonstrate that such a bound holds for any finitely generated subalgebra of  $\text{Coeff } C$ .

Let  $V$  be a generating subspace of  $\text{Coeff } C$ . One can choose  $f_1, \dots, f_n \in C$  and  $M^-, M^+ \in \mathbb{Z}$  such that  $V \subseteq \text{Span}_{\mathbb{k}}(\phi(f_j t^k) \mid 1 \leq j \leq n, M^- \leq k \leq M^+)$ . We can always assume that  $M^- \leq 0$ .

Let

$$\hat{V} = \text{Span}_{\mathbb{k}}(f_j t^k \mid 1 \leq j \leq n, M^- \leq k \leq M^+) \subseteq \hat{C}. \quad (3.3)$$

As  $V \subseteq \phi(\hat{V})$ , it suffices to prove the upper bound on growth of  $\text{Coeff } C$  for the subalgebra generated by  $\phi(\hat{V})$ . We can also restrict ourselves to a smaller conformal algebra generated by  $f_i$ 's; the right-hand side in the theorem's statement does not increase.

Consequently, we change notations and take  $C$  to be the conformal algebra generated by  $f_1, \dots, f_n$ . Put  $V = \phi(\hat{V})$  where  $\hat{V}$  is defined as in (3.3). We shall also use the explicit description of  $C_r$ 's provided in (3.2).

We shall study the growth of  $\dim V^r$  via the growth of certain subspaces of  $\hat{V}^r$  in  $(\hat{C}, \textcircled{0})$ . The inductive statement claims that for every element of  $V^1 + V^2 + \dots + V^r$  one can choose a preimage in the subspace  $\hat{V}_r$  of  $\hat{C}$  spanned by  $ht^k$  such that  $h \in C_r$  and  $M_r^- \leq k \leq M_r^+$ . (These bounds will be calculated within the proof.) Simultaneously, we will prove that  $\hat{V}^r \subseteq \hat{V}_r$ .

When passing from  $r$  to  $r+1$ , the statement automatically holds for elements of  $V^i$ ,  $1 \leq i \leq r$  as  $C_r \subseteq C_{r+1}$  and the bounds can be chosen, so that  $(M_r^-, M_r^+) \subseteq (M_{r+1}^-, M_{r+1}^+)$ . Therefore, to prove the claim one needs only to provide a procedure for choosing the preimage of an element in  $V^{r+1} = V^r V^1$ .

By induction we can consider a larger space, namely  $\phi(\hat{V}_r)V$  and work only with the basis elements of  $\phi(\hat{V}_r)$  and  $V$ .

Let  $gt^{k_1} \in \hat{V}_r$  where  $M_r^- \leq k_1 \leq M_r^+$  and  $g \in C_r$  is a product of  $f_j$ 's. Consider  $f_{j_2} t^{k_2} \in \hat{V}$ . We have from (1.2)

$$gt^{k_1} \textcircled{0} f_{j_2} t^{k_2} = \sum_{j \geq 0} \binom{k_1}{j} (g \textcircled{j} f_{j_2}) t^{k_1+k_2-j}.$$

Notice that  $j \leq N$  for  $N = \max_{i,j} N(f_i, f_j)$  by Lemma 3.4.

Therefore,  $\phi(gt^{k_1} \textcircled{0} f_{j_2} t^{k_2})$  lies in

$$\begin{aligned} & \phi(\text{Span}_{\mathbb{k}}(g \textcircled{j} f_{j_2} t^{k_1+k_2-j} \mid M_r^- \leq k_1 \leq M_r^+, M_1^- \leq k_2 \leq M_1^+, j \leq N)) \subseteq \\ & \subseteq \phi(\text{Span}_{\mathbb{k}}(ht^k \mid h \in C_{r+1}, M_r^- + M^- - N \leq k \leq M_r^+ + M^+)) = \\ & = \phi(\text{Span}_{\mathbb{k}}(ht^k \mid h \in C_{r+1}, M_{r+1}^- \leq k \leq M_{r+1}^+)) \\ & = \hat{V}_{r+1}. \end{aligned} \quad (3.4)$$

The immediate consequence is that  $\dim V^1 + V^2 + \dots + V^{r+1}$  is bounded by the dimension of the subspace of  $\hat{V}_{r+1}$  given in (3.4). It does not exceed  $\dim \hat{V}_{r+1} = (\text{rk } C_{r+1})(M_{r+1}^+ - M_{r+1}^-)$ . If we

define inductively

$$\begin{aligned} M_1^+ &= M^+, & M_{r+1}^+ &= M_r^+ + M^+ \\ M_1^- &= M^-, & M_{r+1}^- &= M_r^- + M^- - N, \end{aligned}$$

we conclude that

$$\dim V^1 + V^2 + \cdots + V^r \leq (M^+ - M^- + N')r \cdot \text{rk } C_r$$

and the theorem follows from the definition of GKdim.  $\square$

REMARK 3.8. A similar result was proven for Lie conformal algebras of growth 0 in [DK, Lemma 2.5].

EXAMPLE 3.9. Let  $C$  be a conformal algebra which is a torsion  $\mathbb{k}[\partial]$ -module. Clearly  $\text{GKdim } C = 0$ , furthermore, it follows from the axiom C2 that any finitely generated subalgebra of  $C$  is finite over  $\mathbb{k}$ .

For any  $f \in C$ , let  $\alpha_i$  be such that  $(\prod_{i=0}^n (\partial + \alpha_i))f = 0$ . Put  $f_j = (\prod_{i=j}^n (\partial + \alpha_i))f$ . Since  $(\partial f)(n) = -nf(n-1)$ , it is clear that the coefficients of  $f_1$  are proportional to either  $f_1(0)$  or  $f_1(-1)$ , namely for all  $n$ ,  $\alpha_0 f_1(n) = n f_1(n-1)$ . By induction, the coefficients of  $f$  are linear combinations of  $f_j(0)$  and  $f_j(-1)$ ,  $1 \leq j \leq n-1$ . Therefore, the coefficient algebra of a finitely generated subalgebra of  $C$  spanned over  $\mathbb{k}$  by some  $g_i$ 's is spanned over  $\mathbb{k}$  by  $\{(g_i)_j(0), (g_i)_j(-1)\}$  and is finite over  $\mathbb{k}$ .

It follows that  $\text{GKdim } \text{Coeff } C = 0$ . This example shows that the inequality in Theorem 3.7 is sometimes strict.

Notice also that the only obstruction to the equality in the proof of Theorem 3.7 are non-zero products of generators whose zeroth coefficient is 0. This does not happen with the standard generators of differential conformal algebras: all such products are of the form  $\tilde{a}$  and if  $\tilde{a}(0) = 0$ , then  $a = 0$ . We conclude with

COROLLARY 3.10. *Let  $A$  be an associative algebra with a locally nilpotent derivation. Then  $\text{GKdim } \text{Diff } A = \text{GKdim } A - 1$ .*

### 3.3. Gelfand-Kirillov dimension for pseudoalgebras

**3.3.1. Definition.** The generalization of Definition 3.1 is straightforward.

Let  $R$  be a finitely generated pseudoalgebra over a cocommutative Hopf algebra  $H$ . For a generating set  $a_1, \dots, a_n$  define  $R_1$  as the  $H$ -span of  $a_i$ 's and  $R_r$  as the  $H$ -span of all  $x$ -products of  $r$  generators. Since any element of  $R$  can be represented as a sum of monomials with coefficients (i.e.

elements of  $H$ ) on the left, we have  $R = \bigcup_r R_r$ . Moreover, for a fixed  $r$  there exist a finite number of ways to put  $r - 1$  brackets in the expression and for any positioning of brackets the number of  $x$ -products of generators is finite due to locality. Therefore,  $R_r$  is finite over  $H$ . Just as before, we have

DEFINITION 3.11. Let  $R$  be a finitely generated pseudoalgebra over a cocommutative Hopf algebra  $H$ . Let  $R_r$  be the  $H$ -span of the products of  $r$  generators. Then

$$\text{GKdim } R = \limsup_{r \rightarrow \infty} \frac{\log \text{rk}_H(R_1 + R_2 + \cdots + R_r)}{\log r}. \quad (3.5)$$

As usual,  $\text{GKdim } R$  does not depend on the original choice of the generating set, the growth of a subalgebra is less than the growth of the larger pseudoalgebra, etc.

EXAMPLE 3.12. It follows from (1.26) that  $\text{GKdim } \text{Cur}_{H'}^H R = \text{GKdim}_{H'} R + (\text{GKdim } H - \text{GKdim } H')$  where  $\text{GKdim}_{H'} R$  stands for the Gelfand-Kirillov dimension of  $R$  regarded as a pseudoalgebra over  $H'$ .

EXAMPLE 3.13. Similarly to Example 3.2, we have  $\text{GKdim } \text{Cend}_n = \text{GKdim } H$  (this follows from (1.28)). More generally,  $\text{GKdim } \text{Cend}_n^\phi = \text{GKdim } H$  for any  $\phi$ . Remarkably, it follows that there exist pseudoalgebras whose growth is strictly less than that of  $\text{Cend}_n$  but who possess no finite representations (take, for instance, a current algebra over an algebra of growth less than  $\text{GKdim } H$  without finite-dimensional representations). In the same manner, one can also obtain a simple pseudoalgebra with the same growth as  $\text{Cend}_n$  but with very different properties.

