A Widely Channel Properties:

- Channel is of infinite bandwidth (no distortion on the signal waveform)
- Channel is corrupted with noise signal $n(t) \sim N(0, \sigma^2_n)$

Geometric representation of signal waveforms:

Consider a set of continuous functions on interval $[a, b]$. This set of functions along with addition and multiplication by scalar form a vector space.

The inner product between two functions is defined as: $\langle f, g \rangle = \int_a^b f(t)g(t) \, dt$. 
\[
\langle S_i(t), S_j(t) \rangle \equiv \int_a^b S_i(t) \cdot S_j(t) \, dt
\]

\[x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \langle x, y \rangle = x^T y = \sum_{i=1}^n x_i \cdot y_i \]

It is easy to see that the following properties hold:

1) \( \langle S_i(t), S_j(t) \rangle = \langle S_j(t), S_i(t) \rangle \quad +i,j \)

2) Let \( a \in \mathbb{R} \) then

\[a \cdot \langle S_i(t), S_j(t) \rangle = \langle a \cdot S_i(t), S_j(t) \rangle = \langle S_i(t), a \cdot S_j(t) \rangle \]

3) \( \langle S_i(t), S_2(t) + S_3(t) \rangle = \langle S_i(t), S_2(t) \rangle + \langle S_i(t), S_3(t) \rangle \)

Proof: \[\langle S_i(t), S_2(t) + S_3(t) \rangle = \int_a^b (S_i(t) , (S_2(t) + S_3(t)) \, dt = \int_a^b S_i(t) \cdot S_2(t) \, dt + \int_a^b S_i(t) \cdot S_3(t) \, dt \]

Distribute with respect to addition

4) A norm of a signal is defined as

\[\| S_i(t) \| = \left( \langle S_i(t), S_i(t) \rangle \right)^{1/2} = \left( \int_a^b S_i(t) \cdot S_i(t) \, dt \right)^{1/2} \]

\[= \sqrt{E_i} \quad E_i - \text{energy of signal} S_i(t) \]

As \( i \): Vector \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \), \( \| x \|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = (x^T x)^{1/2} \)

5) Orthonormal
\( S_1(t) \text{ & } S_2(t) \) they are orthogonal if
\[
\langle S_1(t), S_2(t) \rangle = \int_a^b S_1(t) S_2(t) dt = 0
\]

* In general, the number of signals that exist in \([a,b]\) is infinite.

* However, in a digital communication system we will use a set of \(M\) waveforms where \(M\) is the size of the alphabet \(\sum_{m=1}^{M} S_m(t)\).

* \(N\)-selected waveforms may not be mutually orthogonal.

* We can select \(N \leq M\) waveforms that can be used as a basis for a given signal space.

**Gram-Schmidt Orthogonalization Procedure**

Consider a set of \(M\) waveforms \(\{S_m(t)\}_{m=1}^{M}\).

The task of Gram-Schmidt is to determine an orthogonal basis for the set of signals \(\{S_m(t)\}_{m=1}^{M}\).

1) The first basis waveform can be found in the following way:
\[
\psi_1(t) = S_1(t) \cdot \frac{1}{\sqrt{E_1}}, \quad E_1 = \int_a^b S_1^2(t) dt \quad \text{(energy of } S_1(t)\text{)}
\]

2) The second waveform is given as:
\[
\left( \frac{d_2(t)}{d_2(t)} = S_2(t) - \left( \langle S_2(t), \Psi_1(t) \rangle \right) \cdot \psi_1(t) \right)
\]

\[\psi_2(t) = \frac{d_2(t)}{\sqrt{\epsilon_2}}, \quad \epsilon_2 = \int d_2(t) dt - \text{energy of } d_2(t)\]

\[\chi_1(t) = \frac{\psi_1(t)}{\sqrt{\epsilon_1}}, \quad \epsilon_1 = \int \psi_1(t) dt - \text{energy of } \psi_1(t)\]

\[d_1(t) = S_1(t) - \sum_{i=1}^{\chi^{-1}} \langle S_1(t), \psi_i(t) \rangle \cdot \psi_i(t)\]

The orthonormalization procedure continues until all M waveforms are exhausted.

