Evaluation of Parametric Estimators of Long Memory Processes with Unknown Singularites in the Spectral Density Function

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Abstract

Estimators of long memory processes, with unknown poles in the spectral density function, such as Gegenbauer autoregressive moving average processes (GARMA), have fairly limited established asymptotic theory. We consider two estimators for the GARMA model, namely a Whittle based estimator in the frequency domain and the constrained sum of squares (CSS) estimator in the time domain. We perform extensive Monte Carlo analysis on both estimators, showing that each performs remarkably well in small samples and that the estimator of the pole converges faster than the estimators of other parameters. Further, our findings generally corroborate the established theory of Chung (1996 a,b) for the CSS estimator with one important exception. Our findings suggest that Chung's results are highly dubious when the pole occurs at the origin. This is important, as standard fractional long memory results when the spectrum has a singularity at a frequency equal to 0.

JEL classifications: C12, C13, C14

Key words: long memory; GARMA; CSS estimator; Whittle estimator

I. Introduction

Considerable attention has been given to estimation of the parameters of a long memory process, which is defined here as a process having a pole in the spectral density function, $f(\omega)$, for $\omega \in [0,\pi]$. The majority of the attention has focused on the case where the singularity is known and occurs at the origin. However, there are a plethora of applications where the pole may occur away from the origin, as in long memory seasonal processes. Further, estimation of the position of the pole itself is often of intrinsic interest, since its position can be used to infer the length of a cycle in both physical and macroeconomic data (see Bierens, 2001, for an application to US real GDP).

Obviously, estimation of long memory processes with known pole has proven to be an especially difficult problem, which is complicated by the interest in estimating the long memory cycle. Even in this context, there are numerous models that one can consider. To clarify, we have interest in a special class of models whose spectra satisfy the following,

$$f(\omega) \sim C\,|\,\omega - \nu\,|^{-2\lambda}, \text{ as } \omega \to \nu, \tag{1.1}$$

where '~' denotes asymptotic equivalence of the terms on the right and left hand side of (1.1). A simple example, which we consider extensively, is the Gegenbauer model and the associated Gegenbauer ARMA (GARMA) model, which was introduced by Gray, Zhang, and Woodward (1989). The GARMA model is defined as

$$(1 - 2\eta L + L^2)^{\lambda}\phi(L)(x_t - \mu) = \theta(L)\varepsilon_t, \tag{1.2}$$

where $\phi(L)$ and $\theta(L)$ are p and q order polynomials in L with all zeros outside the unit circle, μ is the mean of x_t, and ε_t is a martingale difference sequence with $E(\varepsilon_t^2)=\sigma^2$. Then, the spectrum of x_t is

$$f(\omega)=\frac{\sigma^2}{2\pi}\left|\frac{\theta(L)}{\phi(L)}\right|^2 2\,|\cos(\omega)-\cos(\nu)\,|^{-2\lambda}, \qquad (1.3)$$

with $\nu=\cos^{-1}(\eta)$, where (1.3) satisfies (1.1) for $\lambda>0$. When $\nu=0$, the result is an ARFIMA(p,2λ,q) process as originally studied by Granger and Joyeux (1980) and Hosking (1981). The process above is covariance stationary provided $\lambda<1/4$ when $\nu\in\{0,\pi\}$ or when $\lambda<1/2$ otherwise.

Several plausible estimators for ν and λ exist including parametric estimators in the time domain (Chung, 1996 a,b) and frequency domain (Giriatis, Hidalgo, and Robinson, 2001) and semi-parametric estimators (Hidalgo and Soulier, 2004, and Hidalgo, 2005) extending the log periodogrom estimators of Geweke and Porter-Hudak (1983) and Robinson (1995). However, a full set of asymptotic results is in relatively short supply, especially for parametric based estimators. Hosoya (1997) established consistency of $\alpha=2\lambda$, when ν is known for a Whittle based estimator. Yajima (1996) established consistency of an estimator of the spectral pole (which includes estimation of ν) based on maximization of the periodogram. For semi-parametric estimators, Hidalgo and Soulier (2004) provide an asymptotic result for α based on (1.2), demonstrating that the distribution for ν unknown is identical to that for ν known. Hidalgo (2005) extends these results and provides an estimator for ν that is asymptotically normal with rate of convergence T^{β}, with $\beta<1$, and T denoting the sample size, whose distribution depends on whether $\nu\in\{0,\pi\}$ or $\nu\in(0,\pi)$. Giriatis, Hidalgo, and Robinson (2001) provide the asymptotic distribution for λ for the parameterized Whittle estimator, and establish rate T convergence for the estimate of ν. Unfortunately, as discussed below, Giriatis et al. are unable to provide a full set of asymptotic results for their estimator of ν. Finally, Chung (1996a,b) claims to have established asymptotic results for η from (1.2), asserting that the constrained sum of squares (CSS) estimator converges at either rate T (if $|\eta|<1$) or T^2 (if $|\eta|=1$) to ratios of functionals of Brownian motion processes, while estimates for the remaining parameters achieve asymptotic normality at the standard rate of $T^{1/2}$.

In the parametric context, therefore, it would seem that a full set of distributional results are only available for Chung's estimator. Unfortunately, the results obtained by Chung have been questioned, since Chung does not establish consistency using uniform convergence arguments. Instead, Chung claims to have established consistency, since the score evaluated at the true parameters is zero, which, as pointed out by Giriatis et al. (2001) is not sufficient.

Given the paucity of results for parametric estimators of long memory processes, this paper is interested in evaluating the properties of the estimator of Giriatis et al. (2001) and the CSS estimator of Chung (1996 a,b). In particular, we wish to compare the finite sample performance of each estimator, and further wish to determine the applicability of Chung's large sample results. This paper demonstrates that both estimators perform remarkably well in small samples, both in terms of bias and root mean squared error (RMSE). In comparison, the CSS estimator tends to do slightly better in terms of mean bias for the parameter associated with the position of the spectral pole. Further, we corroborate the distributional findings of Chung when $\nu\in(0,\pi)$ showing that the CSS estimator of η appears to converge at rate T with an empirical distribution remarkably similar to his established asymptotic distribution even in small samples. However, we are unable to verify the distributional results for the case $|\eta|=1$, and in fact, our results refute his established inference at that point. This casts doubt on the use of Chung's theory in testing for standard fractional integration, as in an ARFIMA model, against a GARMA alternative. Our analysis shows that the use of Chung's distributional results under the null can lead to over-rejection of the true null $\eta=1$, thus causing one to detect cycles that are finite when they, in fact, are infinite.

The rest of the paper is organized as follows. In section 2, we present the two estimators of the GARMA process and present the theoretical results of Chung (1996 a,b) and Giriatis et al. (2001). In section 3, we present the Monte Carlo results, while section 4 concludes.

2. The Parametric Estimators of the GARMA Process

The two estimators considered here maximize functions that are in essence approximations to Gaussian log likelihood functions. Under a Gaussian assumption for ε_t, the constrained sum of squares (CSS) function of the model parameters, $\phi',\theta',\lambda,\eta,\mu,\sigma^2$, is given by:

$$L(\phi',\theta',\lambda,\eta,\mu,\sigma^2) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\varepsilon_t^{\,2}\,. \qquad (2.1)$$

Chung (1996a,b) imposes an initialization assumption, setting all pre-sample observations to zero. With this assumption, maximization of the CSS function with respect to the model parameters produces a set of estimates that is asymptotically equivalent to the maximum likelihood estimates.

