Transmission of signals through linear filters

\[ y(t) = x(t) * h(t) \quad \text{signals} \]

Input and output PSD:
\[ S_y(f) = S_x(f) \cdot |H(f)|^2 \]

Frequency-domain calculation of random signal power
\[ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \]

\[ R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int x(t) \cdot x(t-\tau) dt \quad (\text{autocorrelation}) \]

\[ R_x(\tau) \big|_{\tau=0} = R_x(0) = P_x \]

\[ R_x(f) = \mathcal{F}^{-1} \{ S_x(f) \} = \int_{-\infty}^{+\infty} S_x(f) e^{j \pi f \tau} df \]

\[ P_x = R_x(0) = \int_{-\infty}^{+\infty} S_x(f) df \]

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Example: Consider a random process with PSD given as \( \mathcal{N}(\cdot) \)
$$S_n(t) = N_0 \cdot f(t)$$

→ Calculate the power of the process after it passes through a filter with frequency response:

$$|H(f)| = \frac{1}{\sqrt{1 + \frac{(f)^2}{(f_0)^2}}}$$

$$S_y(f) = S_n(f) \cdot |H(f)|^2 = N_0 \cdot H(f) \cdot H^*(f)$$

$$= N_0 \cdot \frac{1}{1 + j \cdot \frac{f}{f_0}} \cdot \frac{1}{1 - j \cdot \frac{f}{f_0}} = \frac{N_0}{1 + \left(\frac{f}{f_0}\right)^2}$$

Power

$$P_y = \int_{-\infty}^{+\infty} S_y(f) df = \int_{-\infty}^{+\infty} \frac{N_0}{1 + \left(\frac{f}{f_0}\right)^2} df / f_0$$

$$= 2 \cdot N_0 \cdot f_0 \cdot \left[ \tan^{-1} \left( \frac{f}{f_0} \right) \right]_{-\infty}^{+\infty} = \Pi \cdot f_0 \cdot N_0$$

Band Limited Process and Sampling:
**Bandlimited process:** \[ S_X(f) = 0, \quad |f| \geq W \]

- Almost all processes in nature are bandlimited.
- Typical bandwidth (BW) of some processes:
  - Voice: \( W = 4 \text{ kHz} \)
  - Music: \( W = 20 \text{ kHz} \)
  - Video: \( W = 5 \text{ MHz} \)

**Sampling Theorem:** Very important result for comm systems.

- In nature most signals are continuous.

**There are infinitely many points in the waveform.**

- Sampling theorem states that in order to accurately/completely reconstruct the signal \( x(t) \) one does not need to know all the points in the waveform, but just a small fraction of these samples at uniform intervals \( T_s \) satisfying

\[ T_s \leq \frac{1}{2W} \]

Where \( W \) is the bandwidth of the signal (i.e., the highest spectral component in the power spectrum of the signal).
Sampling Theorem: Let $x(t)$ be a random process, which is band-limited, that $S_x(f) = 0$ for $f \geq W$. A realization of the process can be reconstructed from samples taken every $T_s \leq \frac{1}{2W}$ according to the formula:

$$X(t) = \sum_{k=-\infty}^{+\infty} 2W T_s \cdot x(k T_s) \cdot \text{sinc}(2W (t-k T_s)) \quad (1)$$

For the special case $T_s = \frac{1}{2W}$ (Nyquist sampling)

$$X(t) = \sum_{k=-\infty}^{+\infty} x\left(\frac{k}{2W}\right) \cdot \text{sinc}\left(2W \left(t-\frac{k}{2W}\right)\right) \quad (2)$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

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Samples are converted in numbers $\ldots 111000000 \ldots$
Reconstruction:

Quantization:
- Sampling \implies \text{Discretization of the signal in time-domain}
- Quantization \implies \text{Discretization of the signal in amplitude domain}

Consider a sampled signal:

\[ x(t) = S_q(t) \]

\[ t = kT_s \quad (k+1)T_s \quad (k+1)T_s \]

\[ S_q(t) \]

\[ S(t) \]

\[ x_1 = 0.11010 \ldots \]
\[ x_2 = 0.11101 \ldots \]

\[ x \]

\[ \text{In general, signal can take any value along the y-axis within a given range } [S_{\min}, S_{\max}] \]

\[ \text{At the quantizer: We replace the continuous range of amplitude values is going to be replaced with a set of } N \text{ possible levels (discrete values).} \]

\[ \text{As a result, in amplitude domain } S_q(t) \text{ can have only values from a discrete set of values} \]
\[ (S_{\min}, A_1, A_2, \ldots, A_N, A_{\max}) \]

\[ \times \text{Quantization can be:} \]
1) Uniform: \( A_{j+1} - A_j = \Delta \text{ Constant} \)
2) Non-uniform: \( A_{j+1} - A_j \neq \text{ Constant} \)

1) Uniform quantization: Suitable when the signal is uniformly distributed in amplitude.
2) Non-uniform quantization: Suitable when the distribution of signal in amplitude is not uniform, i.e., the signal shows preference/bias for certain value intervals.

Quantization can be:
1) Scalar
2) Vector

1) Scalar quantization: Each output of the sampling device is mapped to the closest available level/rule.
2) Vector quantization: N samples are mapped to a vector.

Examples: Landline PCM (Pulse Code Modulation)

- Signal BW: 4 KHz (Voice)
- Sampling rate: \( f_s = 8 \) K samples/sec
- Quantization bits: 8 bits/sample \( \Rightarrow \) 256 quantization values
- Net data rate: \( R_b = 8 \text{ bits/sample} \times 8 \text{ K samples/sec} = 64 \text{ kbit/s} \)
North America (NA) — TDMA (Time Division Multiple Access)

- **Signal BW**: 4 KHz
- **Sampling rate**: $f_s = 8$ kSamples/sec
- **Quantization bits**: 13 bits/sample => 8192 quantization levels
- **Net data rate**: 8 kSamples/sec x 13 bits/sample => 104 Kbps

**Vocoder** => 7.9 Kbps/sec

**Quantizer**:

![Quantizer Diagram](image)