**3.3.2. Comparing growths of a pseudoalgebra and its annihilation algebra.** Our goal here is to obtain an analogue of Theorem 3.7. Obviously, the proof is going to be less technical as one does not have to worry about negative coefficients. We remark beforehand that the bad case of Example 3.9 exists over any  $H$ : for instance, consider a current extension of the torsion conformal algebra from this example. Thus, the inequality in growth comparison is still strict in some cases.

As above we work with associative pseudoalgebras only, although a similar theorem can be proven for Lie pseudoalgebras as well.

THEOREM 3.14. *For an associative pseudoalgebra  $R$  over a cocommutative Hopf algebra  $H$ ,  $\text{GKdim } \mathcal{A}(R) \leq \text{GKdim } R + \text{GKdim } H$ .*

PROOF. In this proof, we skirt over technical details that are exactly the same as in the proof of Theorem 3.7.

Let  $V$  be a generating subspace of  $\mathcal{A}(R)$ ,  $V = \text{Span}_{\mathbb{k}}(y_j \otimes_H a_j)$ . We can enlarge  $V$  and assume that  $V = W \otimes_H R_1$  where  $W$  is a generating subspace of  $X$  that contains all  $y_j$  and is closed under the action of  $H$  and  $R_1$  generates  $R$ .

REMARK 3.15. When  $H$  is a universal enveloping algebra, one can take  $W$  as the span of all monomials of degree less than  $\max_j \deg y_j$  where  $\deg$  is the standard total degree on  $X$ .

Let  $y \otimes_H b \in W^r \otimes_H (R_1 + R_2 + \cdots + R_r)$ . Then for any  $j$ ,  $(z \otimes_H b)(y_j \otimes_H a_j) = \sum (xS(h_i))y_i \otimes_H (b_{x_i}a_i)$ . As the action of  $S(h_i)$  on  $X$  leaves  $W^r$  invariant (in the case of  $H = U(\mathfrak{g})$ , the action of  $H$  on  $X$  lowers the standard total degree in  $t$ ), we conclude that  $V^1 + V^2 + \cdots + V^{r+1} \subset W^{r+1} \otimes_H (R_1 + R_2 + \cdots + R_{r+1})$  for any  $r$ .

Since  $\dim W^r \otimes_H R_r \leq \dim W^r + \text{rk } R_r$ , we are done. □

EXAMPLE 3.16. As in Corollary 3.10, we can conclude that  $\text{GKdim Diff } A = \text{GKdim } A$ .

## Classification Theorems

The main results of this chapter contain the full classification of conformal algebras of linear growth and, more generally, pseudoalgebras that are algebraically similar to  $\text{Cend}_n$ .

In the case of conformal algebras, the result can be easily formulated:

**THEOREM 4.1.** *Let  $C$  be a simple unital associative conformal algebra of linear growth. Then  $C$  is isomorphic to  $\mathfrak{W}_n$ .*

We also extend this theorem for a larger class of conformal algebras.

In the pseudoalgebra case, growth does not seem to be the right invariant for the classification. Nonetheless, we obtain

**THEOREM 4.2.** *Let  $R$  be a simple unital associative  $H$ -pseudoalgebra,  $H = U(\mathfrak{g})$ . Assume that its maximal unital current subalgebra having the same pseudoidentity as  $R$  is finite as an  $H$ -module and simple. Then  $R$  is isomorphic to either of*

- $\text{Cur End}_m(\mathbb{k})$ ,  $m > 0$ ;
- $\text{Cur}_H^{H'} \text{Cend}_m$ ,  $H' = U(\mathfrak{h})$  a Hopf subalgebra of  $H$ ,  $m > 0$ ;
- $\text{Cur}_H^{H'} \text{Cend}_m^\phi$ ,  $H' = U(\mathfrak{h})$  a Hopf subalgebra of  $H$ ,  $\phi \in H^2(\mathfrak{h}, \mathbb{k})$ ,  $m > 0$ .

### 4.1. Ideals of differential pseudoalgebras

We begin by establishing the structure of ideal lattice of differential pseudoalgebras.

As we are interested in simple unital pseudoalgebras, our objects of study are necessarily differential (Corollary 2.15). For such a pseudoalgebra  $R = \text{Diff } A$ , simplicity easily translates into a property of the  $X^{\text{cop}}$ -algebra  $A$ . Namely, call  $A$   $X^{\text{cop}}$ -simple if it contains no non-zero ideals stable under the action of  $X^{\text{cop}}$ . Then we have

**LEMMA 4.3.**  *$\text{Diff } A$  is simple if and only if  $A$  is  $X^{\text{cop}}$ -simple.*

**PROOF.** Let  $I$  be an  $X^{\text{cop}}$ -stable ideal of  $A$ . Put  $\tilde{I}$  to be the  $H$ -submodule of  $\text{Diff } A$  generated by  $\{\tilde{a} \mid a \in I\}$ . Then by (1.28) and Corollary 2.18,  $\tilde{I}$  is an ideal of  $\text{Diff } A$ .

Conversely, let  $J$  be a non-zero ideal of  $\text{Diff } A$ . For  $b = \sum_K \partial^K \widetilde{a}_K \in J$ , pick  $L$  maximal among indices such that  $a_L \neq 0$  and consider the product  $b_{tL}e$ . It equals  $(-1)^L \widetilde{a}_L$  (see the proof of Theorem 2.12). By induction we obtain that all  $\widetilde{a}_K$  belong to  $J$  (compare this to the proof of Corollary 2.24). As  $\widetilde{a}_1 \widetilde{b} = \widetilde{ab}$ , this implies that  $\widetilde{J} = \{a | \widetilde{a} \in J\}$  is a non-zero ideal of  $A$ . As  $e_x \widetilde{a} = \widetilde{x(a)} \in J$ , this ideal is  $X^{\text{cop}}$ -stable.  $\square$

In fact, we proved a more general result:

LEMMA 4.4. *The lattice of  $X^{\text{cop}}$ -stable ideals of  $A$  is isomorphic to the lattice of ideals of  $\text{Diff } A$ .*

PROOF. Indeed, in notations from the proof of Lemma 4.3, it is clear that  $\widetilde{I} = I$  and  $\widetilde{J} = J$  for any  $X^{\text{cop}}$ -stable ideal  $I$  of  $A$  and any ideal  $J$  of  $\text{Diff } A$ .  $\square$

## 4.2. Semisimple conformal algebras of linear growth

In this section we prove Theorem 4.1 and its several generalizations. We also provide examples of subalgebras of simple and prime conformal algebras.

As follows from Lemma 4.3,  $C$  is simple if and only if  $\text{Coeff } C$  is differentially simple. Following this correspondence, we define a larger subclass of unital associative conformal algebras: we call  $C$  *prime* whenever  $(\text{Coeff } C)_0$  is prime (since being prime is, in some sense, equivalent to being differentially prime). Also, recall that  $C$  is *semisimple* if it does not contain non-zero nilpotent ideals. The latter condition is equivalent to having a semiprime coefficient algebra (see Lemma 4.16).

**4.2.1. Classification of associative algebras of linear growth.** The following theorem was proven in [SSW] (see also [SW]):

THEOREM 4.5. *Let  $A$  be a finitely generated algebra of linear growth. Then*

- (i) *The nil-radical of  $A$ ,  $N(A)$ , is nilpotent.*
- (ii) *If  $A$  is semiprime (i.e. if  $N(A) = 0$ ), then it is a finite module over its center  $Z(A)$  which is also finitely generated.*

Several facts from the original proof of this theorem will be used below as well.

**4.2.2. Prime unital conformal algebras of linear growth.** We are going to classify prime unital conformal algebras of linear growth. It is well known that a differentially simple algebra is necessarily prime (see, e.g. [Po]), hence Theorem 4.1 will follow from such a classification.

Let  $A$  be a finitely generated prime algebra of linear growth. Then by Theorem 4.5, it is a finite module over its center  $Z(A)$ . Moreover, it is easy to see that for any derivation  $\delta$  of  $A$ ,  $Z(A)$  is  $\delta$ -stable: for  $a \in Z(A)$ ,  $0 = \delta([a, b]) = [\delta(a), b] + [a, \delta(b)] = [\delta(a), b]$  for any  $b \in A$ .

Thus, we begin by considering the case of a prime commutative finitely generated algebra.

LEMMA 4.6. *Let  $A$  be a finitely generated prime commutative algebra with a locally nilpotent derivation  $\delta$ ,  $\text{GKdim } A = 1$ . Then either  $A \cong \mathbb{k}[x]$ ,  $\delta = \partial/\partial x$  or  $\delta = 0$ .*

PROOF. Since  $A$  is prime, a non-zero algebraic element of  $A$  must be invertible, hence all its algebraic elements lie in  $\mathbb{k}$ .

Consider two sets of transcendental elements of  $A$ :

$$S_1 = \{x \in A \mid \text{all non-zero } \delta^n(x) \text{ are transcendental}\},$$

$$S_2 = \{x \in A \mid \text{for some } n, \delta^n(x) \neq 0 \text{ and is algebraic}\}.$$

Clearly, both sets are  $\delta$ -stable. Assume both are non-empty. Without loss of generality we can pick  $x_1 \in S_1$ ,  $x_2 \in S_2$  such that  $\delta(x_1) = 0$ ,  $\delta(x_2) = 1$ . As  $\text{tr.deg } A = 1$ ,  $x_1$  and  $x_2$  are algebraically dependent. In any statement of dependence of  $x_2$  over  $\mathbb{k}[x_1]$  the degree in  $x_2$  can be lowered by application of  $\delta$ . Therefore, one of the  $S_i$ 's is empty.