Chung (1996b) asserts that he has calculated the asymptotic distribution of all of the model parameters associated with the stationary GARMA(p,q) process. Theorem 3 of Chung (1996b) establishes the distribution of δ, where $\delta=[\lambda,\phi',\theta']'$. Then the CSS estimator of δ, $\hat{\delta}$, has the following distribution:

$$\sqrt{T}(\hat{\delta}-\delta)\rightarrow N(0,I_{\delta}^{-1}), \tag{2.2}$$

where I_δ is the information matrix, where for example, $I_\lambda=2(\pi^2/3-\pi\nu+\nu^2)$. More controversial, perhaps, is Chung's proposed distribution theory for η. In particular, let W, W_1, and W_2 denote independent standard Brownian motion processes and assume $\lambda\neq 0$. Chung (1996a) proposes that the CSS estimate of η, $\hat{\eta}$, has the following distribution in his Theorem 2:

$$T(\hat{\eta}-\eta)\xrightarrow{d}\frac{\sin(\nu)}{\lambda}\frac{\int_0^1 W_1 dW_2-\int_0^1 W_2 dW_1}{\int_0^1 W^2{}_1(r)dr+\int_0^1 W_2^2(r)dr}, \text{for } |\eta|<1 \tag{2.3}$$

$$T^2(\eta\mp 1)\xrightarrow{d}\pm\frac{1}{2\lambda}\frac{\int_0^1\left[\int_0^1 W(s)ds\right]dW(r)}{\int_0^1\left[\int_0^r W(s)ds\right]^2 dr}, \text{for } \eta=\pm 1. \tag{2.4}$$

Chung (1996b) demonstrates that asymptotically the distribution of η is independent of δ. Further, Chung (1996a) provides numerical calculations for the percentiles of the terms involving the functionals of the Brownian motion processes for (2.3) and (2.4), thus enabling statistical testing. For example, particular interest might lie in testing whether

η=1, such that an ARFIMA process results. Our analysis below demonstrates that it is precisely at this discontinuity where Chung's theory breaks down.

The second estimator we consider is a Whittle type estimator. For a sample size T, let $\tilde{T} = [T/2]$, where [ ] denotes the integer part. Let Δ denote the set of all possible parameter values for δ, and let Q_T denote the set of Fourier frequencies, $\omega_j=2\pi j/T$, j=0,…, $\tilde{T}$. Based on the spectrum defined in (1.3), Giriatis et al. (2001) propose the following estimator,

$$\begin{pmatrix}\hat{\delta}\\ \hat{\nu}\end{pmatrix} = \underset{\Delta\times Q_T}{\arg\min}\, S(\delta,\nu), \text{ where } S(\delta,\nu) = \left[\frac{1}{\tilde{T}}\sum_{j=0}^{T}\frac{I(\omega_j)}{f(\omega_j)}\right], \\ \text{with } I(\omega_j) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} x_t e^{it\omega_j}\right|^2 \tag{2.5}$$

Importantly, note that the estimate of ν is obtained with respect to the discrete set Q_T, and that the true value of ν need not be in this set. Under suitable regularity conditions, Giriatis et al. (2001) establish asymptotic normality for their estimate of δ, and prove that the estimate of ν is consistent. However, a limiting distribution for $\hat{\nu}$ is not available, since the function in (2.5) is not minimized for all values in the interval $[0,\pi]$.

3. The Monte Carlo Results

The previous section introduced the estimators we will employ, along with the distributional results Chung (1996a,b) and Giriatis et al. (2001) presented. In this section, we present Monte Carlo results to assess how the estimators perform both in small and larger samples. In evaluating the distributional results of Chung, we wish to assess the applicability of the distribution theory for δ and η. We are particularly interested in evaluating the ability of Chung's theory to distinguish between a GARMA process, where η<1, and an ARFIMA process, where η=1.

To begin, we must discuss details concerning the estimation algorithms we employ. For the CSS estimator, note that the polynomial $(1\text{-}2\eta L+L^2)^{-\lambda}$ is related to the Gegenbauer polynomials, c_j, as follows (see Gray et al, 1989):

$$(1-2\eta L+L^2)^{-\lambda} = \sum_{j=0}^{\infty} c_j L^j \text{, where, } c_j = \sum_{k=0}^{[j/2]} \frac{(-1)^k \Gamma(\lambda + j - k)(2\eta)^{j-2k}}{\Gamma(d)\Gamma(k+1)\Gamma(j-2k+1)}, \qquad (3.1)$$

where $\Gamma(\ )$ is the gamma function. Note, that the disturbance sequence is (2.1) is a function of the model parameters and in turn the Gegenbauer polynomials. In particular, we have the following expression for ε_t,

$$\varepsilon_t = (1-\phi_1 L - \ldots - \phi_p L^p)(x_t - \mu) - \sum_{j=1}^{t-1} c_j \varepsilon_{t-j} - \theta_1 \sum_{j=0}^{t-2} c_j \varepsilon_{t-j-1} - \ldots - \theta_q \sum_{j=0}^{t-q-1} c_j \varepsilon_{t-j-q}\ . \qquad (3.2)$$

The residuals can be calculated recursively, where the following recursion is used:

$$c_j = 2\eta\left(\frac{\lambda-1}{j}+1\right)c_{j-1} - \left(2\frac{\lambda-1}{j}+1\right)c_{j-2}, \qquad (3.3)$$

with c_0=1, c_1=2$\eta\lambda$. To calculate the model parameters, we utilize a double gradient based procedure employing the Levenberg-Marquandt algorithm, which appears to be necessary given the different rates of convergence of the GARMA model parameters. We initialize a grid for η from -1 to 1 with a step size of 0.10, and estimate the vector δ. Conditional on the estimate of δ, we estimate the parameter η. The procedure continues as we update the value of η along the grid. Once a neighborhood for the maximum value of the CSS function is obtained, we iterate using the two step procedure until the norm of the estimated parameters between steps is sufficiently small.

From a computational perspective, an advantage of the Whittle based estimator of Giriatis et al. (2001) is its relative simplicity. For each Fourier frequency, ω_j, we minimize the function $S(\delta,\omega_j)$ in (2.5) with respect to δ, and track the value of the objective function for j=0,…, $\widetilde{T}$. The estimate of ν, $\hat{\nu}$, is the Fourier frequency associated with the minimum value of the objective function among the $\widetilde{T}$ +1 alternatives. Then, the estimate of δ is the value that minimizes the objective function with the frequency fixed at $\hat{\nu}$. An estimate of η can be obtained through the functional relationship, η=cos(ν).

Using the parametric estimators, we conducted extensive Monte Carlo experiments to determine their relative performance. We considered a total of eight different cases, where six of these are GARMA(0,0) models with the final two being GARMA(1,0) models. For the GARMA(0,0) cases, the true values of η are -1,-0.9995,-0.50,0.50,0.9995, and 1. For $|\eta|$=1, we fix λ=0.20, where λ=0.40 otherwise. For the GARMA(1,0) cases, we fixed λ=0.40 and ϕ=0.80, allowing the true values of η to be 0.50 and 0.9995. Note that the last model has short memory dynamics and is parametrically close to the non-stationary border. We thus anticipate that this model will be the most challenging for the proposed estimators. For each model, we performed 2500 simulations and considered sample sizes of 100, 300, 500, 1000, and 2000 observations. To generate a data series, x_t, we calculated the autocovarinaces of the long memory processes and obtained the Cholesky factorization of the Toeplitz matrix (see Bertelli and Caporin, 2002, for details concerning the autocovariances of long memory processes). This factorization is then multiplied by a sequence of normal random variates of the desired length. For the GARMA(1,0) cases, we then generated data through recursion. Throughout, the mean of x_t is set equal to zero.

