Review of basic concepts in Probability

1) Probability distribution function

Let $X$ denote a continuous random variable.

Cumulative Distribution Function (CDF)

$F_X(x) = P_r[X \leq x]$  

Typical shape of a CDF

Properties of a CDF:

1) $0 \leq F_X(x) \leq 1$, $\forall x$
2) $F_X(-\infty) = 0$
3) $F_X(\infty) = 1$
4) $P_r[a \leq X \leq b] = F_X(b) - F_X(a)$
5) $P_r[X > a] = 1 - F_X(a)$

Example: Consider a R.V. $X$. 

$F_X(x) = \left\{ \begin{array}{ll} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{array} \right.$

$Pr[X \leq 3] = F_X(3) = 1 - e^{-2 \cdot 3} = 0.03375$

$Pr[1 \leq X \leq 3] = F_X(3) - F_X(1)$
Probability Density Function (PDF)

\[ f_{X}(x) = \frac{d}{dx} F_{X}(x) \]

\[ f_{X}(x) \]

\[ F_{X}(x) = \Pr [ X \leq x ] = \int_{-\infty}^{x} f_{X}(x) \, dx \]

\[ \Pr [ a \leq X \leq b ] = \int_{a}^{b} f_{X}(x) \, dx \]

Some properties:

1) \( f_{X}(x) \geq 0, \quad f_{X} \)

2) \( \int_{-\infty}^{\infty} f_{X}(x) \, dx = 1 \)

3) \( F_{X}(x) = \Pr [ X \leq x ] = \int_{-\infty}^{X} f_{X}(x) \, dx \)

4) \( \Pr [ a \leq X \leq b ] = \int_{a}^{b} f_{X}(x) \, dx \)

Example:

\( F_{X}(x) = 1 - \exp(-2x), \quad x \geq 0 \)

\( f_{X}(x) = 2 \cdot \exp(-2x), \quad x \geq 0 \)

Expectation (Mean) and moments:

Consider a function \( h(x) \), where \( x \) is a random variable with a given pdf.

The output (i.e. \( h(x) \)) is a random variable as well.
Expected value of \( X \):

\[
E[h(x)] = \int_{-\infty}^{+\infty} h(x) \cdot f_X(x) \, dx
\]

Some cases of interest:

1) \( h(x) = x \), \( E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x) \, dx \) = mean value of \( X \)

2) \( h(x) = x^2 \), \( E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x) \, dx \) = second moment of \( X \)  (power)

3) \( h(x) = x^n \), \( E[X^n] = \int_{-\infty}^{+\infty} x^n \cdot f_X(x) \, dx \) = nth order moment

4) \( h(x) = (X - E[X])^2 \), \( E[(X - E[X])^2] = \int_{-\infty}^{+\infty} (x - E[X])^2 \cdot f_X(x) \, dx \)

\[ \sigma_X^2 \quad \text{Variance of} \quad \bar{X} \]

\( \sigma_X \): standard deviation of \( X \)

Example: Find mean and standard deviation of \( X \) with pdf \( f_X(x) = 2 \cdot \exp(-2 \cdot x) \), \( x \geq 0 \)

\[
E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x) \, dx = \int_{0}^{+\infty} (x \cdot 2) \cdot e^{-2x} \, dx \quad \text{(1)}
\]

\[
\sigma_X^2 = E[(X - E[X])^2] = \int_{0}^{+\infty} (x - \frac{1}{2})^2 \cdot e^{-2x} \, dx
\]

\[
= \int_{0}^{+\infty} (x^2 - x + \frac{1}{4}) \cdot e^{-2x} \, dx
\]

\[
= \frac{1}{2} \int_{0}^{+\infty} x^2 \cdot e^{-2x} \, dx - \frac{1}{4} \int_{0}^{+\infty} 2x \cdot e^{-2x} \, dx
\]

\[
= \frac{1}{2} \cdot \frac{1}{4} \cdot 4 = \frac{1}{8} \]

\[
= \frac{1}{4} \int_{0}^{+\infty} x \cdot e^{-2x} \, dx - \frac{1}{4} \cdot \frac{1}{2}
\]

\[
= \frac{1}{4} \cdot \frac{1}{4} \cdot 2 = \frac{1}{8}
\]

\[
E[X] = \frac{1}{2} \quad \sigma_X^2 = \frac{1}{8}
\]
\[
\frac{1}{2} - \frac{1}{2} + \frac{1}{y} = \frac{1}{y} \quad \Rightarrow \quad \sigma^2 = \frac{1}{2}
\]

Gaussian PDF (Normal)

\[X \sim N(\mu, \sigma^2)\]

reads \(X\) is a random variable following a normal distribution with mean value \(\mu\) and variance \(\sigma^2\).

\[N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{1}{2\sigma^2} (X-\mu)^2\right]\]

\[\text{E}(X) = \mu\]

\[\text{E}(X - \mu)^2 = \sigma^2\]

\[(\sigma_1)^2 < \sigma^2 \quad \Rightarrow \quad \frac{X}{\sigma_1} \sim N\left(\mu_1, \frac{\sigma^2}{\sigma_1^2}\right)\]

* Normal distribution can model many statistical phenomena due to CLT
It there is a mixture of random variables of arbitrary distribution but of approximately the same power ⇒ the sum tends to be distributed according to a Gaussian distribution.

Random process:

* The majority of that we encounter in communication systems are random due to noise, channel distortions
* The time domain shape of a random signal is not known in advance ⇒ design a system for unknown signals

Two different types of models:

1) Deterministic: No uncertainty in the signal behavior
2) Probabilistic: Physical phenomena that can alter the transmitted signal in many different ways ⇒ we construct probabilistic models that describe the random processes

Random signals have two properties
1) They are functions of time
2) They are random in the sense that before the outcome of the process is observed the actual shape of the process cannot be predicted.
Stationary Process:

* At any point in time each sample process takes a value.

* The value taken by a sample process is a random variable described by its PDF, \( f_{X_t}(x) \).

* In general, the PDF \( f_{X_t}(x) \) will depend on the observation time.

* In stationary processes these PDFs \( f_{X_t}(x) \) DO NOT DEPEND ON TIME.
\[ E \left[ X(t_1) \right] = \int_{-\infty}^{+\infty} x \cdot f_{X_t}(x) \, dx = E \left[ X(t_2) \right] \]

\[ E \left[ X^2(t_1) \right] = \int_{-\infty}^{+\infty} x^2 \cdot f_{X_t}(x) \, dx = E \left[ X^2(t_2) \right] \]

Two first-order moments are time-invariant \( \Rightarrow \) W.S.S. process

**Ergodic Process:** Consider a strict stationary process \( X(t) \)