SP1. Recall the following definition: A subset $S \subset \mathbb{R}^n$ is open if either $S$ is empty or $S$ is a union of open balls of the form $U_\epsilon(p) = \{x \in \mathbb{R}^n : |x - p| < \epsilon\}$.

Show that $S$ is open in $\mathbb{R}^n$ if and only if for every $q \in S$, there exists $\delta > 0$ such that $U_\delta(q) \subset S$.

SP2. Recall Definition D2: $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if for every open set $V \subset \mathbb{R}^m$, the set $f^{-1}(V)$ is open in $\mathbb{R}^n$.

(a) Give examples of three different functions $f : \mathbb{R} \to \mathbb{R}$, each with a different type of discontinuity (a jump discontinuity, a removable discontinuity, and a discontinuity where $f$ becomes unbounded), and show explicitly in each case how Definition (D2) fails.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}^2$ that is not continuous, and show explicitly how it fails Definition (D2).

(c) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is not continuous, and show explicitly how it fails Definition (d2).

SP3. True or False? Prove your answer.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the image of $f$ is open.

(b) If $\mathbb{R}$ is endowed with the discrete topology, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ is continuous.

(c) If $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then the set of all $p \in \mathbb{R}$ such that $f(p)$ lies in the unit disk $x^2 + y^2 < 1$ is open in $\mathbb{R}$.

SP4. In each case, determine whether $X$ and $Y$ are homeomorphic, and prove your answer.

(a) $X$ is the open unit disk, $x^2 + y^2 < 1$, in $\mathbb{R}^2$, and $Y$ is $\mathbb{R}^2$.

(b) $X$ is the upper half plane $y > 0$ in $\mathbb{R}^2$, and $Y$ is $\mathbb{R}^2$.

(c) $X$ is $\mathbb{R}^2$ with the standard topology, and $Y$ is $\mathbb{R}^2$ with the discrete topology.

(d) $X = \mathbb{Z}$, and $Y = \mathbb{Q}$, both with the subspace topology as subsets of $\mathbb{R}$. 
SP5. Show that the following two definitions of continuity of \( f : \mathbb{R}^n \to \mathbb{R}^m \) are equivalent.

(d1) \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous at \( p \in \mathbb{R}^n \) if, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \in U_{\delta}(p) \), \( f(x) \) is in \( U_{\epsilon}(f(p)) \). \( f \) is continuous if it is continuous at every \( p \in \mathbb{R}^n \).

(d2) \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous if for every open set \( V \subset \mathbb{R}^m \), the set \( f^{-1}(V) \) is open in \( \mathbb{R}^n \).

SP6. Read the example in Appendix A, Volume I, page 459, of a topological space that is not Hausdorff. Explain this space and its open sets in your own words, with illustrative and varied examples of open sets. Then show why this space is not Hausdorff.

SP7. Let \((u, v)\) be the coordinates on \( S_2 - \{N\} \) given by stereographic projection from the north pole, and let \((\tilde{u}, \tilde{v})\) be the coordinates on \( S_2 - \{S\} \) given by stereographic projection from the south pole. Find the transition function between these coordinate charts; that is, express \( \tilde{u} \) and \( \tilde{v} \) as functions of \( u \) and \( v \). How is the transition function expressed in terms of the complex coordinates \( \alpha = u + iv \) and \( \tilde{\alpha} = \tilde{u} + i\tilde{v} \)? Express \( \tilde{\alpha} \) as a function of \( \alpha \), without referring to the real and complex parts.

SP8. Find additional coordinate chart(s) on the torus \( \mathbb{R}^2/\mathbb{Z}^2 \) so that they, together with the two charts \( U \) and \( V \) we found in class, provide an atlas for the torus.

b) Find the transition function from the chart \( U \) we found in class to one of your new charts.

SP9. Find an atlas of coordinate charts for the real projective space \( \mathbb{R}P^2 \), and determine the transition function for each pair of intersecting charts.

b) Show that \( \mathbb{R}P^n \), the set of all 1-dimensional subspaces of \( \mathbb{R}^{n+1} \), can be given the structure of an \( n \)-dimensional manifold. Display the open parametrized sets of \( \mathbb{R}P^n \) and the coordinate charts, and give the transition function between two arbitrary charts in your atlas.

SP10. This problem concerns coordinate transformations defined by linear functions and Jacobians of linear functions on \( \mathbb{R}^n \).

(a) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = x^2 - y^2 \).