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**Example:** Apply GS to the following set of signals:

\[I = [0, 3]\]

\[\epsilon_1 = \int_0^2 dt = 2 \quad ; \quad \psi_1(t) = \frac{1}{\sqrt{2}} \cdot S_1(t)\]

\[\epsilon_2 = \int_1^2 dt = 2 \quad ; \quad \psi_2(t) = \left( S_2(t) - \frac{1}{2} \cdot \psi_1(t) \right) \cdot S_2(t)\]

\[d_1(t) = S_2(t) - \langle S_2(t), \psi_1(t) \rangle \cdot \psi_1(t)\]
\[
\langle S_2(t), \psi_1(t) \rangle = \int_0^2 S_2(t) \cdot \psi_1(t) \, dt = \int_0^2 \left( \frac{1}{\sqrt{2}} \right) \, dt + \int_1^2 \left( -\frac{1}{\sqrt{2}} \right) \, dt = 0 \quad (\text{since} \ S_2(t) \text{ and } \psi_1(t) \text{ are orthogonal})
\]

\[d_2(t) = S_2(t)\]

\[E_2 = \int_0^2 d_2(t) \, dt = \int_0^2 1 \, dt = 2 \quad \Rightarrow \quad \psi_2(t) = \frac{S_2(t)}{\sqrt{2}} = \frac{n}{\sqrt{2}} \cdot S_2(t)\]

\[k = 3 \quad \Rightarrow \quad d_3(t) = S_3(t) = \langle S_3(t), \psi_1(t) \rangle \cdot \psi_1(t) - \langle S_3(t), \psi_2(t) \rangle \psi_2(t)\]

\[\langle S_3(t), \psi_1(t) \rangle = 0\]

\[\langle S_3(t), \psi_2(t) \rangle = -2 \cdot \frac{1}{\sqrt{2}} = -\sqrt{2}\]

\[= S_3(t) + \sqrt{2} \cdot \psi_2(t)\]

\[E_3 = \int_0^3 d_3(t) \, dt = \int_0^3 \left( S_3(t) + \sqrt{2} \cdot \psi_2(t) \right) \, dt = 4\]

\[\psi_3(t) = S_3(t) + \sqrt{2} \cdot \psi_2(t)\]
\[
\begin{align*}
&\quad \text{(k=1)} \quad d_u(t) = S_u(t) = \sum_{k=1}^{3} \frac{N}{N-1} <S_u(t), \Psi_k(t)> \Psi_k(t) \\
&<S_u(t), \Psi_1(t)> = \int_0^3 S_u(t) \Psi_1(t) \, dt = \int_0^2 \frac{1}{\sqrt{2}} \, dt = \sqrt{2} \\
&<S_u(t), \Psi_2(t)> = 0 \\
&<S_u(t), \Psi_3(t)> = \int_2^3 \, dt = 1
\end{align*}
\]

Therefore, \[d_u(t) = \sqrt{2} \cdot \Psi_1(t) - \Psi_3(t)\]

\[S_u(t) \text{ is a linear combination of } \Psi_1(t), \Psi_2(t) \text{ and } \Psi_3(t)\]

\[S_u(t) = \sqrt{2} \cdot \Psi_1(t) + \Psi_3(t)\]

The dimension of the signal set is \(N=3\), \(N=4\)

Once you know the basis \(\Psi_1(t), \Psi_2(t), \ldots, \Psi_N(t)\) (e.g., \(N=3\)), how can we express the signals \(\{S_m(t)\}_{m=1}^{N}\)?

A: \[S_m(t) = \sum_{h=1}^{N} \sum_{n=1}^{N} S_{mn} \Psi_n(t)\]

Where \[S_{mn} = <S_m(t), \Psi_n(t)> = \int_{0}^{b} S_m(t) \Psi_n(t) \, dt\]

Example:

\[S_1(t) = S_{11}(\Psi_1(t)) + S_{12}(\Psi_2(t)) + S_{13}(\Psi_3(t))\]

\[<S_1(t), \Psi_1(t)> = \int_0^3 S_1(t) \Psi_1(t) \, dt = \int_1^2 \frac{1}{\sqrt{2}} \, dt\]
\[
S_{21} = \langle S_2(t), \Psi_1(t) \rangle = 0 \\
S_{22} = \langle S_2(t), \Psi_2(t) \rangle = \frac{2}{\sqrt{2}} \\
S_{23} = \langle S_2(t), \Psi_3(t) \rangle = \frac{1}{\sqrt{2}} \\
S_{24} = \langle S_2(t), \Psi_4(t) \rangle = 0
\]

\[
S_{31} = \langle S_3(t), \Psi_1(t) \rangle = 0 \\
S_{32} = \langle S_3(t), \Psi_2(t) \rangle = 0 \\
S_{33} = \langle S_3(t), \Psi_3(t) \rangle = 0 \\
S_{34} = \langle S_3(t), \Psi_4(t) \rangle = 0
\]

\[
S_{41} = \langle S_4(t), \Psi_1(t) \rangle = 1 \\
S_{42} = \langle S_4(t), \Psi_2(t) \rangle = 0 \\
S_{43} = \langle S_4(t), \Psi_3(t) \rangle = 0 \\
S_{44} = \langle S_4(t), \Psi_4(t) \rangle = 1
\]