Study Project: Extending the algebra of limits

Recall the following theorem on algebra of limits:
Suppose X and Y are convergent sequences such that $\lim X = A$ *and* $\lim Y = B$*, and let* $c \in \mathbb{R}$.

Then $X+Y$, $X-Y$, XY, *and* cX *converge, and*
$\lim X+Y = A+B$, $\lim X-Y = A-B$, $\lim XY = AB$, *and* $\lim cX = cA$.
If, in addition, $y_n \neq 0$ $\forall n$ *and* $B \neq 0$, *then* X/Y *converges and* $\lim X/Y = A/B$.

1. A positive and an infinite limit: the product

Suppose $\lim(x_n) = L > 0$ and $\lim(y_n) = \infty$.

a) Does the theorem on algebra of limits guarantee that $\lim(x_n y_n) = \infty$?

b) Fill in the blanks below to show that $\lim(x_n y_n) = \infty$.

> Suppose $\lim(x_n) = L > 0$ and $\lim(y_n) = \infty$.
> Let $M > 0$.
> To show that $\lim(x_n y_n) = \infty$, we must find $K \in \mathbb{N}$ such that ____________________________.
> Let $\varepsilon = L/2$.
> Since $\lim(x_n) = L$, there exists $K_1 \in \mathbb{N}$ such that ___________________________________.
> Therefore, $x_n > L/2$ for all $n \geq K_1$.
> Since $\lim(y_n) = \infty$, there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$, _______ > __________.
> Let $K =$ ___________.
> Then for $n \geq K$ it follows that $x_n y_n > L/2 \cdot$ ______ $= M$.
> We conclude, by the definition of tending to infinity, that $\lim(x_n y_n) = \infty$.

c) Modify the proof above to show that if $\lim(x_n) = L < 0$ and $\lim(y_n) = \infty$ then $\lim(x_n y_n) = -\infty$.

d) Can the proof be adjusted to make a conclusion about $\lim(x_n y_n)$, given that $\lim(x_n) = 0$ and $\lim(y_n) = \infty$?

2. A positive and an infinite limit: the sum

Suppose $\lim(x_n) = 5$ and $\lim(y_n) = \infty$.

a) Does the theorem on algebra of limits guarantee that $\lim(x_n + y_n) = \infty$?

b) Fill in the blanks below to show that $\lim(x_n + y_n) = \infty$.

Suppose $\lim(x_n) = 5$ and $\lim(y_n) = \infty$.
Let $M > 0$.
To show that $\lim(x_n + y_n) = \infty$, we must find $K \in \mathbb{N}$ such that ________________________.
Let $\varepsilon = 1$.
Since $\lim(x_n) = 5$, there exists $K_1 \in \mathbb{N}$ such that _________________________________.
Therefore, $x_n > 4$ for all $n \geq K_1$.
Since $\lim(y_n) = \infty$, there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$, _______ > __________.
Let $K =$ ___________.
Then for $n \geq K$ it follows that $x_n + y_n > 4 +$ ______ ______ M.
Therefore, by the definition of tending to infinity, $\lim(x_n + y_n) = \infty$.

c) Suppose (x_n) converges and $\lim(y_n) = \infty$. Modify the proof above, without splitting it into separate cases, to show that $\lim(x_n + y_n) = \infty$.

3. Formalizing the observations

Using your results in Parts 1 and 2, extend the theorem on algebra of limits to draw conclusions about products and sums of sequences, where one sequence converges and the other tends to infinity. In your extension, be sure to clearly state the hypothesis and the conclusion. ■