(i) Find the derivative of \( f \) in the direction \((3/5, 4/5)\) at the point \((1, 1)\) in the original coordinate system \((x, y)\).

(ii) Now consider the change of coordinates \((u, v) = (2x, 2y)\) on \( \mathbb{R}^2 \). Calculate the same directional derivative as in (i) in this new coordinate system. What are the components of the direction vector in the \((u, v)\) coordinate system?

(iii) Now rotate the \((x, y)\) coordinate system into the new coordinates \((\tilde{x}, \tilde{y}) = (-y, x)\). Calculate the same directional derivative as in (i) in this new coordinate system. What are the components of the direction vector in the \((\tilde{x}, \tilde{y})\) coordinates?
(b) The two coordinate changes in (ii) and (iii) above are examples of linear changes of variables. Now consider a general linear function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that is not necessarily invertible. From linear algebra, we know that $f$ can be written in matrix form as

$$f(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Find the Jacobian of $f$ at an arbitrary point $(x, y)$. What do you notice?

(c) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear function. Again, linear algebra tells us that $f$ can be expressed in matrix form. Find the Jacobian of $f$ in terms of the matrix of the linear transformation. What conclusion can you make about Jacobians of linear functions?

SP11. Transition functions on tangent bundles

(a) Write down two intersecting coordinate charts on the tangent bundle of the projective space $\mathbb{RP}^4$, and calculate the transition function between them.

(b) Write down two intersecting coordinate charts on the tangent bundle, $T(\mathbb{R}^2/\mathbb{Z}^2)$, of the torus, and calculate the transition function between them. What interesting fact do you notice here?

(c) Write down the two intersecting coordinate charts on the tangent bundle, $T(S^2)$, of the sphere, obtained from stereographic projection. Calculate the transition function between them.

SP12. Let $M$ be a differentiable manifold with cotangent bundle $T^*M$. Write down the transition function between two arbitrary intersecting coordinate charts on $T^*M$.

SP13. Consider the vector field $\frac{\partial}{\partial x^1}$, expressed on the open subset $U = \{[x^1 : x^2 : x^3] : x^1 \neq 0\}$ of $\mathbb{RP}^2$. Extend this vector field to all of $\mathbb{RP}^2$ to create a global vector field on $\mathbb{RP}^2$. Are there any points on $\mathbb{RP}^2$ where this vector field is zero?

SP14. Find the Lie bracket of the vector fields $X = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$ and $Y = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Do these vector fields commute? Do the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute? What is their Lie bracket?

SP15. Explain how the directional derivative of the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^4$, where $f(x, y, z) = x^2 y - y^2 z$, in the direction $(-1, 2, 1)$ at the point $(3, 1, 4)$, expressed in $(x, y, z)$ coordinates, can be viewed as

(a) a tangent vector acting on a function,

(b) and as a cotangent vector acting on a vector.
SP16. Sections of the cotangent and tangent bundles
(a) Define a cotangent vector field on a manifold $M$.

(b) Consider the cotangent vector field $\omega = ydx - xdy$ and the tangent vector field $v = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Calculate $\omega(v)$.

(c) Change coordinates on $\mathbb{R}^2$ to $(s, t) = (-y, x^3)$. What is the expression of $\omega$ in the new coordinate system? How is $v$ written in the new coordinates?

(d) Calculate $\omega(v)$ in the coordinate system $(s, t)$ and check that the answer agrees with the answer in (a).

SP17. Computing push forward and pull back maps
Denote by $S^2$ the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Let $f : S^2 \to \mathbb{R}$ be the height function that yields the $z$-value of a point on the sphere. Then, in local coordinates $(x, y)$ on the upper hemisphere and coordinate $u$ on $\mathbb{R}$, $u(x, y) = \sqrt{1 - x^2 - y^2}$. Let $v = \frac{\partial}{\partial x}|_p + 3\frac{\partial}{\partial y}|_p$ at the point $p = (1/2, 1/2)$.

(a) Find the vector $f_*(v)$.

(b) Verify that in this example, $f_*(v)(\alpha) = v(\alpha \circ f)$ for any $\alpha \in C^\infty_{f(p)} \mathbb{R}$.

(c) Find the cotangent vector $f^*(du|_{f(p)})$.

(d) Verify that in this example, $f^*(du|_{f(p)})(v) = f_*(v)(du|_{f(p)})$

SP18. Composition of push forward and pull back maps
Let $M, N$, and $Q$ be manifolds, and let $f : M \to N$ and $g : N \to Q$ be smooth functions.

