Capstone Connection: Notation for derivatives

In differential calculus, two notational systems for calculating derivatives are used, one initiated by Leibnitz and the other by Newton. Each has useful purposes in different contexts. Here we explore the meaning of limits used to calculate derivatives using each notational system.

Suppose we have a differentiable function $y = f(x)$. From calculus, we know that the derivative of f at a point p is the limit, which may be written in two ways as

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

Isaac Newton used the notation $f'(p)$ to denote this limit and the notation $f'(x)$ to denote the derivative function, which yields the derivative of f at an arbitrary point *x.* In the notation of Gottfried Wilhelm Leibniz, the derivative function is denoted as $\frac{df}{dx}$ or $\frac{dy}{dx}$.

Newton's notation is more compact, while Leibnitz' notation is more suggestive of the process of taking a derivative. In the following contexts, we explore the meaning and practicality of each.

1. Differentiation as a linear operator

In the theory of linear operators, taking the derivative of a function $y = f(x)$ at a point p is regarded as a *linear operator* on the function. An operator Ω that acts on elements of a vector space V is said to be *linear* if it satisfies the properties $\Omega(v+w) = \Omega(v) + \Omega(w)$ and $\Omega(kv) = k\Omega(v)$ for all v, w in V and for all constants k.

a) Using the definition of the derivative and of a linear operator, prove that the operation of taking the derivative of a function at a point p is a linear operator on the set of functions that are differentiable at p (this set of functions is a vector space, but you do not need to prove this here).

b) Explain how the Leibnitz notation $\frac{d}{dx} f(x)$ may be thought of as an operator acting on a function. $\frac{d}{dx}$ is called a *differential operator*. Could Newton's notation also be used to denote a the action of a differential operator on a function? If so, illustrate how you might do this.

c) Suppose we have differentiable functions f and g. Show how to use both Leibnitz' notation and Newton's notation to express the fact that the derivative of $f+g$ is the sum of the derivative of f and the derivative of g. What do you see as the advantages of each notational system in this situation?

2. Extensions of the notation

Variations of the Leibnitz and Newton notations for the derivative are used also to denote partial derivatives of functions of several variables and higher-order derivatives.

a) Suppose we wish to state in mathematical notation that the partial derivative of the function $\sin(xy)\cos(2\pi x)-3x\cos(\pi x/2)$ with respect to y is $x\cos(xy)\cos(2\pi x)$. How would you denote this symbolically? Did you use an extension of Leibnitz' notation or Newton's notation for a derivative? Is the other notation just as convenient?

b) Suppose f and g are functions of a single variable x and we wish to state in mathematical notation that the second derivative of f divided by the third derivative of g is equal to the tenth derivative of g divided by the square of the first derivative of f. How would you denote this? Is the alternate notation just as convenient?

3. Differentials

In the field of differential geometry and in the study of differential equations, one is interested in the *differential* of a function of several variables. For example, the differential of $f(x,y)$ is written

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \qquad \text{or} \qquad df=f_x\,dx+f_y\,dy\,.$$

a) In each denotation of df above, which pieces of the notation refer to limits? Explain carefully what limit is being taken and what information the limit holds about the behavior of f.

b) What do you think the symbols dx and dy refer to in the differential df? Are these also limits? Are they associated with any limit that is being taken? Give well thought-out reasons for your answers. ■