Table 1 reports the results of the mean bias and RMSE for each model and both the time domain estimator of Chung (CSS) and the frequency based estimator of Giriatis et al (GHR). To elucidate the interpretation of the results, the estimator that yields the smallest bias/RMSE in absolute value for a given sample size has been presented in bold type. For both estimators, the absolute value of the mean bias associated with η is remarkably small, with a value that decreases rapidly with the small size. The CSS estimator tends to outperform the GHR estimator slightly in terms of the mean bias of η. This likely results from the fact that the true value of η is not typically in the discrete parameter space for the GHR estimator (except when ν is 0 or π). In contrast, when $\nu \neq 0$, the GHR estimator tends to dominate in mean bias for λ. It is curious to note, that while the mean bias in λ is manageable using the CSS estimator, for every case except for the GARMA(1,0) model with ϕ=0.80, the bias is positive, indicating that the use of this estimator may result in a slight exaggeration of the persistence of the process. In terms of RMSE, for $|\eta| \neq 1$, the CSS estimator tends to dominate for η, λ, and in the cases of the GARMA(1,0) model, for ϕ as well. It should be noted that the RMSE for both estimators of λ and ϕ are generally quite similar, and compare favorably with the computed asymptotic standard deviations of these parameters from Chung's (1996b) Theorem 3 with one exception. In particular, performance of the estimators for the GARMA(1,0) model with η=0.9995, λ=0.40, and ϕ=0.80 tends to be quite poor. For sample sizes less than 2000, the CSS and GHR procedures can result in a mean bias for λ of -0.2023 and -0.3017 respectively. A similar picture emerges for ϕ, where the mean bias of ϕ can be as large as 0.0992 for the CSS estimator, while the GHR estimator tends to underestimate ϕ with a mean bias that is typically quite large in absolute value. The results for the GARMA(1,0) cases illustrate that the mean bias in λ tends to be inversely related to the mean bias in ϕ, especially for the CSS estimator. In other words, as is well known to researchers using parametric estimators in the ARFIMA context, it can be difficult to distinguish high frequency components from low frequency components (see Nielsen and Frederisksen, 2005).

Table 2 compares the empirical distribution of the estimates of η for both estimators to the percentiles calculated by Chung (1996a) corresponding to his asymptotic distribution theory when $|\eta|<1$. Below the reported sample size, we present the percentiles of the distribution of the statistic $T(\hat{\eta}-\eta)$ calculated from Chung's (1996a) equation 19 and Table 1, along with the empirical distribution of the same quantity resulting from both the CSS and GHR estimators. We are primarily interested in the CSS estimator, noting two things regarding the GHR estimator. First, the empirical distribution of $T(\hat{\eta}-\eta)$, for the GHR estimator, largely confirms the established convergence rate of T as shown by Giriatis et al (2001). Second, we note that for the GHR estimator, the underlying parameter space is discrete. Consider for example, the empirical distribution of $T(\hat{\eta}-\eta)$ when the true values of η/λ are 0.9995/0.40 for a sample size equal to 1000. While the CSS estimator has a continuous parameter space with an infinitely large domain, the GHR estimator has a domain taking on 501 possible values, where the closest value in the domain to the true value is 0.9995066. To illustrate, consider Figure 1, which depicts the empirical cumulative distribution function for the CSS/GHR estimates of η based on this parameterization. The figure shows the discrete nature of the GHR estimator, which exhibits large mass at the value of 0.9995066. The CSS estimator smooths the distribution, with both estimators performing well.

Throughout, Table 2 shows that the CSS estimator of η has an empirical distribution that is well approximated by the asymptotic distribution provided by Chung (1996 a,b). For the case where $|\eta|$=0.9995, we see that there is a slight skewness with fatter tails, especially for the smaller samples, than implied by Chung's results. Nonetheless, the values of the empirical percentiles are quite close to the reported percentiles of Chung, especially for the 2.5% and 5.0% levels, which are important for statistical testing. Thus, the results of Table 2 provide support for the asymptotic distribution results for η established by Chung, while reinforcing that both estimators of η are consistent, converging at rate T.

We turn now to Table 3, which depicts the empirical distribution for both estimators standardized by both T and T^2 when the true value of $|\eta|$ is 1. Table 3a presents the results for η=1, with Table 3b presenting the results for the case where η=-1. Immediately we see, similar to the theoretical semi-parametric results reported by Hidalgo (2005), that the empirical distribution for the estimate of η from the GHR estimator is truncated at 1. In contrast, estimates of η outside the parameter space are plausible for the CSS estimator, and thus, similar to the asymptotic results of Chung (1996a), we see that the empirical distribution of $T^2(\hat{\eta}-1)$ can take on both positive and negative values. Regarding Chung's asymptotic results, we can not rule out boundedness of $T^2(\hat{\eta}-1)$. For example, from Table 3, the 5th percentile of $T^2(\hat{\eta}-1)$ takes on the values -382.49, -253.80, -246.83, -286.06, and -199.01 for sample sizes ranging from 100 to 2000. The results of Table 3b, with η=-1, yield similar conclusions, where the distribution of $T(\hat{\eta}+1)$ is truncated for the GHR estimator, while the CSS estimator attains a rate of convergence that is plausibly faster than T and not out of line with a rate equal to T^2.

Turning to the specific values of the empirical percentiles, Table 3 shows that the empirical distribution of $T^2(\hat{\eta}-1)$ does not match the proposed asymptotic distribution of Chung (1996a). For example, in Table 3a, the value of the empirical 1st percentile for $T^2(\hat{\eta}-1)$ can be more than 11 times larger the value established by Chung. In other words, the empirical distribution is significantly more skewed left than the results of Chung would imply. Further, the empirical distribution takes on fewer positive entries than the proposed asymptotic distribution. For example, the value associated with the 99th percentile from Chung's asymptotic distribution for $T^2(\hat{\eta}-1)$ is 31.13. Based on a sample size of 300, this implies that when the true value of η is unity, 1% of all estimates of this parameter will be at least 1.00035. In contrast, the empirical distribution suggests that 1% of all estimates of η will exceed 1.00018 when the true value is 1. While values of η in excess of 1 are mathematically possible based on equation (1.2), for positive λ, the Gegenbauer polynomials typically become explosive, and thus the disparity from the empirical and proposed theoretical distributions can be drastic. For values of η equal to 1.00035 and 1.00018 and setting λ=0.20, the 1000th values of the Gegenbauer coefficients calculated from (3.1) are $4.1391x10^8$ and $2.3003x10^5$ respectively. While it should be pointed out that only 300 values of the Gegenbauer coefficients are relevant here given Chung's truncation assumption, the results nonetheless call into question the proposed theoretical results at this point.