Consider now the case  $A = \mathbb{k} + S_1$ . Assume that there exists an element with a non-zero derivation. Without loss of generality we can consider  $x$  and  $y$  such that  $\delta(x) = y$ ,  $\delta(y) = 0$ . Just as above, consider a statement of dependence of  $x$  over  $\mathbb{k}[y]$ . Application of  $\delta$  lowers the degree in  $x$  (though it increases the degree in  $y$ ), a contradiction. Therefore,  $\delta$  kills all transcendental elements.

The remaining case is  $A = \mathbb{k} + S_2$ . Choose  $x$  such that  $\delta(x) = 1$ . Let  $y$  be an arbitrary element with  $\delta(y) \in \mathbb{k}$ . Then  $x - y(\delta(y))^{-1} \in \mathbb{k}$  and  $y \in \mathbb{k}[x]$ . For an arbitrary  $y \in S_2$ , by induction on the minimal  $n$  such that  $\delta^n(y) \in \mathbb{k}$ , we also obtain  $y \in \mathbb{k}[x]$ .  $\square$

REMARK 4.7. The final part of the proof of the above lemma can be also deduced from a result in [Wr].

COROLLARY 4.8. *Let  $A$  be a finitely generated differentiably simple commutative algebra of growth 1 with a locally nilpotent derivation. Then  $A \cong \mathbb{k}[x]$ .*

PROOF. Indeed, if  $A \not\cong \mathbb{k}[x]$ , it must be simple. Therefore,  $A$  is a field of transcendental degree 1 and can not be finitely generated.  $\square$

LEMMA 4.9. *Let  $A$  be a finitely generated prime algebra with a locally nilpotent derivation  $\delta$ ,  $\text{GKdim } A = 1$ . Then  $A$  is either isomorphic to  $\text{End}_n(\mathbb{k}[x])$ ,  $\delta = \partial/\partial x$ , or  $A$  can be embedded into an algebra  $B$  such that  $\delta$  extends to an inner derivation of  $B$  determined by a nilpotent element.*

PROOF. As mentioned above,  $A$  is a finite module over its center  $Z(A)$  which is finitely generated and has linear growth. Clearly,  $Z(A)$  is prime and  $\delta$ -stable.

**Case 1.**  $Z(A) = \mathbb{k}[x]$  and  $\delta|_{Z(A)} = \partial/\partial x$ .

Consider subalgebra  $A_0 = \ker \delta$ . We begin by demonstrating that  $A_0$  generates  $A$  as a module over  $Z(A)$ . More precisely, every  $a \in A$  is of the form  $\sum_1^n \frac{x^i}{i!} a_i$ ,  $a_i \in A_0$ , where  $n$  is such that  $\delta^n(a) = 0, \delta^{n-1}(a) \neq 0$ . Indeed, with the inductive assumption  $\delta(a) = \sum_1^{n-1} \frac{x^i}{i!} b_i$ ,  $b_i \in A_0$ , consider  $a_0 = a - \sum_1^{n-1} \frac{x^{i+1}}{(i+1)!} b_i$ . Since  $\delta(a_0) = 0$ , we see that  $a$  is also a polynomial in  $x$  over  $A_0$ . Moreover, this polynomial form is unique for any  $a \in A$ . Indeed, if  $\sum_1^n x^i a_i = 0, a_i \in A$ , applying a necessary number of derivations shows that the coefficient at the highest power is 0. In particular, this implies that  $A = \mathbb{k}[x] \otimes A_0$ .

Fix a subset  $\{a_i\}$  of  $A_0$  that generates  $A$  as a module over  $Z(A)$ . Any product of elements of  $A_0$  is a linear combination  $\sum p_i(x)a_i$  over  $Z(A)$  with the derivation  $\sum (\partial p_i(x)/\partial x)a_i = 0$ . This implies  $A_0 = \text{Span}_{\mathbb{k}}(a_i)$  is finite dimensional.

Clearly any ideal of  $A_0$  can be lifted to  $A$ , thus  $A_0$  is prime as well and therefore simple over  $\mathbb{k}$  [Rw, 2.1.15]. Hence,  $A = \text{End}_n(\mathbb{k}[x])$  and  $\delta = \partial/\partial x$ .

**Case 2.**  $\delta|_{Z(A)} = 0$ .

Let  $F$  be the field of fractions of  $Z(A)$  and consider the finite-dimensional simple  $F$ -algebra  $B = F \otimes_{Z(A)} A$ . Clearly,  $\delta$  extends to a derivation of  $B$  and is, therefore, an inner nilpotent derivation,  $\delta = \text{ad } a$  [Ja]. We may take  $a$  to be nilpotent (it is enough to pick any  $a$  such that  $\delta = \text{ad } a$  and take its nilpotent part, since the semisimple part must commute with all elements of  $B$ ). □

PROOF OF THEOREM 4.1. If we strengthen the condition of Lemma 4.9 to  $A$  being differentiably simple, by Corollary 4.8 we will have to consider Case 1 of the above lemma only. Thus  $\text{Diff } A$  is isomorphic to  $\mathfrak{W}_n$ . □

REMARK 4.10. It follows that there exist no finitely generated simple associative current conformal algebras of linear growth. However, this (quite unexpected) result can be deduced directly from Theorem 4.5. Indeed, if  $\text{Cur } A$  is such an algebra, then  $A$  must be simple and have linear growth. So should its center, hence it is a field of transcendental degree 1.

In the more general framework of prime conformal algebras, the second case of Lemma 4.9 merits further consideration. We notice first that by a change of pseudoidentity, one can discount the twisting on the coefficient algebra introduced by an inner derivation:

LEMMA 4.11. *Let  $C$  be a differential conformal algebra over algebra  $A$  with an inner derivation determined by a nilpotent element. Then  $C \cong \text{Cur } A$ .*

PROOF. We have  $\delta = \text{ad } r$  for a nilpotent  $r$ . Clearly,  $A[t, t^{-1}; \delta]$  is isomorphic to the polynomial algebra  $A[s, s^{-1}]$  via the mapping  $t - r \rightarrow s$ . To prove that corresponding differential algebras  $C$  and  $\text{Cur } A$  are isomorphic as well, we first show that the formal distribution  $e' = \sum (t - r)^n z^{-n-1}$  belongs to  $C$ . Indeed, in this case, the conformal subalgebra of  $C$  generated by  $e'$  and  $\tilde{a} \circledast e', a \in A$ , is isomorphic to  $\text{Cur } A$ .

Let  $m$  be the degree of nilpotency of  $r$ . Since  $\delta(r) = 0$ ,  $t$  and  $r$  commute; therefore,

$$e' = \sum_{k=0}^{m-1} \frac{1}{k!} \partial^k \left( \sum r^k t^n z^{-n-1} \right) \in \text{Diff } A.$$

Conversely,  $(\tilde{r} \circledast e') \circledast e' = \tilde{r}$  lies in the above subalgebra, hence, so does  $e$ . Thus, this subalgebra coincides with  $C$ .  $\square$

COROLLARY 4.12. *Let  $C$  be a simple unital associative conformal algebra that is finite over  $\mathbb{k}[\partial]$ . Then  $C \cong \text{Cur } \text{End}_n(\mathbb{k})$ .*

PROOF. Let  $C = \text{Diff } A$  where  $A$  is differentiably simple. Since  $A$  is finite, it must be simple [B1]; thus  $A \cong \text{End}_n(\mathbb{k})$  and all its derivations are inner. The rest follows from Lemma 4.11.  $\square$

REMARK 4.13. The above result also follows from the classification of simple Lie conformal algebras that are finite over  $\mathbb{k}[\partial]$  in [DK]. In fact, in this line of proof one does not require unitality; though, we still get only  $\text{Cur } \text{End}_n(\mathbb{k})$  as an answer [K2]. This shows that every simple associative conformal algebra that is finite over  $\mathbb{k}[\partial]$  is unital.

Of course, in general not every associative conformal algebra is unital. Moreover, one can not simply “adjoin” a pseudoidentity as in the ordinary case. However, in every known case, a conformal algebra can be embedded into a unital one.

CONJECTURE 4.14. *Every  $\partial$ -torsion free associative conformal algebra can be embedded into a unital conformal algebra.*

Another question is: if such embeddings  $C \rightarrow C'$  exist for a given conformal algebra  $C$ , what is the lower bound on  $\text{GKdim } C'$ ?

Clearly, when  $C$  has a faithful representation that is finite over  $\mathbb{k}[\partial]$ ,  $C$  embeds into  $\text{Cend}_n$ . Moreover, when  $C$  is finite itself it can be embedded into a finite conformal algebra (this follows

from the classification and so far no direct proof is known). So, in such cases it is always possible to find  $C'$  with  $\text{GKdim } C' = \text{GKdim } C$ .

We can now translate the statement of Lemma 4.9 into the language of conformal algebras:

**PROPOSITION 4.15.** *Let  $C$  be a prime unital finitely generated associative conformal algebra with  $\text{GKdim } C = 1$ . Then  $C$  is isomorphic to either  $\mathfrak{W}_n$  or a subalgebra of a current algebra over a finitely generated prime algebra.*

**PROOF.** By Lemma 4.9,  $(\text{Coeff } C)_0$  is either  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[x]$  with a nilpotent derivation given by  $\partial/\partial x$  or a subalgebra of a prime algebra with an inner derivation determined by a nilpotent element. In the first case,  $C$  is isomorphic to  $\mathfrak{W}_n$  and in the second case it is a subalgebra of a current algebra by Lemma 4.11. In general, this current algebra might be infinitely generated but by construction one may choose an appropriate prime current subalgebra.  $\square$

Essentially, this classification is the best one can hope for; this is explained in Subsection 4.2.4.

