Set of signals \( \{ s_1(t), s_2(t), \ldots, s_M(t) \} \)

Obtained \( \Rightarrow \) Set of orthonormal signals \( \{ \psi_1(t), \psi_2(t), \ldots, \psi_N(t) \} \) \( N \leq M \)

Use Gram-Schmidt process

Once the basis signals are known, every signal \( s_m(t) \) can be expressed as a linear combination of basis signals, i.e.: \( s_m(t) = \sum_{n=1}^{N} a_{m,n} \psi_n(t) \)

Dimensionality of the signaling scheme

\( [\psi_1(t), \psi_2(t), \ldots, \psi_N(t)] \)

Signal \( S_m(t) \) \( \Leftrightarrow \) Signal \( s_m(t) \)

Vector representation

\( S_m = [s_1 \ s_2 \ \cdots \ s_M]^T \)

* The energy of signal \( S_m(t) \), namely \( E_m \)

\[ E_m = \int_{-b}^{b} s_m^2(t) \, dt = \int_{-b}^{b} \left( \sum_{n=1}^{N} a_{m,n} \psi_n(t) \right)^2 \, dt \]

\[ = \int_{-b}^{b} \left( \sum_{n=1}^{N} \sum_{p=1}^{N} a_{m,n} a_{m,p} \psi_n(t) \psi_p(t) \right)^2 \, dt \]

Aside: \( (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \)
\[ \sum_{h=1}^{N} \frac{1}{s_{mh}^2} = \sum_{h=1}^{T} s_{mh}^2 = \| s_{m} \|_2^2 \]

* The energy of signal \( s_{m}(t) \) is equal to the Euclidean norm of the vector representation \( s_{m} \).

\[ \langle s_{m}(t), s_{n}(t) \rangle = \int_{a}^{b} s_{m}(t) \cdot s_{n}(t) dt \]

**Correlation of signals**

\[ s_{m}(t), s_{n}(t) \]

\[ \sum_{h=1}^{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} s_{mi} \cdot \psi_{i}(t) \right) \cdot \left( \sum_{k=1}^{N} s_{mj} \cdot \psi_{j}(t) \right) \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} s_{mi} \cdot s_{mj} \]

\[ = \sum_{i=1}^{N} s_{mi} \cdot s_{mi} = s_{m}^T \cdot s_{m} \]

- Inner product of two functions \( s_{m}(t) \) and \( s_{n}(t) \) is equal to the inner product of their vector representations \( s_{m}, s_{n} \).

**Pulse Amplitude Modulation (PAM)**

PAM - information is conveyed by the amplitude of transmitted signal.

Remember - Signaling is through AWGN channel with infinite BW

**Baseband Signaling**: 
* The simplest PAM is binary PAM.

- Binary PAM modulation is performed if the following:
  \( (0) \rightarrow A \cdot \psi(t) = S_1(t) \)
  \( (1) \rightarrow -A \cdot \psi(t) = S_2(t) \)

- This is XID 4 signaling is referred to as antipodal signaling.

- Binary PAM \( \rightarrow \) one symbol per bit

- Generalization of PAM: \( M \)-ary PAM

\( M \)-ary PAM: Combines several bits and assigns one of \( M \) possible amplitude levels.

**Example:** Consider the following mapping.

\( b_1 \) \( b_2 \) \( S(t) \)
\begin{align*}
0 & \quad 0 & \quad 3A \cdot \psi(t) & \quad S_3(t) \\
0 & \quad 1 & \quad -A \cdot \psi(t) & \quad S_2(t) \\
1 & \quad 0 & \quad -A \cdot \psi(t) & \quad S_3(t) \\
1 & \quad 1 & \quad -3A \cdot \psi(t) & \quad S_4(t)
\end{align*}

**Diagram:**

- 4-PAM
- \( M = 4 \)
- \( T \): Symbol duration
In general, for PAM signals:

\[ S_m(t) = (A_m \cdot g_{\tau}(t)) \quad \text{for} \quad m = 1, \ldots, M \]

\[ 0 \leq t \leq T \]

- \( A_m \) one of \( M \) possible amplitudes
- \( g_{\tau}(t) \) pulse

Vector representation of PAM signal:

- Dimension of PAM is equal to \( N = 1 \)
- Basis function in PAM: \( \Psi(t) = \frac{g_{\tau}(t)}{\sqrt{\varepsilon_g}} \) with
  \[ \varepsilon_g = \int_0^T g_{\tau}(t) dt \]

\[ S_m(t) = (\varepsilon_g A_m) \Psi(t) \quad \text{for} \quad m = 1, \ldots, M \]

- In general, to limit the BW necessary for transmission
  \( g_{\tau}(t) \) has rounded edges

Energy of the baseband PAM signal:

\[ E_m = \int_0^T S_m^2(t) dt = \int_0^T (A_m \cdot g_{\tau}(t))^2 dt \]

\[ = A_m^2 \cdot \int_0^T g_{\tau}^2(t) dt = A_m^2 \cdot \varepsilon_g \]
where $E_g$ is the energy of the pulse signal $g(t)$.

Different PAM signals will have energy

**Bandpass PAM**:

To obtain bandpass PAM signals, we multiply with a sinusoidal carrier.

Message signal is baseband PAM.

\[
\text{Baseband signal} \quad \rightarrow \quad \text{Carrier frequency} \quad \uparrow \quad \text{PAM signal}
\]

\[
\text{For PAM signal } \quad S_m(t) \cdot \cos(2\pi f_c t)
\]

**Therefore**: (Bandpass PAM)

\[
U_{m}(t) = S_m(t) \cdot \cos(2\pi f_c t) = A_m \cdot g(t) \cdot \cos(2\pi f_c t)
\]

Energy of bandpass PAM signal

\[
E_m = \int_0^T U_{m}(t) \, dt = \int_0^T A_m \cdot g(t) \cdot \cos(2\pi f_c t) \, dt
\]

\[
= \frac{A_m^2}{2} \int_0^T \left[ g(t) \right] \, dt + \frac{A_m^2}{2} \int_0^T g(t) \cos(2\pi f_c t) \, dt
\]

\[
\approx \frac{A_m^2}{2} \cdot E_g
\]
Assume that \( f_c \gg W \) (\( W \) is the highest frequency component of the spectrum of \( g_T(t) \)).

\[ S_m(t) = A_m \sqrt{E_g} \cdot \Psi(t) \]

\[ \Psi(t) = \frac{g_T(t)}{\sqrt{E_g}} \]

\[ m = 1, \ldots, M = 2^K \]

K: number of bits per symbol

\[ S_m(t) \leftarrow S_m = A_m \sqrt{E_g} \]

Band pass PAM:

\[ S_m(t) = A_m \cdot g_T(t) \cdot \cos(2\pi ft) \]

\[ \Psi(t) = \frac{g_T(t) \cdot \cos(2\pi ft)}{\sqrt{E_g}} \]

As is:

\[ \int_0^T g_T^2(t) \cos^2(2\pi ft) \, dt \]

\[ U_m(t) = A_m \sqrt{\frac{E_g}{2}} \cdot \Psi(t) \]

\[ S_m = A_m \cdot \sqrt{\frac{E_g}{2}} \]

\[ \int_0^\infty \cos^2(2\pi ft) \, dt \]

\[ \int_0^\infty (1 + \cos^2(2\pi ft)) \, dt \]

\[ = 1 \]
\[-\frac{1}{2} \int f(t) \, dt \quad \Rightarrow \quad \frac{1}{2} \int f(t) \, dt \quad \text{with} \quad f(t) \gg M\]