Given our failure to confirm the only proposed distributional results for η associated with the parametric estimators when $|\eta|=1$, an open ended question remains. How can one test the null hypothesis that $\eta=1$? This is potentially a vital research question, since one may be interested in determining the length of a cycle, which is infinite when the pole occurs at the origin and is finite otherwise. To this end, we consider the proposed tests of Chung (1996 a,b) in Table 4 for $|\eta|=1$, when the true value of η is 1 or -1. The hypothesis can be tested by constructing confidence intervals about the estimate of η. If the value of unity lies within the confidence interval, we fail to reject the null hypothesis that the process is an ARFIMA process. The left hand side of Table 4 reports the empirical size based on the 95% and 99% confidence intervals when the true of η is 1, while right hand side of Table 4 presents the same results when $\eta=-1$. Note, that the confidence intervals are constructed here using the distributional results with T^2 rate of convergence when $|\eta|=1$ (see equation 2.4). The table shows the implementation of Chung's test will result in a rejection of the true null hypothesis too frequently. Consider the case where the generated data are ARFIMA processes with $\eta=1$. The rejection rates do decrease as the sample size increases, but even with 2000 observations and a 5% test, we reject the true null 17.6% of the time. The rejection rates of the true null $\eta=1$ can be larger than 20%. Moving to the 99% confidence intervals (e.g. a 1% test), we still see that the rejection rates exceed 13%. Clearly, these results show that although the test can be used as a rough guide, its implementation will lead one to reject the true hypothesis, $\eta=1$, about 18% of the time. Throughout, the results are slightly worse when $\eta=-1$.

Although suggesting that the limiting distribution is correct, to be fair, Chung (1996a) seems to be somewhat aware of the problems associated with the distribution theory when η=1, reporting that in small samples, "the test statistic $T^2(\hat{\eta}-1)$ is so much inflated by the T^2 factor that the probability of erroneously rejecting that the true null hypothesis η=1 tends to be higher than the significance level" (page 249). In fact, Chung (1996b) estimates a GARMA(0,1) model to quarterly US WPI inflation, obtaining a value of η=0.9850. Instead of reporting the 95% confidence intervals under the null that η=1, Chung instead reports the more liberal 95% confidence intervals based on the hypothesis that $|\eta|$<1. It might seem, therefore, that a conservative approach would involve the use of both sets of confidence intervals. However, recall from above that values of η in excess of unity are plausible and thus the distribution theory for $|\eta|$<1 is not applicable given a complex value for ν (see equation 2.3). While it is quite clear that more work is needed in establishing an appropriate test, one possibility might be the implementation of a one sided test for η=1 versus the alternative η<1, where any estimated value in excess of unity is taken as an automatic failure to reject. As an example, we implemented this approach for the generated ARFIMA data with the sample sizes ranging from 100 to 2000 observations using the test statistic from (2.3). The test statistic, which assumes rate T convergence, yielded rejection rates of 6.64%, 5.28%, 4.84%, 5.44%, and 4.36% respectively.

4. Conclusions

The vast majority of interest in long memory models has centered on a class of models with a pole in the spectrum whose position is known. It is clear, however, that there is emerging interest in models with unknown poles, where, for example, the length of a cycle in physical and macroeconomic data can be inferred from the position of the pole. One model, in particular, that has received a great deal of attention is the Gegenbauer autoregressive moving average (GARMA) model, which generalizes fractional integration, while allowing for short memory dynamics through ARMA components.

In this paper, we consider the properties of two parametric estimators of the GARMA model whose objective functions are approximations to the Gaussian log likelihood function in the time domain (the CSS estimator) and in the frequency domain (the Whittle based estimator). Although, in the frequency domain, asymptotic theory for estimates associated with the position of the pole is not fully established, Giriatis et al. (2001) provided rate T convergence of their estimator of the pole, while showing that the remaining parameters converge at rate $T^{1/2}$ to a normal distribution. Chung (1996 a,b) asserts that the CSS estimator achieves rate T^2 convergence when the singularity occurs at 0 or π, converging at rate T otherwise. Further, Chung (1996 a,b) claims to have established an exact limiting distribution for the CSS estimator of the parameter η, which is the cosine of the frequency associated with the pole. In contrast, Giriatis et al. are unable to provide an exact limiting distribution for their estimator of the pole given its discrete nature. We perform extensive Monte Carlo analysis with sample sizes ranging from 100 to 2000 observations. Our results show that both estimators perform remarkably well, while the CSS estimator tends to do slightly better in terms of RMSE. Of note, the mean bias and RMSE indicate that the estimator of η does indeed converge faster than the estimators of the other parameters as the theory of both Chung and Giriatis et al. documents.

There is considerable doubt regarding the inference proposed by Chung (1996 a,b), since he establishes his distributional results without using uniform convergence arguments. Our results generally support the asymptotic theory of Chung when the singularity occurs away from 0 or π. However, our results refute his established distribution theory otherwise. This is particularly important since standard fractional integration occurs when the pole is at the origin. Thus, it is important to consider the extent to which Chung's established distribution theory can be used to test if a process is distributed as an ARFIMA process against a GARMA alternative. To this end, we apply the test proposed by Chung to fractional processes. Our results indicate that when one uses Chung's proposed distribution theory under the null (with rate T^2 convergence), rejection rates can exceed the nominal size of the test by as much as 16%. Awaiting additional theoretical results, we propose as one possibility the use of the more liberal test statistic established by Chung for the case where $|\eta|<1$, which involves standardizing the test statistic by T rather than T^2.

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FIGURES

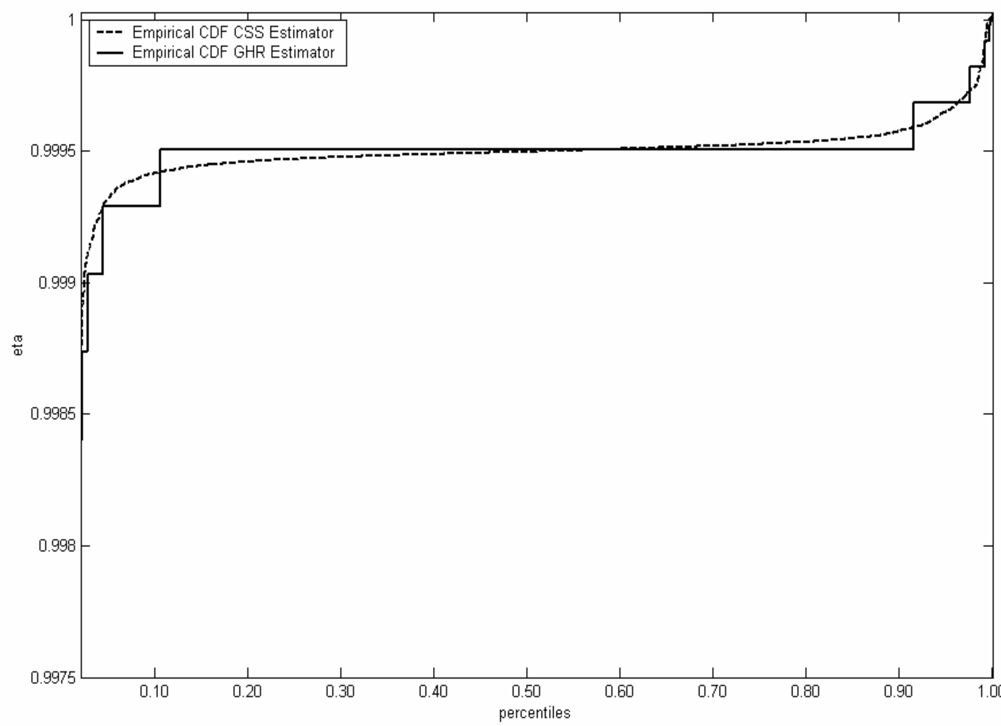

Figure 1

Empirical CDF for the CSS and GHR Estimators

of η, with η=0.9995 (1000 OBS with 2500 Simulations).