**4.2.3. Classification of semisimple conformal algebras of linear growth.** We begin by translating semisimplicity of conformal algebras of linear growth into a condition for its coefficient algebra.

**LEMMA 4.16.** *Let  $\text{Diff } A$  be an associative conformal algebra of zero or linear growth. Then  $\text{Diff } A$  is semisimple if and only if  $A$  is semiprime.*

**PROOF.** By construction in the proof of Lemma 4.3, an ideal  $J$  of  $\text{Diff } A$  is nilpotent if and only if the corresponding ideal  $\bar{J}$  of  $A$  is nilpotent. Indeed, if for any  $a_0, \dots, a_n, (\dots (\tilde{a}_0 \textcircled{0} \dots) \textcircled{0} \tilde{a}_n = 0$ , then  $a_0 \cdot \dots \cdot a_n = 0$  and, conversely, if  $(\bar{J})^n = 0$ , then  $(\dots (\tilde{a}_0 \textcircled{m_1} \dots) \textcircled{m_n} \tilde{a}_n = 0$  as  $\bar{J}$  is  $\delta$ -stable.

Thus, if  $\text{Diff } A$  contains a nilpotent ideal, so does  $A$ .

Conversely, if  $A$  is not semiprime, its nilradical  $N(A)$  is  $\delta$ -stable [Rw, 2.6.28] and nilpotent (Theorem 4.5, (i) for the case of linear growth). Hence,  $\widetilde{N(A)}$  is nilpotent.  $\square$

Thus, we are able to classify semisimple conformal algebras of linear and zero growth.

We need an easy (and probably known) technical lemma first.

**LEMMA 4.17.** *Let  $A = \bigoplus_i A_i$  be a finite direct sum of unital associative algebras  $A_i$ . Then a derivation  $\delta$  on  $A$  restricts to each  $A_i$ .*

PROOF. Let  $e_i$  be the identity of  $A_i$ . Since  $\delta(e_i e_j) = e_i \delta(e_j) + \delta(e_i) e_j = 0$  and the summands lie in  $A_i$  and  $A_j$  respectively, we have  $\delta(e_i) e_j = 0$  (for any  $j \neq i$ ). Thus,  $\delta(e_i) \in A_i$ . Now, as  $\delta(e_i) = \delta(e_i^2) = 2\delta(e_i)$ , it follows that  $\delta(e_i) = 0$ . Consequently,  $\delta(A_i) = \delta(A_i e_i) = \delta(A_i) e_i \subset A_i$ .  $\square$

Classification of semisimple unital conformal algebras of  $\text{GKdim} \leq 1$  follows:

LEMMA 4.18 ([K2]). *Let  $C$  be a semisimple unital finitely generated associative conformal algebra that is finite over  $\mathbb{k}[\partial]$ . Then  $C \cong \bigoplus_{i=1}^k \text{Cur End}_{n_i}(\mathbb{k})$ .*

PROOF. The proof is the same as for Corollary 4.12. Let  $C = \text{Diff } A$ . By Lemma 4.16, we have  $A \cong \bigoplus_{i=1}^k \text{End}_{n_i}(\mathbb{k})$ . The nilpotent derivation that leads to  $C$  is inner, hence its effects can be removed by a change of pseudoidentity as in Lemma 4.11.  $\square$

THEOREM 4.19. *Let  $C$  be a semisimple unital finitely generated associative conformal algebra,  $\text{GKdim } C = 1$ . Then  $C$  embeds into a direct sum of a current algebra over a semiprime algebra of zero or linear growth and  $\mathfrak{W}_n$  for some  $n$ .*

PROOF. Let  $C = \text{Diff } A$  be determined by a nilpotent derivation  $\delta$ . By Lemma 4.16,  $A$  is semiprime. It follows from various lemmas in [SSW] that  $A$  splits as  $A = B \oplus F$  where  $B$  is semiprime Goldie and  $F$  finite-dimensional.

We have  $Q = Q(B) = \bigoplus_i Q_i$ , a semisimple Artinian quotient algebra of  $B$ . By Lemma 4.17,  $\delta$  restricts to  $B$ . The standard construction of  $Q(B)$  implies that  $\delta$  can be extended to  $Q$  and we can again restrict it to  $Q_i$  (though it is no longer locally nilpotent at this stage). We obtain a system of prime ideals  $P_i = B \cap \bigoplus_{i \neq j} Q_j$ . Clearly,  $P_i$  is  $\delta$ -stable, hence so is  $B/P_i$ . As  $B \hookrightarrow \bigoplus_i B/P_i$  with the action of  $\delta$  preserved, we have  $\text{Diff } B \hookrightarrow \bigoplus_i \text{Diff } B/P_i$ . Also,  $\text{Diff } A = \text{Diff } B \oplus \text{Diff } F$ .

Thus,  $\text{Diff } A$  embeds into a direct sum of prime conformal algebras of linear growth and a  $\mathbb{k}[\partial]$ -finite semisimple conformal algebra. We have two types of components in this sum: some come from subalgebras of  $\mathfrak{W}_m$ , hence their sum can be viewed as a subalgebra of  $\mathfrak{W}_n$  for some  $n$ . Others are subalgebras of prime current algebras of growth not exceeding 1. Thus, their sum is a subalgebra of a semiprime current algebra.  $\square$

**4.2.4. Examples of subalgebras of prime unital conformal algebras.** In this subsection we discuss what kinds of conformal algebras can live inside typical examples of prime conformal algebras.

The most innocently looking one is a current algebra over a prime algebra.

The proof of Lemma 4.11 relies on the simple fact that to each locally nilpotent derivation of Coeff  $C$  corresponds a pseudoidentity and a canonical basis of  $C$ , hence the effect of the derivation can be “untwisted.” This is not necessarily true for all subalgebras of  $C$ .

REMARK 4.20. Let  $C'$  be a unital conformal subalgebra of a unital current algebra  $C$  with the same pseudoidentity. Then  $C'$  itself is current. Indeed, let  $a = \sum_0^n \partial^k \widetilde{a}_k \in C'$ . Then  $(-1)^n n! \widetilde{a}_n = a \circledast e \in C'$ , hence, by induction all  $\widetilde{a}_k \in C'$ .

However, when pseudoidentities of the conformal algebra and its subalgebra are different, the approach in the above remark can not be used. Such a situation arises in the setting of Proposition 4.15: let  $C'$  be a subalgebra of  $C = \text{Diff } A$  where the derivation on  $A$  is inner,  $\delta = \text{ad } a$ . If  $\widetilde{a} \notin C'$ , the change of pseudoidentity prescribed by Lemma 4.11 can not be performed inside  $C'$ , thus  $C'$  becomes a possibly non-current subalgebra of a current algebra  $C$ . Consider the following

EXAMPLE 4.21. Let  $A = \text{Mat}_2(\mathbb{k}[x])$  and  $\delta = \text{ad } e_{12}$  be a locally nilpotent derivation on  $A$ . Remark that  $\text{Diff } A$  is current by Lemma 4.11. Let  $B = \text{Mat}_2(x\mathbb{k}[x])$  with the identity adjoined.  $B$  is  $\delta$ -stable, so  $\text{Diff } B \subset \text{Diff } A$ ; however, we will show below that  $\text{Diff } B$  is not current for any choice of pseudoidentity.

More generally, it turns out that whenever the derivation is external with respect to  $C'$ ,  $C'$  can not be current. The following statement is, in some sense, a converse of Lemma 4.11.

LEMMA 4.22. *Let  $C$  be an associative conformal algebra such that for different choices of pseudoidentities,  $C \cong \text{Cur } A$  and  $C \cong \text{Diff } B$  for a nilpotent derivation  $\delta$ . Then  $\delta$  can be made inner on  $A$ , i.e., there exists  $a \in A$  such that the conformal algebra  $\text{Diff } A$  is determined by  $\text{ad } a$  is isomorphic to  $\text{Diff } B$  with the isomorphism preserving the pseudoidentity. Moreover,  $a$  is nilpotent.*

PROOF. We fix the following notations:  $\widetilde{a}$ ,  $a \in A$ , is the canonical basis of  $\text{Cur } A$  and  $\widetilde{e}$  is its pseudoidentity. For  $\text{Diff } B$ ,  $\bar{b}$  is the canonical basis and  $\bar{e}$  the pseudoidentity.

We also identify the elements of  $\text{Cur } A$  and  $\text{Diff } B$  via the given isomorphism. Thus, we have  $\bar{e} = \sum \partial^i \widetilde{e}_i$  for some  $e_i \in A$ . For an arbitrary  $b \in B$ ,  $\bar{b} = \sum \partial^i \widetilde{b}_i$ . Since  $\bar{b} = \bar{b} \circledast \bar{e}$ , we have  $\bar{b} = (\sum \partial^i \widetilde{b}_i) \circledast (\sum \partial^i \widetilde{e}_i) = \sum \partial^i \widetilde{b}_0 e_i$ . Thus,  $b_i = b_0 e_i$ .

In particular this implies that if  $b \neq 0$ , then  $b_0 \neq 0$ . Moreover,  $(bb')_0 = b_0 b'_0$  as  $\overline{bb'} = \bar{b} \circledast \bar{b}'$ . This establishes a map  $B \rightarrow A$ ,  $b \mapsto b_0$ . Now we have to show that it extends to a map of given conformal algebras.