(a) Show that for any $p \in M$, the push forward map $(g \circ f)_*$ on $T_p M$ is equal to the composition $g_* \circ f_*$. 

(b) Show that for any $p \in M$, the pull back map $(g \circ f)^*$ on $T^*_{(g \circ f)(p)} Q$ is equal to the composition $f^* \circ g^*$.

SP19. The pull back map in local coordinates
Let $f : M \to N$ be a smooth function between manifolds $M$ and $N$ with local coordinates $(x^1, ..., x^m)$ near $p \in M$ and $(y^1, ..., y^n)$ near $f(p) \in N$. We have shown that the push forward map $f_* : T_p M \to T_{f(p)} N$ is the Jacobian $df_p$ in local coordinates. In this problem, we find the matrix that implements the pull back map in local coordinates.

(a) For each cotangent vector $dy^j|_{f(p)}$, the pull back, $f^*(dy^j|_{f(p)})$ is a cotangent vector in $T^*_p M$. Write $f^*(dy^j|_{f(p)})$ as a linear combination of the basis vectors $dx^i|_p$ in $T^*_p M$.

(b) Now let $l = \sum_{j=1}^n \alpha_j dy^j|_{f(p)}$ be an arbitrary cotangent vector in $T^*_f N$. Then $f^*(l)$ is a cotangent vector in $T^*_p M$, which has the form $f^*(l) = \sum_{i=1}^m \beta_i dx^i|_p$. Find the matrix that takes the coefficients $\alpha_j$ to the coefficients $\beta_i$. [Hint: using part (a), this takes only a line or two.]
(c) Let \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) be given by \((x, y, z) = (u^2, v^2, uv)\). Using your results above, calculate the pull back, \( f^* \Gamma \), of the cotangent vector field \( \Gamma = dx + dy + dz \) on \( \mathbb{R}^3 \).

(d) Verify that for the vector field \( V = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \) on \( \mathbb{R}^2 \), it is true that \( (f^* \Gamma)(V) = \Gamma(f_*(V)) \).

SP20. Working with inner product tensors
(a) Find a 3-tensor on \( \mathbb{R}^3 \) that gives the signed volume of the parallelepiped determined by three ordered vectors in \( \mathbb{R}^3 \). [Hint: In Calculus III, you learned a formula for calculating this volume.]

(b) Does the 2-tensor define an inner product on \( \mathbb{R}^2 \)?
   (i) \( R = e_1^1 \otimes e_1^1 + 2e_1^2 \otimes e_2^1 + 2e_2^2 \otimes e_1^1 + 5e_2^3 \otimes e_2^2 \)

   (ii) The tensor \( S \) whose matrix in the standard basis is \( \begin{pmatrix} -3 & 4 \\ 4 & 0 \end{pmatrix} \).

   (iii) \( T = 2e_1^1 \otimes e_1^1 - 2e_1^2 \otimes e_2^1 + e_2^2 \otimes e_1^1 + 2e_2^3 \otimes e_2^2 \)

(c) For each tensor in part (b) that is an inner product, calculate the length of the vector \( v = \frac{3}{5} e_1 + \frac{4}{7} e_2 \). Then find the angle between the vectors \( e_1 \) and \( e_2 \). Are \( e_1 \) and \( e_2 \) orthogonal?

SP21. The volume form
Let \{\( e_1, e_2, e_3 \)\} be the standard basis of \( \mathbb{R}^3 \) with dual basis \{\( e^1, e^2, e^3 \)\} of \( \mathbb{R}^3^* \). Then the volume form with respect to the dot product is the alternating 3-tensor \( \Omega = e^1 \wedge e^2 \wedge e^3 \). Let \( B \) be the ordered basis \( B = (v_1, v_2, v_3) \) of \( \mathbb{R}^3 \), where \( v_1 = 2e_1 + e_2 \), \( v_2 = e_1 - e_3 \), and \( v_3 = 3e_1 + e_2 + 4e_3 \).

(a) Write the volume form \( \Omega \) using the elements of the dual basis \{\( v^1, v^2, v^3 \)\}.

(b) Find the signed volume of the ordered triple of vectors \((w_1, w_2, w_3)\), where \( w_i = e_i + v_i \) for \( i = 1, 2, 3 \) in two ways: first by using the expression of \( \Omega \) in the standard basis and then by using the expression of \( \Omega \) you found in part (a). Verify that your answers are the same. Is \((w_1, w_2, w_3)\) positively or negatively oriented?