Table 1

Bias and RMSE

Semiparametric and CSS estimates for the GARMA model parameters

Model 1: GARMA (0,0) η=1.00, λ=0.20

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	**-0.00644**	-0.00659	**0.00564**	-0.03109	0.02620	**0.02366**	**0.04436**	0.05907	0.03898
300	**-0.00089**	-0.00093	**0.00311**	-0.01106	0.00575	**0.00427**	**0.02510**	0.02779	0.02251
500	**-0.00027**	-0.00028	**0.00228**	-0.00630	0.00174	**0.00120**	**0.01942**	0.01994	0.01743
1000	**-0.00006**	-0.00008	**0.00173**	-0.00320	**0.00033**	0.00037	**0.01305**	0.01329	0.01233
2000	**-0.00002**	-0.00002	0.00210	**-0.00158**	**0.00013**	0.00017	0.00951	**0.00917**	0.00872

Model 2: GARMA (0,0) η=-1.00, λ=0.20

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	0.00724	**0.00624**	**0.00498**	-0.03160	0.02768	**0.02480**	**0.04475**	0.05930	0.03898
300	**0.00090**	0.00091	**0.00306**	-0.01025	0.00520	**0.00393**	**0.02515**	0.02759	0.02251
500	0.00031	**0.00027**	**0.00249**	-0.00670	0.00189	**0.00117**	**0.01909**	0.02052	0.01743
1000	**0.00007**	0.00009	**0.00183**	-0.00342	**0.00036**	0.00043	**0.01343**	0.01381	0.01233
2000	0.00002	**0.00002**	**0.00167**	-0.00176	**0.00016**	0.00017	0.00935	**0.00917**	0.00872

Model 3: GARMA (0,0) η=0.9995, λ=0.40

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	**-0.00070**	-0.00118	0.01696	**0.00227**	**0.00290**	0.00438	**0.04702**	0.05459	0.03958
300	**-0.00008**	-0.00018	0.01057	**0.00999**	**0.00049**	0.00083	**0.02735**	0.03162	0.02285
500	**-0.00002**	-0.00006	**0.00809**	0.00918	**0.00024**	0.00036	**0.02127**	0.02446	0.01770
1000	**1.18E-06**	-1.99E-06	0.00562	**-0.00107**	**0.00011**	0.00014	0.01489	**0.01325**	0.01252
2000	**1.59E-06**	2.22E-06	0.00495	**-0.00007**	**0.00005**	0.00006	0.01081	**0.00921**	0.00885

Model 4: GARMA (0,0) η=-0.9995, λ=0.40

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	**0.00089**	0.00120	0.01566	**0.00180**	**0.00389**	0.00415	**0.04533**	0.04583	0.03958
300	**0.00009**	0.00014	0.01049	**0.00929**	**0.00060**	0.00066	**0.02759**	0.03089	0.02285
500	**0.00002**	0.00006	**0.00823**	0.00839	**0.00025**	0.00037	**0.02143**	0.02432	0.01770
1000	1.63E-06	**1.43E-06**	0.00553	**-0.00123**	**0.00011**	0.00013	0.01490	**0.01314**	0.01252
2000	6.80E-07	**-3.76E-06**	0.00452	**-0.00022**	**0.00005**	0.00006	0.01058	**0.00926**	0.00885

Model 5: GARMA (0,0) η=0.50, λ=0.40

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	**0.00102**	-0.00587	0.02535	**0.00602**	**0.02982**	0.04370	**0.08052**	0.10500	0.06752
300	**0.00029**	0.00051	**0.01620**	-0.01930	**0.01023**	0.01136	**0.04583**	0.04796	0.03898
500	**0.00001**	0.00152	0.01245	**0.00467**	**0.00599**	0.00793	**0.03606**	0.03873	0.03020
1000	**-0.00002**	-0.00080	0.00754	**0.00472**	**0.00301**	0.00380	**0.02484**	0.02682	0.02135
2000	**-0.00002**	0.00036	0.00521	**0.00397**	**0.00157**	0.00196	**0.01704**	0.06297	0.01510

Notes: See notes for Table 1 below.

Table 1 (cont).

Model 6: GARMA (0,0) η=-0.50, λ=0.40

Sample Size	CSS Bias η	GHR Bias η	CSS Bias λ	GHR Bias λ	CSS RMSE η	GHR RMSE η	CSS RMSE λ	GHR RMSE λ	Asymptotic Std Dev λ (Chung)
100	**-0.00002**	0.00621	0.02950	**-0.02176**	**0.03132**	0.04286	**0.08115**	0.10387	0.06752
300	**-0.00006**	-0.00016	**0.01535**	-0.01945	**0.01024**	0.01131	**0.04472**	0.04899	0.03898
500	**0.00002**	-0.00151	0.01112	**0.00245**	**0.00626**	0.00799	**0.03464**	0.03742	0.03020
1000	**-0.00005**	0.00063	0.00736	**0.00456**	**0.00313**	0.00385	**0.02393**	0.02595	0.02135
2000	**-0.00001**	-0.00035	0.00460	**0.00328**	**0.00156**	0.00199	0.01656	**0.00199**	0.01510

Model 7: GARMA (1,0) η=0.50, λ=0.40, ϕ=0.80

Parameter Biases

Sample Size	CSS η	GHR η	CSS λ	GHR λ	CSS ϕ	GHR ϕ	Asymptotic Std Dev λ (Chung)
100	**-0.00058**	-0.00901	**0.01937**	-0.05005	-0.02735	**-0.02068**	0.06779
300	-0.00036	**-0.00005**	**0.01320**	-0.02692	-0.01035	**-0.00357**	0.03914
500	**-0.00016**	0.00119	0.00999	**-0.00106**	**-0.00683**	-0.00724	0.03032
1000	**-0.00001**	-0.00086	0.00710	**0.00249**	-0.00348	**-0.00319**	0.02144
2000	**-0.00001**	0.00037	0.00655	**0.00488**	**-0.00146**	-0.00150	0.01516

RMSEs of the Parameters

Sample Size	CSS η	GHR η	CSS λ	GHR λ	CSS ϕ	GHR ϕ	Asymptotic Std Dev ϕ (Chung)
100	**0.03235**	0.04947	**0.07735**	0.11326	**0.07617**	0.07828	0.06024
300	**0.01075**	0.01152	**0.04481**	0.05360	0.03854	**0.03769**	0.03478
500	**0.00641**	0.00784	**0.03435**	0.03796	**0.02918**	0.03093	0.02694
1000	**0.00334**	0.00408	**0.02422**	0.02584	**0.01951**	0.02004	0.01905
2000	**0.00161**	0.00201	**0.01736**	0.01867	**0.01369**	0.01394	0.01347