We also remark that as  $\bar{b} = \bar{e} \circledast \bar{b}$ , we see that  $e_0 b_0 = b_0$ .

According to (1.30),  $\overline{\delta(b)} = -\bar{e} \circledast \bar{b}$  for any  $b \in B$ . We will now calculate  $\bar{e} \circledast \bar{b}$  in  $\text{Cur } A$ :

$$\begin{aligned}
\bar{e} \circledast \bar{b} &= \left( \sum \partial^i \tilde{e}_i \right) \circledast \left( \sum \partial^i \widetilde{b_0 e_i} \right) = \\
&= \tilde{e}_0 \circledast \left( \sum \partial^i \widetilde{b_0 e_i} \right) - \tilde{e}_1 \circledast \left( \sum \partial^i \widetilde{b_0 e_i} \right) = \\
&= \sum \partial^i (\tilde{e}_0 \circledast \widetilde{b_0 e_i}) + \sum i \partial^{i-1} (\tilde{e}_0 \circledast \widetilde{b_0 e_i}) - \sum \partial^i \tilde{e}_1 \circledast \widetilde{b_0 e_i} = \\
&= \sum i \partial^{i-1} \widetilde{e_0 b_0 e_i} - \sum \partial^i \widetilde{e_1 b_0 e_i}
\end{aligned} \tag{4.1}$$

(the last equality is valid, as we are working with a current algebra: products of positive orders of basis elements are 0).

On the other hand,

$$\overline{\delta(b)} = \sum \partial^i \widetilde{\delta(b)_0 e_i}. \tag{4.2}$$

Comparing the coefficients at  $\partial^0$  in (4.1) and (4.2), we see that  $\delta(b)_0 e_0 = -e_0 b_0 e_1 + e_1 b_0 e_0$ . Since  $\delta(b)_0 e_0 = \delta(b)_0$ , we see that  $\delta(b)_0 = \text{ad}(e_1) b_0$ . In particular, this implies that  $\text{ad}(e_1)$  is nilpotent.

Consider now a differential algebra  $\text{Diff } A$  over  $A$  determined by  $\text{ad}(e_1)$ . By the above, we obtained a injective map  $(B, \delta) \rightarrow (A, \text{ad}(e_1))$  of differential algebras; therefore,  $\text{Diff } B$  embeds into  $\text{Diff } A$  with the embedding given by  $\bar{b} \mapsto \widetilde{b_0}$ .

It remains to show that such an embedding is surjective. Since  $\text{Cur } A \cong \text{Diff } A$ , it is possible to express  $\tilde{a}$ ,  $a \in A$ , as  $\tilde{a} = \sum_j \partial^j \widetilde{a_j}$ , where  $a_j \in B$ . Hence,  $\tilde{a} = \sum_{i,j} \partial^{i+j} \widetilde{(a_j)_i}$ . As  $C$  is free over  $\mathbb{k}[\partial]$ ,  $a = (\widetilde{a_0})_0$ . This show that the constructed map is an isomorphism.

For our purposes, we need also to show that  $e_1$  is nilpotent. Substitute  $\bar{b} = \bar{e}$  in (4.1). The coefficient at  $\partial^j$  in the last line is  $\widetilde{e_{j+1}} - \widetilde{e_1 e_j}$ . Since  $\bar{e} \circledast \bar{e} = 0$ , we obtain by induction that  $e_j = (e_1)^j$  for  $j \geq 1$ . As the expression for  $\bar{e}$ ,  $\sum \partial^i \tilde{e}_i$ , is a finite sum, we see that  $e_1$  is nilpotent. This completes the proof.  $\square$

**COROLLARY 4.23.** *Let  $A'$  be a subalgebra of  $A$  and  $a \in A$  a nilpotent element. Then the conformal algebra  $\text{Diff } A'$  determined by  $\text{ad } a$  is current if and only if there exists  $a' \in A'$ ,  $\text{ad } a = \text{ad } a'$ .*

**PROOF.** If such  $a'$  exists, the statement follows from the proof of Lemma 4.11.

Otherwise, assume that  $\text{Diff } A'$  is current for some choice of pseudoidentity. By Lemma 4.22,  $\text{Diff } A'$  is determined by an inner derivation (for the same choice of pseudoidentity). Hence, there exists such  $a'$ .  $\square$

This shows that the subalgebra in Example 4.21 is indeed never current.

REMARK 4.24. Such examples can be constructed with ease. In particular, this means that the classification in Proposition 4.15 is the best possible for the case of prime conformal algebras.

Just as subalgebras of current algebras, subalgebras of  $\mathfrak{W}_n$  also appear naturally. However, this happens either when such a subalgebra acts on a given finite module or in the context of Theorem 4.19. In either case the pseudoidentities of  $\text{Cend}_n$  and its subalgebra coincide. Thus, we essentially speak of subalgebras of  $\text{End}_n(\mathbb{k}) \otimes \mathbb{k}[x]$  (with the same identity).

Such subalgebras are too general to describe (cf. Theorem 4.5), even in the prime case [SW]. The only known result is that on differentiably simple subalgebras, i.e. Theorem 4.1 and Corollary 4.12, and here the resulting conformal subalgebras are isomorphic to either  $\mathfrak{W}_m$  or  $\text{Cur End}_m(\mathbb{k})$ .

The case of simple subalgebras is obviously important for conformal representation theory. Kac's conjecture [K2] describes all subalgebras of  $\text{Cend}_n$  that act irreducibly on the standard module  $E_n$ . The unital conformal algebras from the list are the ones described above; therefore, we can say that results in this section confirm Kac's conjecture for unital algebras.

### 4.3. Simple unital pseudoalgebras

In this section we classify finitely generated simple unital pseudoalgebra satisfying certain conditions. The motivation is to find pseudoalgebras similar to the most important simple pseudoalgebras,  $\text{Cend}_n$ .

**4.3.1. Small  $X^{\text{cop}}$ -algebras.** Let  $A$  be an  $X^{\text{cop}}$ -algebra. We introduce a filtration on  $A$ :

$$\mathbb{F}^m A = (\mathbb{F}_m X)^\perp = \{a \in A \mid \mathbb{F}_m X(a) = 0\}. \quad (4.3)$$

Because of (1.14) and (1.29), the filtration (4.3) respects multiplication in  $A$ :

$$(\mathbb{F}^l A)(\mathbb{F}^m A) \subset \mathbb{F}^{l+m} A. \quad (4.4)$$

REMARK 4.25. Since the annihilator of every element of  $A$  is non-zero,  $A = \bigcup_m \mathbb{F}^m A$ .

We say that a non-zero  $a \in A$  has *degree*  $m$  if  $a \in \mathbb{F}^m A \setminus \mathbb{F}^{m-1} A$ .

Remark that the action of  $t_i$  lowers the degree. Namely, we have  $t_i(\mathbb{F}^{m+1} A) \subset \mathbb{F}^m A$ , otherwise  $t_i^{m+1} \mathbb{F}^{m+1} A \neq 0$ . Notice also the following useful properties:

LEMMA 4.26. (i) If  $\deg a = m$ , then for any  $M$  such that  $|M| = m$ ,  $t^M(a) \in \mathbb{F}^0 A$ .

(ii) If  $\deg a = m$ , there exists  $M$  with  $|M| = m$  such that  $t^M(a) \neq 0$ .

(iii)  $\deg(ab) \leq \deg(a) + \deg(b)$ .

PROOF. (i), (ii) follow immediately from (4.3) (recall that  $t^M \in F_{|M|-1} X$ ); whereas (iii) is a reformulation of (4.4).  $\square$

By definition, for  $H$  (or, rather, for  $H^{op}$ ) this filtration coincides with the canonical one. Here every filtration component is finite-dimensional.

LEMMA 4.27. *Let  $A$  be an  $X^{cop}$ -algebra such that  $\dim F^0 A < \infty$ . Then every filtration component is finite-dimensional.*

PROOF. For any  $a \in F^{m+1} A \setminus F^m A$ , there exists  $t_i$  such that  $t_i(a) \in F^m A \setminus F^{m-1} A$ . Hence,

$$F^{m+1} A = \bigcup_{i=1}^n t_i^{-1}(F^m A). \quad (4.5)$$

Notice also that filtration (4.3) can be introduced for any  $X$ -module  $M$  as long as all its elements have non-zero annihilators. In particular, this is true for submodules  $M_i$  of  $A$  defined by  $M_i = \{a \in A \mid t_i a = 0\}$ .

Consider the particular case of  $n = 1$  (i.e.  $X = \mathbb{k}[[t]]$ ). To demonstrate the statement of the lemma, we induct on  $m$ . For any two elements  $a, b \in t^{-1}(F^m A) = F^{m+1} A$  such that  $t(a) = t(b)$ , we have  $a - b \in F^0 A$ . Thus  $\dim F^{m+1} A \leq (\dim F^m A)(\dim F^0 A) \leq (\dim F^0 A)^{m+1}$  and  $F^m A$  is finite-dimensional for all  $m$ .

Now we induct on  $n$ . Let  $a, b \in t_i^{-1}(c)$  for some  $i$  and  $c \in F^m A$ . Then  $a - b \in F^m M_i$ . Hence (4.5) implies:

$$\begin{aligned} \dim F^{m+1} A &\leq \sum_{i=1}^m \dim t_i^{-1}(F^m A) \\ &\leq \sum_{i=1}^m (\dim F^m A)(\dim F^m M_i). \end{aligned}$$

Since by induction  $\dim F^m M_i < \infty$ , we see that  $F^{m+1} A$  is finite-dimensional.  $\square$

REMARK 4.28. Our results in Section 4.2 on the classification of conformal algebras can be reinterpreted as follows: when  $H = \mathbb{k}[\partial]$ , every finitely generated  $X^{cop}$ -simple  $X^{cop}$ -algebra of GKdim not exceeding 1 has finite filtration components.