(c) Let \( A \) be the linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) given by the matrix

\[
\begin{pmatrix} -3 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.
\]

Find the signed volume of the ordered triple of vectors \((u_1, u_2, u_3)\) where \( A(u_i) = v_i \) for \( i = 1, 2, 3 \). Is \((u_1, u_2, u_3)\) positively or negatively oriented?
SP22. Vectors and dual vectors via the inner product
(a) Let $\langle \cdot, \cdot \rangle$ be the dot product on $\mathbb{R}^3$, and let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ with dual basis $\{e^1, e^2, e^3\}$ of $\mathbb{R}^3$. Let $v$ be the vector $v = 4e_1 + 2e_2 - e_3$.

(i) Find $\alpha(v) = \langle v, \cdot \rangle$. Write it in terms of the dual basis $\{e^1, e^2, e^3\}$ as $\alpha(v) = \sum_{i=1}^3 c_i e^i$. (That is, find the constants $c_i$.)

(ii) Let $w$ be the vector $w = e_1 - e_2 + 3e_3$. Verify your calculation in part (a) by checking that $\langle v, w \rangle$ yields the same result as $\alpha(v)(w)$.

(b) Now let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^3$ given by the tensor $e^1 \otimes e^1 + 2e^1 \otimes e^2 + 2e^2 \otimes e^1 + 5e^2 \otimes e^2$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and $\{e^1, e^2, e^3\}$ is its dual basis. As before, let $v$ be the vector $v = 4e_1 + 2e_2 - e_3$.

(i) Find $\alpha(v) = \langle v, \cdot \rangle$. Write it in terms of the dual basis $\{e^1, e^2, e^3\}$ as $\alpha(v) = \sum_{i=1}^3 c_i e^i$. (That is, find the constants $c_i$.)

(ii) Let $w$ be the vector $w = e_1 - e_2 + 3e_3$. Verify your calculation in part (a) by checking that $\langle v, w \rangle$ yields the same result as $\alpha(v)(w)$.

SP23. Curvature and torsion of a helix
Let $c$ be the helix $c(u) = (R \cos(u), R \sin(u), Mu)$ with radius $R$ and slope of inclination $M$, defined for $u \geq 0$.

(a) Write the helix $c$ in terms of the arclength parameter, $s$.

(b) Find the curvature of the helix at an arbitrary point on the curve.

(c) Find the torsion of the helix at an arbitrary point on the curve.

(d) Does there exist a helix that has constant curvature 1 and constant torsion 1? If so, find its equation. If not, explain why.

SP24. Angular velocity and Frenet frames
(a) A rigid body moving along a curve in $\mathbb{R}^3$ at constant speed has an angular velocity vector $\omega$, called the Darboux vector, which satisfies $t' = \omega \times t$, $n' = \omega \times n$, and $b' = \omega \times b$, where $(t, n, b)$ is the Frenet frame along the curve. Show that $\omega = \tau t + \kappa b$.

(b) Find the angular velocity $\omega$ for a circle in the plane, the helix $h(u) = (R \cos(u), R \sin(u), Mu)$, and the twisted cubic $c(u) = (u, u^2, u^3)$. Describe any interesting features that you notice in the results of these calculations.
(a) For vectors \( A \) and \( B \) in \( \mathbb{R}^3 \), let \( P(A,B) \) denote the parallelogram determined by \( A \) and \( B \), and let \( \Pi(A) \) denote the projection of \( A \) to the \((x,y)\)-plane along the \( z \)-axis. In class we showed that the Gaussian curvature \( K(p) \) of a surface \( S \) embedded in \( \mathbb{R}^3 \) is equal to

\[
K(p) = \frac{\text{Signed Area}[P(\nu_s V, \nu_s W)]}{\text{Signed Area}[P(V, W)]},
\]

where \( V \) and \( W \) are independent vectors in \( T_{\nu}(S) \) and \( \nu \) is the Gauss map. We then made the claim that this ratio is equal to

\[
K(p) = \frac{\text{Signed Area}[P(\Pi(\nu_s V), \Pi(\nu_s W))]}{\text{Signed Area}[P(\Pi(V), \Pi(W))]},
\]

provided that the denominator is not zero.