Model 8: GARMA (1,0) η=0.9995, λ=0.40, ϕ=0.80

Parameter Biases

Sample Size	CSS η	GHR η	CSS λ	GHR λ	CSS ϕ	GHR ϕ	Asymptotic Std Dev λ (Chung)
100	**-0.01416**	-0.02782	**-0.20227**	-0.30174	**0.09923**	0.10871	0.12635
300	**-0.00223**	-0.00796	-0.15377	**-0.11110**	**0.09992**	-0.11546	0.07295
500	**-0.00065**	-0.00337	-0.12132	**-0.01833**	**0.07958**	-0.24442	0.05651
1000	**-0.00009**	-0.00066	-0.08719	**0.00541**	**0.06242**	-0.13533	0.03996
2000	**0.00000**	-0.00004	**0.01609**	0.01929	**-0.02224**	-0.10542	0.02825

RMSEs of the Parameters

Sample Size	CSS η	GHR η	CSS λ	GHR λ	CSS ϕ	GHR ϕ	Asymptotic Std Dev ϕ (Chung)
100	**0.03787**	0.05880	**0.24362**	0.33197	0.22420	**0.20178**	0.19154
300	**0.00981**	0.01753	**0.19453**	0.24230	**0.15666**	0.41871	0.11058
500	**0.00466**	0.00822	**0.16658**	0.20506	**0.14172**	0.48856	0.08566
1000	**0.00162**	0.00255	**0.12891**	0.14197	**0.12002**	0.33196	0.06057
2000	**0.00006**	0.00037	**0.04347**	0.10770	**0.07160**	0.27240	0.04283

Notes (Table 1): CSS refers to the constrained sum of squares estimator of Chung (1996 a,b), while GHR refers to the Whittle based estimator of Giriatis et al. (2001). The mean bias and RMSE are obtained from 2500 simulations where the true model is a GARMA model whose coefficients appear in the heading. The asymptotic standard errors for λ have been calculated using Chung (1996b).

Table 2

Percentiles of the Distribution for η

Model: GARMA (0,0) η=0.9995, λ=0.40

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-1.6393	-1.2678	-0.6890	-0.3858	-0.1686	0.0467	0.0670	0.0917	0.1202	0.1398
$T(\eta$-0.9995) GHR	-3.0917	-1.7213	-0.7385	-0.7385	-0.1473	0.0500	0.0500	0.0500	0.0500	0.0500
Sample Size 300										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-1.0015	-0.6283	-0.3194	-0.1942	-0.0959	0.0660	0.1148	0.1488	0.1657	0.1822
$T(\eta$-0.9995) GHR	-1.4934	-0.9021	-0.4420	-0.4420	-0.1132	0.0842	0.1500	0.1500	0.1500	0.1500
Sample Size 500										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-0.7217	-0.4314	-0.2746	-0.1552	-0.0752	0.0710	0.1372	0.2024	0.2467	0.2604
$T(\eta$-0.9995) GHR	-0.7366	-0.7366	-0.3815	-0.3815	-0.1053	0.0921	0.2105	0.2105	0.2500	0.2500
Sample Size 1000										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-0.4601	-0.3677	-0.2093	-0.1216	-0.0718	0.0791	0.1520	0.2293	0.3454	0.4935
$T(\eta$-0.9995) GHR	-0.7630	-0.4671	-0.2105	-0.2105	0.0066	0.0066	0.1842	0.1842	0.3224	0.4210
Sample Size 2000										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-0.3693	-0.2758	-0.1893	-0.1175	-0.0637	0.0636	0.1256	0.2364	0.3428	0.4612
$T(\eta$-0.9995) GHR	-0.4211	-0.4211	-0.1941	-0.1941	-0.1941	0.0131	0.2006	0.2006	0.3684	0.5164

Table 2b

Model: GARMA (1,0) η=0.9995, λ=0.40, ϕ=0.80

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-21.7628	-17.4257	-11.7336	-8.9017	-4.8172	0.0148	0.0521	0.0900	0.1362	0.1687
$T(\eta$-0.9995) GHR	-20.4891	-18.7362	-13.9341	-9.8225	-6.4070	0.0921	0.0921	0.2105	0.2105	0.2500
Sample Size 300										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-20.6200	-15.1730	-9.4449	-4.8006	-0.4528	0.0756	0.1470	0.1721	0.1901	0.2008
$T(\eta$-0.9995) GHR	-25.7864	-23.2911	-18.6654	-12.6542	-7.7763	0.0842	0.0842	0.0842	0.1500	0.1500
Sample Size 500										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-18.3603	-14.2201	-2.2334	-0.3337	-0.1210	0.1089	0.2298	0.2600	0.2715	0.2758
$T(\eta$-0.9995) GHR	-20.4891	-18.7362	-13.9341	-9.8225	-6.4070	0.0921	0.0921	0.2105	0.2105	0.2500
Sample Size 1000										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-0.8881	-0.5791	-0.3007	-0.1591	-0.0767	0.1293	0.3415	0.5051	0.5115	0.5141
$T(\eta$-0.9995) GHR	-16.0548	-14.9357	-9.9239	-5.1992	-0.4671	0.0066	0.1842	0.1842	0.3224	0.4210
Sample Size 2000										
$T(\eta$-0.9995) Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta$-0.9995) CSS	-0.3893	-0.2797	-0.1890	-0.1177	-0.0639	0.0664	0.1351	0.2617	0.4211	0.5961
$T(\eta$-0.9995) GHR	-7.2946	-6.1906	-0.4211	-0.1941	-0.1941	0.0131	0.2006	0.2006	0.3684	0.5164

Notes: See notes for Table 2 below.

Table 2c

Model: GARMA (0,0) η=0.50, λ=0.40

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-15.4781	-11.0576	-6.95743	-4.222	-2.19716	2.32601	4.40435	6.82301	9.68296	11.7663
$T(\eta$-0.50) GHR	-13.1875	-13.1875	-7.4221	-7.4221	-1.8246	3.5827	3.5827	8.7785	13.7424	18.4547
Sample Size 300										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-11.6332	-8.9867	-6.753	-4.0548	-2.2088	2.4581	4.6887	7.2587	11.3436	13.3134
$T(\eta$-0.50) GHR	-11.0112	-11.0112	-5.4739	-5.4739	0.0000	0.0000	5.4081	5.4081	10.7480	16.0175
Sample Size 500										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-12.157	-10.2145	-6.3531	-4.1076	-2.0878	2.2638	4.1712	6.5504	9.7285	12.1072
$T(\eta$-0.50) GHR	-14.648	-9.1232	-9.1232	-3.6363	-3.6363	1.8116	7.2198	7.2198	12.5873	12.5873
Sample Size 1000										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-13.7771	-10.6317	-6.7873	-4.1047	-2.3753	2.2403	4.2293	6.6596	9.7020	11.6747
$T(\eta$-0.50) GHR	-12.7499	-12.7499	-7.2727	-7.2727	-1.8149	3.6232	3.6232	9.0414	9.0414	14.4395
Sample Size 2000										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-13.5445	-10.7726	-7.1169	-4.0953	-2.2009	2.1097	3.8463	6.3299	11.4778	13.2061
$T(\eta$-0.50) GHR	-14.5453	-9.0827	-9.0827	-3.6298	-3.6298	1.8133	7.2464	7.2464	12.6696	12.6696