However, this is not true in general and, by itself, limiting GKdim  $A$  does not imply any similarity of  $\text{Diff } A$  to  $\text{Cend}_n$  as we will see immediately below. Remark, first, that for a current pseudoalgebra  $\text{Cur } A$ , the filtration is trivial:  $F^0 A = A$ . Unlike the case of conformal algebras ( $H = \mathbb{k}[\partial]$ ), for larger  $H$  there exist non-finite finitely generated simple current algebras  $\text{Cur } A$  such that  $\text{GKdim } A \leq \text{GKdim } H$ , see e.g., Example 1.38. Moreover, consider the following example:

EXAMPLE 4.29. Let  $H = U(\mathfrak{g})$  where  $\mathfrak{g}$  is the three-dimensional abelian algebra. Then  $X = \mathbb{k}[[t_1, t_2, t_3]]$  and  $X = X^{cop}$ . Let  $A = \mathbb{k}[x, y]$  with the action of  $X$  defined by  $t_1 = \partial/\partial x, t_2 = 0, t_3 = 0$ . Then  $F^0 A = \mathbb{k}[y]$  and we obtain an  $X^{cop}$ -algebra that is smaller than  $H$  but has a non-trivial filtration with infinite filtration components. One can replace  $A$  with the Weyl algebra  $A_1 = \mathbb{k}\langle x, y \mid xy - yx = 1 \rangle$  (see Example 1.38) and obtain an  $X^{cop}$ -simple algebra smaller than  $H$  that has a non-trivial filtration with infinite filtration components.

Thus, pseudoalgebras similar to  $\text{Cend}_n$  should not be described by a simple combinatorial condition such as a bound on  $\text{GKdim}$ ; although, some sort of a growth restriction should be imposed. In particular, as we just saw, filtration components should be finite.

We arrive at the following definition:

DEFINITION 4.30. An  $X^{cop}$ -algebra  $A$  is called *small* if  $\dim F^n A < \infty$  for all  $n$ .

Thus, Lemma 4.27 can be reformulated as

LEMMA 4.31. *Let  $A$  be an  $X^{cop}$ -algebra. If  $F^0 A$  is finite-dimensional, then  $A$  is small.*

REMARK 4.32. It will follow from Theorem 4.45 that for a simple finitely generated pseudoalgebra  $\text{Diff } A$  with a small  $A$ ,  $A$  has Gelfand-Kirillov dimension not exceeding  $\text{GKdim } H = \dim \mathfrak{g}$ . This is a generalization of the converse of the first statement in Remark 4.28

**4.3.2. Digression: simplicity conditions for small  $X^{cop}$ -algebras.** In the next subsection we will show at first that under certain conditions  $A$  can be encoded by an associative algebra  $(F^0 A)$  and a certain Lie algebra acting on it. These conditions will be automatically satisfied when  $A$  is small and  $X^{cop}$ -simple and  $F^0 A$  is simple. These two statements about simplicity of  $A$  or  $F^0 A$  are closely related.

Indeed, let  $A$  be a small  $X^{cop}$ -algebra. Clearly, if  $J$  is a non-zero proper  $X^{cop}$ -stable ideal of  $A$ ,  $J \cap F^0 A$  is a non-zero ideal of  $F^0 A$  by Lemma 4.26,(ii).

EXAMPLE 4.33. Let  $A = \text{End}_2(\mathbb{k})$  with an inner derivation  $\delta = \text{ad}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ . Then  $F^0 A = \mathbb{k} + \mathbb{k}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ . Therefore,  $A$  may be simple while  $F^0 A$  is not.

This does not happen when  $F^0 A$  semisimple. In this case there exist non-zero idempotents  $a, b \in F^0 A$  such that  $a(F^0 A)b = 0$ . If  $aAb = 0$ , then  $AaA$  is a proper  $X^{cop}$ -stable ideal of  $A$ , and  $A$  is not  $X^{cop}$ -simple. However, if there exists  $c \in A$  such that  $acb \neq 0$ , then by applying a suitable element of  $X$ , we may assume that  $acb \in F^0 A$ . But as  $acb = a(acb)b = 0$ , we obtain a contradiction.

The above example suggests that the case of an inner derivation is different from the general one. This is also true on the pseudoalgebra level, where the action of an inner derivation can be changed to a trivial one by changing the pseudoidentity (the conformal version is contained in Lemma 4.11). This leads us to conjecture that whenever the action of  $X^{cop}$  is external in some sense,  $A$  is  $X^{cop}$ -simple if and only if  $F^0 A$  is simple. When  $X$  is cocommutative, “external” should be understood in the sense of [Kh]:  $X \not\rightarrow \text{Der } A_f$ , where  $A_f$  is the Martindale quotient of  $A$ .

**4.3.3. Simple small  $X^{cop}$ -algebras.** For the rest of this subsection,  $A$  will always stand for a simple small  $X^{cop}$ -algebra with  $F^0 A$  simple (although simplicity is not always necessary for the statements below to hold).

Assume now that  $A$  satisfies the following technical condition:

CONDITION 4.34. As an  $F^0 A$ -module,  $F^1 A$  is generated by 1 and elements  $b_i$ ,  $1 \leq i \leq r$ , where  $r \leq n$ , such that  $t_j(b_i) = \delta_{ij}$ .

REMARK 4.35. If  $F^1 A$  is generated over  $F^0 A$  by elements  $b_i$ ,  $i \in \mathcal{J}$ , where  $\mathcal{J} \subset \{1, \dots, n\}$  such that  $t_j(b_i) = \delta_{ij}$ , we can always renumerate  $t_i$ 's, so that Condition 4.34 holds.

Moreover, we will show below (Lemma 4.41) that in such a case  $A$  should be an algebra over  $\mathbb{k}[[t_1, \dots, t_r]]^{cop}$ , i.e. that  $\mathbb{k}[[t_1, \dots, t_r]] \subset X$  is closed under comultiplication.

We will show in Theorem 4.45 that simplicity of  $A$  implies the above condition (if, of course,  $A \neq F^0 A$ ). The proof is simple but lengthy, hence we delay it and turn to demonstrating the consequences of Condition 4.34. If it holds the structure of  $A$  is remarkably nice. Namely,

LEMMA 4.36. *If Condition 4.34 holds,  $A$  is generated by  $F^1 A$ . In particular,  $A$  is finitely generated.*

LEMMA 4.37. *If Condition 4.34 holds,  $\text{Span}(F^0 A, b_i)$  is a Lie subalgebra  $\mathfrak{b}$  of  $A^{(-)}$ .*

LEMMA 4.38. *If Condition 4.34 holds,  $[\mathfrak{b}, F^0 A] \subset F^0 A$ .*

The proofs of the last two statements are immediate and the proof of the first comes down to solving a system of linear differential equations.

PROOF OF LEMMA 4.38. For any  $a \in F^0 A$  and any  $j$ ,  $t_j([b_i, a]) = [t_j(b_i), a] = 0$ . □

We can go even further and provide a complete description of  $\mathfrak{b}$ . By Lemma 4.38,  $\text{ad } b_i$  is a derivative of  $F^0 A$  which is a finite-dimensional simple algebra. Hence, it is inner, i.e.  $\text{ad } b_i = \text{ad } c_i$

[Ja]. We can replace  $b_i$  with  $b_i - c_i$ , then the span of  $b_i$ 's will act trivially on  $F^0 A$ . It follows that for any  $b_i, b_j$ , we have  $[b_i, b_j] \in \text{Span}_{\mathbb{k}}(b_i) + Z(F^0 A)$ . More explicitly,

LEMMA 4.39. *If Condition 4.34 holds, we can choose  $b_i$ 's, so that for any  $1 \leq i, j \leq r$ ,*

$$[b_i, b_j] = \sum_{k=1}^r c_{ij}^k b_k + a_{ij}, \quad \text{where } c_{ij}^k, a_{ij} \in \mathbb{k}, \quad (4.6)$$

and  $[b_i, F^0 A] = 0$  for all  $i$ .

We can now turn to Lemmas 4.37 and 4.36.

PROOF OF LEMMA 4.37. For any  $a \in F^1 A$  and any  $j$ , by (1.13)  $t_j([a, b_i]) = [t_j(a), b_i] + [a, t_j(b_i)]$  mod  $F^0 A$ . Hence, by Lemma 4.38,  $t_j([a, b_i]) \in F^0 A$  for all  $j$ .  $\square$

PROOF OF LEMMA 4.36. Let  $B$  be the subalgebra of  $A$  generated by  $\mathfrak{b}$  as defined in the statement of Lemma 4.37.

We will prove the following three statements simultaneously:

- (i) For all  $m$ ,  $F^m A \subset B$ ;
- (ii) For all  $m$  and any  $k > r$ ,  $t_k F^m A \subset F^{m-2} A$ ;
- (iii) For any collection  $\{c_i\}_{i=1}^r$  of elements from  $F^{m-1} A$  such that  $t_i(c_j) = t_j(c_i)$  for all  $i, j$ , there exists  $c \in F^m A$  such that  $t_i(c) = c_i$ .

Clearly, (i) will imply the statement of the Lemma.