Show that this claim is true. To do this, let \( L \) be a plane in \( \mathbb{R}^3 \) that is not perpendicular to the \((x,y)\)-plane, and let \( X, Y, V, \) and \( W \) be vectors in \( L \) such that \( X \) and \( Y \) are independent and \( V \) and \( W \) are independent. Show that

\[
\frac{\text{Signed Area}[P(X, Y)]}{\text{Signed Area}[P(V, W)]} = \frac{\text{Signed Area}[P(\Pi(X), \Pi(Y))]}{\text{Signed Area}[P(\Pi(V), \Pi(W))]} \tag{1}
\]

This can be done explicitly in terms of the components of the vectors and the volume forms on the two planes, but this method can become tedious. A less computational approach is to consider the linear transformation \( \Pi \) from \( L \) to the \((x,y)\)-plane, \( M \), and determine how it scales volume. Since \( \Pi \) is not a transformation from one vector space to itself, we cannot directly use our earlier result that \( \Pi \) scales volume by a “determinant”. To understand how \( \Pi \) scales volume, let \( V \) and \( W \) be independent vectors in \( L \), and denote the volume forms on \( L \) and \( M \) by \( \sigma_L \) and \( \sigma_M \), respectively. Show that the ratio of the signed volume of \( P(\Pi(V), \Pi(W)) \) to the signed volume of \( P(V, W) \) does not depend on the two vectors chosen. Then show how this result implies equation (1).

(b) Let \( S \) be the graph of \( g : \mathbb{R}^2 \to \mathbb{R} \) and therefore the image of \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) where \( f(s, t) = (s, t, g(s, t)) \). In class we found that

\[
K(p) = \frac{\partial(\nu^1 \circ f)}{\partial s} \frac{\partial(\nu^2 \circ f)}{\partial t} - \frac{\partial(\nu^2 \circ f)}{\partial s} \frac{\partial(\nu^1 \circ f)}{\partial t}.
\]

We also found that \( (\nu \circ f)(s, t) \) is the normalized cross product of the vectors \( (1, 0, \frac{\partial g}{\partial s}|_{f(p)}) \) and \( (0, 1, \frac{\partial g}{\partial s}|_{f(p)}) \), expressed in the standard basis of coordinate vectors in the coordinate system \((x, y, z)\).

Complete the calculation to find \( K(p) \) in terms of the function \( g \) and its partial derivatives.

(c) Use your formula in part (b) to calculate following:

(i) Find the Gaussian curvature of the parabolic cylinder \( z = x^2 \) in \( \mathbb{R}^3 \) at an arbitrary point on the surface. Explain geometrically why your answer seems to be correct.

(ii) Find the Gaussian curvature of the paraboloid \( z = x^2 + y^2 \) in \( \mathbb{R}^3 \) at an arbitrary point on the surface. Where on the surface is the curvature the greatest?
(iii) Find a surface $S$ in $\mathbb{R}^3$ and a point $p$ on $S$ where $K(p)$ is negative. Give the equation of the surface and the coordinates of $p$.

SP26. Isometries and local isometries
As always, justify and explain your answers.

(a) Are the cylinders $x^2 + y^2 = 25$ and $x^2 + y^2 = 400$ isometric? Is there a cylinder with some positive real radius that is isometric to the plane?

(b) Are the spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 = 400$ isometric? Is there a sphere of some positive real radius that is isometric to the plane?

(c) Do the Euclidean plane and the cylinder $x^2 + y^2 = 9$ have the same first fundamental form? Do they have the same second fundamental form? The same Gaussian curvature? The same mean curvature?

SP27. More on isometries and local isometries
Study Definition 3.6.9 and Section 4.1 in Baer. Work out Exercises 4.1, 4.2, 4.4, 4.5, and 4.6 on p. 151.

SP28. Christoffel symbols
Work out the algebraic calculation to derive the formula for the Christoffel symbol of the first kind, given in Formula (5) in Spivak Volume II, Chapter 4, Part D on p. 186(+2).

SP29. The Riemann Curvature Tensor
On a Riemannian manifold $(M, g)$, we obtained quantities $R^i_{jkl}$ and called them components of the Riemann Curvature Tensor.

(a) How did the $R^i_{jkl}$ arise in our work, and what significance do they have?

(b) Are the $R^i_{jkl}$ functions defined on the manifold $M$? If not, then what are they as mathematical objects?

(c) What does it mean to say that the $R^i_{jkl}$ are components of a tensor? How is this tensor defined? What significance does it hold for the manifold $M$?

(d) Refer to Spivak, Volume II, Chapter 4, Part D, proof of Proposition 5 (p. 191 in the 2nd printing). What does Spivak mean to say that to show there exists a tensor as written in the proposition, one must just compute that the components transform “correctly”? Why must they transform as stated in the proof in order to prove the proposition?