Table 2d

Model: GARMA (0,0) η=0.50, λ=0.40, ϕ=0.80

	0.005	0.001	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-14.1109	-10.7445	-6.9486	-4.28355	-2.34761	2.2261	4.4758	7.49062	10.6812	12.7716
$T(\eta$-0.50) GHR	-19.0983	-13.1875	-7.4221	-7.4221	-1.8246	3.5827	3.5827	8.7785	13.7424	22.8969
Sample Size 300										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-13.1868	-11.1885	-7.2356	-4.8396	-2.3884	2.0475	4.1253	6.4882	10.2628	12.8625
$T(\eta$-0.50) GHR	-11.0112	-11.0112	-5.4739	-5.4739	0.0000	0.0000	5.4081	5.4081	10.7480	16.0175
Sample Size 500										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-14.5129	-10.6831	-6.7606	-4.4167	-2.1636	1.9749	3.9664	6.5643	10.1527	13.4670
$T(\eta$-0.50) GHR	-14.648	-14.648	-9.12316	-3.63633	-3.63633	1.8116	7.21977	7.21977	12.5873	12.5873
Sample Size 1000										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-12.6192	-9.8416	-6.5348	-3.9720	-2.1391	2.1065	4.1168	6.7738	9.7826	13.2590
$T(\eta$-0.50) GHR	-12.7499	-12.7499	-7.2727	-7.2727	-1.8149	3.6232	3.6232	9.0414	9.0414	14.4395
Sample Size 2000										
$T(\eta$-0.50) Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta$-0.50) CSS	-15.5426	-11.1772	-6.6565	-3.8722	-2.0461	2.0429	3.7523	6.8691	10.2117	13.4884
$T(\eta$-0.50) GHR	-14.5453	-9.08266	-9.08266	-3.62979	-3.62979	1.81325	7.2464	7.2464	12.6696	12.6696

Notes: See notes for Table 2 below.

Table 2e

Model: GARMA (0,0) η=-0.50, λ=0.40

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta+0.50)$ Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta+0.50)$ CSS	-15.4781	-11.0576	-6.95743	-4.222	-2.19716	2.32601	4.40435	6.82301	9.68296	11.7663
$T(\eta+0.50)$ GHR	-18.4547	-13.7424	-8.77853	-3.58268	-3.58268	1.82463	7.42207	7.42207	13.1875	13.1875
Sample Size 300										
$T(\eta+0.50)$ Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta+0.50)$ CSS	-12.7572	-11.0549	-6.3613	-4.4357	-2.2225	2.1546	3.9756	6.485	9.5661	12.6257
$T(\eta+0.50)$ GHR	-16.0175	-10.748	-5.4081	-5.4081	0.0000	0.0000	5.4739	5.4739	11.0112	16.6094
Sample Size 500										
$T(\eta+0.50)$ Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta+0.50)$ CSS	-14.2564	-10.1997	-6.7394	-4.127	-2.082	2.2793	4.0491	6.121	10.0732	13.3915
$T(\eta+0.50)$ GHR	-12.5873	-12.5873	-7.2198	-7.2198	-1.8116	3.6363	3.6363	9.1232	9.1232	14.6480
Sample Size 1000										
$T(\eta+0.50)$ Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta+0.50)$ CSS	-14.5265	-11.1608	-7.1794	-4.5863	-2.2939	2.3728	4.1862	6.6117	9.5438	12.0477
$T(\eta+0.50)$ GHR	-14.4395	-9.0414	-9.0414	-3.6232	-3.6232	1.8149	7.2727	7.2727	12.7499	12.7499
Sample Size 2000										
$T(\eta+0.50)$ Chung	-9.1755	-7.7856	-6.0124	-4.7004	-3.3991	3.3991	4.7004	6.0124	7.7856	9.1755
$T(\eta+0.50)$ CSS	-12.9660	-9.6244	-6.6175	-4.3339	-2.4920	0.3043	4.4406	6.6573	10.9873	12.2147
$T(\eta+0.50)$ GHR	-12.6696	-12.6696	-7.2464	-7.2464	-1.8133	3.6298	3.6298	9.0827	14.5453	14.5453

Table 2f

Model: GARMA (0,0) η=-0.9995, λ=0.40

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
Sample Size 100										
$T(\eta+0.9995)$ Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta+0.9995)$ CSS	-0.1513	-0.1284	-0.0933	-0.0685	-0.0471	0.1962	0.4686	0.8167	1.7049	2.6856
$T(\eta+0.9995)$ GHR	-0.0500	-0.0500	-0.0500	-0.0500	-0.0500	0.1473	0.7385	0.7385	1.7213	3.0917
Sample Size 300										
$T(\eta+0.9995)$ Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta+0.9995)$ CSS	-0.1778	-0.1719	-0.1489	-0.1209	-0.0689	0.0974	0.2297	0.3826	0.6576	1.0513
$T(\eta+0.9995)$ GHR	-0.1500	-0.1500	-0.1500	-0.1500	-0.0842	0.1132	0.4420	0.4420	0.9021	0.9021
Sample Size 500										
$T(\eta+0.9995)$ Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta+0.9995)$ CSS	-0.2670	-0.2523	-0.2085	-0.1449	-0.0710	0.0824	0.1611	0.2979	0.5178	0.6598
$T(\eta+0.9995)$ GHR	-0.2500	-0.2500	-0.2105	-0.2105	-0.0921	0.1053	0.3815	0.3815	0.7366	0.7366
Sample Size 1000										
$T(\eta+0.9995)$ Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta+0.9995)$ CSS	-0.4974	-0.3595	-0.2231	-0.1372	-0.0727	0.0724	0.1322	0.2113	0.3692	0.5366
$T(\eta+0.9995)$ GHR	-0.4210	-0.3224	-0.1842	-0.1842	-0.0066	-0.0066	0.2105	0.2105	0.4671	0.7630
Sample Size 2000										
$T(\eta+0.9995)$ Chung	-0.3350	-0.2843	-0.2195	-0.1716	-0.1241	0.1241	0.1716	0.2195	0.2843	0.3350
$T(\eta+0.9995)$ CSS	-0.4809	-0.3578	-0.2318	-0.1469	-0.0649	0.0698	0.1271	0.2350	0.3580	0.4779
$T(\eta+0.9995)$ GHR	-0.5164	-0.3684	-0.2006	-0.2006	-0.0131	-0.0131	0.1941	0.1941	0.4211	0.6677

Notes: CSS/GHR refers to the CSS and Whittle based estimators of Chung (1996 a,b) and Giriatis et al. (2001) respectively. The first row beneath the sample size provides the percentiles of the asymptotic distibution calculated from Chung (1996a). The empirical distribution using the CSS estimator and GHR estimator follow.