Remark first that by Condition 4.34, for  $m = 1$  (i) and (ii) hold automatically. As for (iii), for  $m = 1$  we let  $c = \sum_i c_i b_i$ .

We first demonstrate (ii): let  $a \in F^m A$ . For any  $j$ ,  $t_j t_k(a) = t_k t_j(a) \in t_k F^{m-1} A \subset F^{m-3} A$  and we are done.

Now assume by induction that (iii) holds for  $m - 1$ . Let  $\{c_i\}$  be a collection of elements from  $F^m A$ . Here and below we will always take  $i \leq r$ . By additivity of action of  $X$  we may assume  $\deg c_i = m$  for all  $i$ . When  $X$  is cocommutative,  $t_i$ 's act simply as derivatives, thus (iii) comes down to solving a system of linear differential equations. Moreover, by induction, first we can pass to gr  $A$ , i.e., solve the system modulo  $F^{m-1} A$ . By Lemma 4.39 it is equivalent to assuming that  $b_i$ 's commute with each other and elements of  $F^0 A$ . Then the result is classical.

By (1.13), if we consider the action of  $t_i$ 's modulo  $F^{m-1} A$ , there is no difference between the general and the cocommutative case (i.e. the solution for  $c$  obtained above is valid modulo  $F^{m-1} A$ ). Hence, there exists an element  $c' \in F^{m+1} A$  such that  $t_i(c') = c_i + d_i$  where  $d_i \in F^{m-1} A$ . Let  $d$  be an element from  $F^m A$  such that  $t_i(d_i) = d$ . Then for  $c = c' - d$ ,  $t_i(c) = c_i$ .

We turn to (i). Assume by induction that  $F^m \subset B$ . For an arbitrary  $a$  of degree  $m + 1$ , put  $t_i(a) = a_i \in F^m A$ . The collection  $\{a_i\}$  satisfies the conditions of (iii); therefore, we can produce an element  $c \in F^{m+1} A$  such that  $t_i(c) = a_i$ . By the explicit construction of  $c$  above,  $c \in B$  as  $b_i \in B$  and  $a_i, d \in F^m A$ . Thus, (ii) implies that  $t_j(a - c) \in F^{m-1} A$  for all  $j$  and  $a - c \in F^m A$ . Therefore,  $a \in B$ .  $\square$

COROLLARY 4.40. *If for  $a \in A$  and all  $i \leq r$ ,  $t_i(a) = 0$ , then  $a \in F^0 A$ .*

PROOF. By Lemma 4.36,  $a$  is the sum of monomials of the type  $cb_{i_1} \dots b_{i_m}$ ,  $c \in F^0 A$ . Denote  $\deg a$  by  $m$ . Pick a monomial of degree  $m$ , say, it ends with  $b_j$ .

We may rewrite (1.13) as  $\Delta(t_j) = 1 \otimes t_j + t_j \otimes 1 + \sum_k t_k \otimes y_{jk} +$  summands that have first terms of degree greater than 1, where  $y_{jk} \in X$  has no constant terms. Then it is clear that monomials of the highest degree in  $t_j(a)$  come from monomials in  $a$  of degree  $m$ . Hence,  $\deg t_i(a) = m - 1$ .

It follows that if  $a$  satisfies the statement of the corollary,  $a \in F^1 A$ . Condition 4.34 forces  $a \in F^0 A$ .  $\square$

We conclude that  $A$  is generated by a simple associative algebra  $F^0 A$  and a Lie subalgebra of  $A^{(-)}$  that acts trivially on  $F^0 A$ . Our goal now is to describe this subalgebra, i.e., to explain the structural constants  $c_{ij}^k$  in (4.6).

Let  $i < j \leq r$ . Clearly,  $c_{ij}^k = t_k(b_i b_j - b_j b_i)$  for  $k \leq r$ . For  $k > r$ , we put  $c_{ij}^k = 0$ . Consider now the structural constants of  $\mathfrak{g}$ :  $[\partial_j, \partial_i] = \sum d_{ji}^k \partial_k$ . By definition,  $d_{ji}^k = \langle t_k, [\partial_j, \partial_i] \rangle$ .

Since  $\partial_i \partial_j$  is an element of the standard basis of  $H$  (1.12),  $\langle t_k, \partial_i \partial_j \rangle = 0$ , and we have  $d_{ji}^k = \langle t_k, \partial_j \partial_i \rangle = \langle t_{k(1)}, \partial_j \rangle \langle t_{k(2)}, \partial_i \rangle$ . Therefore, the only summand of  $\Delta(t_k)$  proportional to  $t_j \otimes t_i$  is  $d_{ji}^k t_j \otimes t_i$  (and if  $d_{ji}^k = 0$ , there is no such summand). Remark also that  $\Delta(t_k)$  has no summand proportional to  $t_i \otimes t_j$ , otherwise  $\langle t_k, \partial_i \partial_j \rangle \neq 0$ .

Thus, comparing expressions for  $c_{ij}^k$  and  $d_{ji}^k$ , we see that  $c_{ij}^k = d_{ji}^k$ .

LEMMA 4.41. *If Condition 4.34 holds,  $\text{Span}(\partial_1, \dots, \partial_r)$  is a Lie subalgebra of  $\mathfrak{g}$ .*

In the same way as above it is not difficult to show that for  $k > r$  in (1.13), whenever both  $t^{K_j}, t^{L_j} \in \mathbb{k}[[t_1, \dots, t_r]]$ ,  $c_j = 0$ . Thus, by induction, using the formula for  $\Delta(t_k)$  stated in the proof of Corollary 4.40, we have

LEMMA 4.42. *For  $k > r$ ,  $t_k$  acts as 0 on  $A$ .*

We can also strengthen the statement of Lemma 4.26(ii):

COROLLARY 4.43. *If  $\deg a = m$ , there exists a unique  $M$  with  $|M| = m$  such that  $t^M(a) \neq 0$ .*

Now denote the subalgebra  $\text{Span}(\partial_1, \dots, \partial_r)$  by  $\mathfrak{h}$ . We can pass to  $H^{op}$  and consider its Lie subalgebra also spanned by  $\partial_1, \dots, \partial_r$ ; denote it by  $\mathfrak{h}^{op}$ . Taking into account that  $a_{ij}$ 's in (4.6) need not be 0, we obtain

LEMMA 4.44. *If Condition 4.34 holds,  $b_i$ 's generate a Lie subalgebra of  $A$  isomorphic either to  $\mathfrak{h}^{op}$ ,  $\mathfrak{h} \subset \mathfrak{g}$ , or its non-trivial 1-dimensional abelian extension.*

THEOREM 4.45. *Let  $A$  be an  $X^{cop}$ -simple small  $X^{cop}$ -algebra such that  $F^0 A$  is simple. Then  $A$  is isomorphic to either of*

( $\star$ )  $\text{End}_k(\mathbb{k})$  with a trivial  $X^{cop}$ -action;

( $\star\star$ )  $\text{End}_k(\mathbb{k}) \otimes U(\mathfrak{h}^{op})$ , where  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{g}$ , and the action of  $X^{cop}$  is determined by the action of  $U(\mathfrak{h})^*$ ;

( $\star\star\star$ )  $\text{End}_k(\mathbb{k}) \otimes (U(\widehat{\mathfrak{h}^{op}})/(1 - c))$ , where  $\widehat{\mathfrak{h}^{op}}$  is a 1-dimensional abelian extension  $1 \rightarrow \mathbb{k}c \rightarrow \widehat{\mathfrak{h}^{op}} \rightarrow \mathfrak{h}^{op} \rightarrow 1$  of  $\mathfrak{h}^{op}$  for a Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , and the action of  $X^{cop}$  is determined by the action of  $U(\mathfrak{h})^*$ .

Moreover, in the last two cases  $A$  is a simple  $(U(\mathfrak{h})^*)^{cop}$ -algebra.

PROOF.  $F^0 A = \text{End}_k(\mathbb{k})$  for some  $k$ . If  $A \neq F^0 A$ , assume that Condition 4.34 holds. Then Lemma 4.44 implies that  $F^1 A$  is isomorphic to either  $\text{End}_k(\mathbb{k}) \otimes \mathfrak{h}$  or  $(\text{End}_k(\mathbb{k}) \otimes \widehat{\mathfrak{h}})/(1 - c)$ , and by Lemma 4.36,  $F^1 A$  generates all of  $A$ .

Therefore, there exists a natural surjective map  $\phi$  of either the algebra of type ( $\star\star$ ) or ( $\star\star\star$ ) onto  $A$ , which is an isomorphism on the first filtration component. Notice that these algebras are small and satisfy Condition 4.34. Notice also that  $\phi$  must commute with the action of  $X^{cop}$ , in particular, it preserves the filtration (4.3). To prove injectivity, let  $a$  be the element of least degree such that  $\phi(a) = 0$ . By statement (iii) of the proof of Lemma 4.36, there exists  $c \in A$  such that  $t_i(c) = t_i\phi(a)$ ,  $i \leq r$ . Let  $c'$  be a preimage of  $c$  under  $\phi$ . Then by Corollary 4.40,  $c' - a$  lie in the zero component and  $\phi(a) \in \phi(c') + F^0 A$ , a contradiction.

The last claim of the Theorem follows from Lemma 4.42.

It remains to show that Condition 4.34 is valid for simple small  $X^{cop}$ -algebras such that  $F^1 A \neq \emptyset$  and  $F^0 A$  is simple.