Table 3 a

Percentiles of Distribution for η with $|\eta|=1$

Model: GARMA (0,0) η=1.000, λ=0.20

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
$T^2(\eta-1)$ Chung	**-244.44**	**-185.29**	**-121.23**	**-81.00**	**-75.13**	**10.14**	**15.44**	**21.53**	**31.18**	**39.85**
Sample Size 100										
$T^2(\eta-1)$ CSS	-1779.07	-1326.23	-719.39	-382.49	-123.84	7.36	10.11	13.30	16.65	21.11
$T(\eta-1)$ CSS	-17.79	-13.2623	-7.19	-3.82	-1.24	0.07	0.10	0.13	0.17	0.21
$T^2(\eta-1)$ GHR	-1556.72	-1236.93	-702.24	-314.17	-177.13	0.00	0.00	0.00	0.00	0.00
$T(\eta-1)$ GHR	-15.57	-12.37	-7.02	-3.14	-1.77	0.00	0.00	0.00	0.00	0.00
Sample Size 300										
$T^2(\eta-1)$ CSS	-2992.32	-2052.81	-698.42	-253.80	-89.10	7.31	9.54	12.47	15.82	18.05
$T(\eta-1)$ CSS	-9.9744	-6.8427	-2.33	-0.85	-0.30	0.02	0.03	0.04	0.05	0.06
$T^2(\eta-1)$ GHR	-2377.90	-1594.15	-709.68	-493.03	-177.59	0.00	0.00	0.00	0.00	0.00
$T(\eta-1)$ GHR	-7.93	-5.31	-2.37	-1.64	-0.59	0.00	0.00	0.00	0.00	0.00
Sample Size 500										
$T^2(\eta-1)$ CSS	-2510.10	-1666.20	-557.87	-246.83	-88.70	7.04	9.35	11.80	15.25	18.70
$T(\eta-1)$ CSS	-5.02	-3.33	-1.12	-0.49	-0.18	0.01	0.02	0.02	0.03	0.04
$T^2(\eta-1)$ GHR	-1971.30	-1262.25	-710.25	-315.75	-177.65	0.00	0.00	0.00	0.00	0.00
$T(\eta-1)$ GHR	-3.94	-2.52	-1.42	-0.63	-0.36	0.00	0.00	0.00	0.00	0.00
Sample Size 1000										
$T^2(\eta-1)$ CSS	-2072.14	-1429.79	-592.49	-286.06	-86.90	6.84	9.21	11.98	14.82	17.29
$T(\eta-1)$ CSS	-2.07	-1.43	-0.59	-0.29	-0.09	0.01	0.01	0.01	0.01	0.02
$T^2(\eta-1)$ GHR	-2841.10	-1598.40	-710.50	-315.80	-177.60	0.00	0.00	0.00	0.00	0.00
$T(\eta-1)$ GHR	-2.84	-1.60	-0.71	-0.32	-0.18	0.00	0.00	0.00	0.00	0.00
Sample Size 2000										
$T^2(\eta-1)$ CSS	-1821.48	-1288.00	-493.68	-199.01	-68.11	6.66	8.93	11.10	14.23	18.19
$T(\eta-1)$ CSS	-0.91	-0.64	-0.25	-0.10	-0.03	0.00	0.00	0.01	0.01	0.01
$T^2(\eta-1)$ GHR	-2388.20	-1598.80	-710.60	-315.80	-177.60	0.00	0.00	0.00	0.00	0.00
$T(\eta-1)$ GHR	-1.19	-0.80	-0.36	-0.16	-0.09	0.00	0.00	0.00	0.00	0.00

Notes: CSS/GHR refer to the CSS and Whittle based estimators of Chung (1996 a,b) and Giriatis et al. (2001) respectively. The first row beneath the sample size provides the percentiles of the asymptotic distibution calculated from Chung (1996a). The empirical distribution using the CSS and GHR estimators follow. The generated data are distributed as ARFIMA processes such that η=1, and the pole in the specturm occurs at the origin.

Table 3b

Percentiles of the Distribution for η with $|\eta|=1$

Model: GARMA (0,0) η=-1.000, λ=0.20

	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
$T^2(\eta+1)$ Chung	**244.44**	**185.29**	**121.23**	**81.00**	**75.13**	**-10.14**	**-15.44**	**-21.53**	**-31.18**	**-39.85**
Sample Size 100										
$T^2(\eta+1)$ CSS	-22.02	-17.11	-13.25	-10.10	-7.16	147.14	432.61	859.84	1369.95	1841.63
$T(\eta+1)$ CSS	-0.22	-0.17	-0.13	-0.10	-0.07	1.47	4.33	8.60	13.70	18.42
$T^2(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	177.13	314.17	702.24	1236.93	1909.83
$T(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	1.77	3.14	7.02	12.37	19.10
Sample Size 300										
$T^2(\eta+1)$ CSS	-18.20	-16.39	-11.47	-9.00	-6.70	109.76	300.03	692.32	1569.82	2871.65
$T(\eta+1)$ CSS	-0.06	-0.05	-0.04	-0.03	-0.02	0.37	1.00	2.31	5.23	9.57
$T^2(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	177.59	493.03	709.68	1594.15	2377.90
$T(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	0.59	1.64	2.37	5.31	7.93
Sample Size 500										
$T^2(\eta+1)$ CSS	-19.04	-15.47	-11.77	-8.78	-6.76	115.30	325.05	682.76	1240.06	2792.51
$T(\eta+1)$ CSS	-0.04	-0.03	-0.02	-0.02	-0.01	0.23	0.65	1.37	2.48	5.59
$T^2(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	78.95	315.75	710.25	1262.25	1971.30
$T(\eta+1)$ GHR	0	0	0	0	0	0.1579	0.6315	1.4205	2.5245	3.9426
Sample Size 1000										
$T^2(\eta+1)$ CSS	-19.82	-15.67	-11.45	-9.07	-6.64	97.11	255.53	625.08	1557.42	2365.19
$T(\eta+1)$ CSS	-0.02	-0.02	-0.01	-0.01	-0.01	0.10	0.26	0.63	1.56	2.37
$T^2(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	177.60	493.40	710.50	1598.40	2841.10
$T(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	0.18	0.49	0.71	1.60	2.84
Sample Size 2000										
$T^2(\eta+1)$ CSS	-15.63	-13.35	-10.32	-8.62	-6.35	81.31	249.40	555.92	1451.99	2230.24
$T(\eta+1)$ CSS	-0.01	-0.01	-0.01	0.00	0.00	0.04	0.12	0.28	0.73	1.12
$T^2(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	177.60	315.80	710.60	1263.20	1973.80
$T(\eta+1)$ GHR	0.00	0.00	0.00	0.00	0.00	0.09	0.16	0.36	0.63	0.99

Notes: CSS/GHR refer to the CSS and Whittle based estimators of Chung (1996 a,b) and Giriatis et al. (2001) respectively. The first row beneath the sample size provides the percentiles of the asymptotic distibution calculated from Chung (1996a). The empirical distribution using the CSS and GHR estimator follow. The generated data are distributed as GARMA processes such that η=-1, and the pole in the specturm occurs at π.

Table 4
Rejection Rates of the Null Hypothesis $|\eta|$=1
using Chung's Confidence Intervals under the null $|\eta|$=1

	GARMA (0,0) Process η=1, λ=0.20		**GARMA (0,0) Process η=-1, λ=0.20**	
	99% CI η=1	95% CI η=1	99% CI η=-1	95% CI η=-1
Sample Size				
100	0.1780	0.2184	0.1808	0.2268
300	0.1468	0.1864	0.1704	0.2236
500	0.1492	0.1848	0.1724	0.2180
1000	0.1460	0.1812	0.1528	0.1964
2000	0.1328	0.1756	0.1424	0.1844

Notes: The table records the proportion of rejection of the true null hypothesis $|\eta|$=1 using the 95% and 99% confidence intervals of Chung (1996a) calculated among 2500 simulations. The data are generated with $|\eta|$=1.