Remark that we can always change the basis of  $\mathfrak{g}$ , hence, the generating set of  $X$ . Thus we will abandon the notation  $t_1, \dots, t_n$  that stands for a fixed generating set and will work with elements of  $X$  of degree 1. Let  $T_0 = \{t \in X \mid \deg t = 1, t(F^1 A) = 0\}$ .

Clearly, by definition of the filtration on  $A$ , for any  $t \notin T_0$  there exists  $b \in F^1 A$ ,  $t(b) \neq 0$ . Moreover, if  $t \notin T_0$ , there exists  $b_t$  such that  $t(b_t) = 1$ . Indeed, let  $b \in F^1 A$  be such that  $t(b) \neq 0$ . Then since  $F^0 A$  is simple, there exist  $s_{1k}, s_{2k}$  such that  $t(\sum_k s_{1k} b s_{2k}) = \sum_k s_{1k} t(b) s_{2k} = 1$ .

In some sense  $b_t$ , as defined above, is unique. Put  $N(t) = \{b \mid t(b) = 0\}$ . Then for any  $b$ ,  $b - t(b)b_t \in N(t)$  and  $\text{rk } F^1 A/N(t) = 1$ .

Pick an arbitrary element  $t_1 \in X$  of degree 1,  $t_1 \notin T_0$ . Let  $b'_1$  be such that  $t_1(b'_1) \neq 0$ , and, inductively,  $b'_j$  an element from  $\bigcap_{i=1}^{j-1} N(t_i)$  such that there exists  $t_j$  for which  $t_j(b'_j) \neq 0$ . In this way we obtain sequences  $b'_1, \dots, b'_m$  and  $t_1, \dots, t_m$ . The process terminates when  $\bigcap_{i=1}^m N(t_i) = F^0 A$ , i.e. when it is annihilated by all  $t$ 's. We may assume that  $t_i(b'_i) = 1$ . Now let  $b_m = b'_m$  and, inductively,  $b_i = b'_i - \sum_{j>i} t_j(b'_i)b_j$ . In this way we obtain  $b_i$ 's such that  $t_j(b_i) = \delta_{ij}$  for  $i, j \leq m$ .

For an arbitrary  $b \in F^1 A \setminus F^0 A$ , consider the difference  $b - \sum_{i=1}^m t_i(b)b_i$ . As it is killed by all  $t_i$ 's, by construction it lies in  $F^0 A$ , hence  $1, b_1, \dots, b_m$  is the basis of  $F^1 A$  over  $F^0 A$ . Notice that this is also true when we consider  $F^1 A$  as a right  $F^0 A$ -module.

For any  $t \notin T_0$  consider the operator  $\sum_{i=1}^m t(b_i)t_i$  on  $F^1 A$ . Using the left and right bases constructed above, it is easy to see that it acts exactly like  $t$ .

As in the proof of Lemma 4.36, one can show that  $T_0 F^m A \subset F^{m-2} A$ . Hence, on  $F^2 A$ ,  $t_j t = t_j(\sum_i t(b_i)t_i)$  for any  $j$ . We can calculate  $t_j(\sum_i t(b_i)t_i)(cb_j^2)$ , where  $c \in F^0 A$ , in two ways: either by, applying  $t_i$ 's first and then multiplying the results by coefficients  $t(b_i)$ , or directly by applying  $\Delta(t)$ . It follows from the discussion immediately preceding Lemma 4.41 that  $\Delta(t)(cb_j^2) = t(cb_j)b_j + cb_j t(b_j)$ . By comparing the results we see that  $t(b_j)$  must commute with  $c$ . Therefore,  $t \in \text{Span}(t_1, \dots, t_m)$  and we are done.  $\square$

REMARK 4.46. I am unaware of another algebraic classification where objects in a given class are parametrized by  $\bigcup_{\mathfrak{h} \subset \mathfrak{g}} H^2(\mathfrak{h}, \mathbb{k})$  for a finite-dimensional Lie algebra  $\mathfrak{g}$ . For an arbitrary finite group  $G$ , elements of  $\bigcup_{H \leq G} H^2(H, \mathbb{k})$  correspond to indecomposable modular categories over  $\text{Rep } G$  [Os] but this result can not be carried over to the case of  $\text{Rep } \mathfrak{g}$ .

Clearly, if a simple pseudoalgebra  $\text{Diff } A$  is finite as an  $H$ -module,  $A$  is finite, hence it must be of type  $(\star)$ . Since  $X^{\text{cop}}$  acts trivially on  $A$ ,  $\text{Diff } A$  is necessarily a current algebra.

COROLLARY 4.47. *Let  $R$  be a simple differential  $H$ -pseudoalgebra that is finite as an  $H$ -module. Then  $R = \text{Cur End}_n(\mathbb{k})$ ,  $n > 0$ .*

Now we will describe  $H$ -pseudoalgebras  $\text{Diff } A$  when  $A$  is either of the type  $(\star\star)$  or  $(\star\star\star)$  as defined in Theorem 4.45. So, let  $\mathfrak{h}$  be a Lie subalgebra of  $\mathfrak{g}$ , and  $H' = U(\mathfrak{h})$ . We can consider

$H'$ -pseudoalgebra  $R' = \text{Diff}_{H'} A$ . Recall that in Example 1.38 we introduced a notation for such pseudoalgebras:  $\text{Cend}_n^\phi$  (see also Remark 1.39). A simple comparison of (1.26) and (1.30) shows that  $\text{Diff}_H A = \text{Cur}_{H'}^H R'$ . Therefore, we conclude:

**COROLLARY 4.48.** *Let  $R = \text{Diff} A$  be a simple differential  $H$ -pseudoalgebra such that  $A$  is small and  $F^0 A$  is simple. Then either  $R = \text{Cur} \text{End}_n(\mathbb{k})$  or  $R = \text{Cur}_{U(\mathfrak{h})}^H \text{Cend}_n^\phi$  for a Lie subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ,  $\phi \in H^2(\mathfrak{h})$ , and  $n > 0$ .*

**4.3.4. Proof of Theorem 4.2.** Let  $\text{Diff} A$  be a simple associative pseudoalgebra. In the previous subsection we classified its underlying algebra  $A$  satisfying certain conditions. Our goal here is to “translate” this condition into one for pseudoalgebras. We need to study the properties of current subalgebras of  $\text{Diff} A$ .

**LEMMA 4.49.** *Let  $\text{Diff} A$  be a unital associative pseudoalgebra with the pseudoidentity  $e$ . Then among its current unital subalgebras whose pseudoidentity is  $e$ ,  $\text{Cur} F^0 A$  is maximal.*

**PROOF.** Let  $R$  be a unital subalgebra of  $\text{Diff} A$  with pseudoidentity  $e$ . Pick  $a = \sum_I \partial^I \tilde{a}_I \in R$ . For  $J$  maximal such that  $a_J \neq 0$ ,  $a_{t^J} e \in R$ . By induction, all  $\tilde{a}_I \in R$ .

Let  $R$  be a current unital subalgebra of  $\text{Diff} A$  with pseudoidentity  $e$  (i.e.  $R = \text{Cur} B$  and  $e = \tilde{1}$  with regard to the canonical  $H$ -basis of  $\text{Cur} B$ ). Let  $a = \sum_I \partial^I \tilde{a}_I$  be an element of the canonical basis of  $\text{Cur} B$ . For a non-zero  $J$ , maximal such that  $a_J \neq 0$ ,  $a_{t^J} e = 0$ , hence  $a_J = 0$ , and  $a = a_0$ . Since for  $I > 0$ ,  $e_{t^I} a = 0$ , we conclude that  $a_0 \in F^0 A$  and  $a \in \text{Cur} F^0 A$ .  $\square$

Therefore, if  $\text{Diff} A$  is simple and its maximal unital current subalgebra with the same pseudoidentity is simple and finite,  $A$  is small and  $X^{\text{cop}}$ -simple, and  $F^0 A$  is simple.

By definition, elements  $\tilde{a}$  when  $a \in F^0 A$  form a (unital) current subalgebra of  $\text{Diff} A$ . Hence, by Corollary 4.47 and Corollary 4.48, the pseudoalgebras satisfying the conditions of Theorem 4.2 are precisely the ones listed there.

It remains to show that pseudoalgebras from that list satisfy the conditions of the Theorem. Simplicity follows from Lemma 4.3. The maximal unital current subalgebra is finite simple by Lemma 4.49. This completes the proof of Theorem 4.2.

**4.3.5. Small pseudoalgebras.** Finiteness of filtration components of  $A$  should be somehow translated into a finiteness condition for  $\text{Diff} A$ . With this in mind, we propose the following

**DEFINITION 4.50.** A unital differential associative pseudoalgebra is called *small* if all its unital current subalgebras are finite as  $H$ -modules.

CONJECTURE 4.51. *A unital pseudoalgebra  $\text{Diff } A$  is small if and only if  $A$  is a small  $X^{\text{cop}}$ -simple algebra.*

Together with the conjectural statement in subsection 4.3.2, this will imply a stronger version of Theorem 4.2:

COROLLARY 4.52. *A simple unital pseudoalgebra such that all its unital current subalgebras are finite over  $H$  is either of the pseudoalgebras from Theorem 4.2.*

The proof of this conjecture will require understanding what unital current subalgebras a differential pseudoalgebra may contain and, more generally, the description of the structure of unital subalgebras of a differential pseudoalgebra. The latter is hard to ascertain, as there are some counterintuitive examples: for instance, as we noticed in the previous section, a current conformal algebra may contain a non-current unital subalgebra (Example 4.23).

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