

## **A Longitudinal Study of Invention and Understanding in Children's Multidigit Addition and Subtraction\***

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Thomas P. Carpenter, University of Wisconsin-Madison

Megan L. Franke, University of California at Los Angeles

Victoria R. Jacobs, California State University-San Marcos

Elizabeth Fennema, University of Wisconsin-Madison

Susan B. Empson, University of Texas at Austin

This 3-year longitudinal study investigated the development of 82 children's understanding of multidigit number concepts and operations in Grades 1-3. Students were individually interviewed 5 times on a variety of tasks involving base-ten number concepts and addition and subtraction problems. The study provides an existence proof that children can invent strategies for adding and subtracting and illustrates both what that invention affords and the role that different concepts may play in that invention. About 90% of the students used invented strategies. Students who used invented strategies before they learned standard algorithms demonstrated better knowledge of base-ten number concepts and were more successful in extending their knowledge to new situations than were students who initially learned standard algorithms.

An understanding of most fundamental mathematics concepts and skills develops over an extended period of time. Although cross-sectional studies can provide snapshots of the development of these concepts at particular points in time, longitudinal studies provide a more complete perspective; however, relatively few studies have traced the development of fundamental mathematics concepts in children over more than a single year. In this paper we report the results of a 3-year longitudinal study of the growth of children's understanding of addition and subtraction involving multidigit numbers. We focus particularly on children's construction of invented strategies for adding and subtracting multidigit numbers. The overarching goal of the study was to investigate the role that invented strategies may play in developing an understanding of multidigit addition and subtraction concepts and procedures. We trace the development and use of invented addition and subtraction strategies and examine the relation of these strategies to the development of fundamental knowledge of base-ten number concepts and the use of standard addition and subtraction algorithms. Finally, we consider what the use of invented strategies affords by way of avoiding systematic errors and extending knowledge of basic multidigit operations to new problem situations.

### **BACKGROUND**

There is mounting evidence that children both in and out of school can construct methods for adding and subtracting multidigit numbers without explicit instruction in specific procedures (Carpenter & Fennema, 1992; Carraher, Carraher, & Schliemann, 1987; Cobb & Wheatley, 1988; Fuson & Burghardt, 1993; Hiebert & Wearne, 1996; Kamii, 1989; Labinowicz, 1985; Nunes, 1992; Olivier, Murray, & Human, 1990; Saxe, 1988). It is hypothesized that these invented strategies can play a central role in making problem solving a focus of learning arithmetic procedures and in helping students develop number sense and understanding of multidigit operations (Carpenter et al.,

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1994). Children employ a number of strategies for solving multidigit problems at varying levels of sophistication. Many of these strategies are constructed by children individually or collectively, without direct instruction by the teacher (for a more complete discussion of children's strategies, see Fuson et al., 1997). Initially, children may use individual counters to model addition or subtraction operations. As they gain understanding of base-ten numbers, they begin to use various materials representing tens rather than individual counters. Over time, many children construct procedures for adding and subtracting multidigit numbers without using physical materials of any kind. These are the strategies that are the focus of this study, and we use the term *invented strategy* to refer to them.

Fuson et al. (1997) provided a comprehensive analysis of invented strategies children use to solve multidigit addition and subtraction problems. The following protocols for finding the sum  $38 + 26$  illustrate distinctions among three primary classes of invented strategies identified in this analysis:

*Sequential:* "Thirty and twenty is fifty and the eight makes fifty-eight. Then six more is sixty-four."

*Combining units separately:* "Thirty and twenty is fifty, and eight and six is fourteen. The ten from the fourteen makes sixty so it's sixty-four."

*Compensating:* "That's like forty and twenty-four, and that's sixty-four."

In the sequential strategy, the sum is kept as a running total. In the combining-units-separately strategy, the tens and ones are added separately then combined. In the compensating strategy, the numbers are adjusted to simplify the calculation. Similar strategies are used for subtraction, although they are somewhat more difficult to implement. The following examples illustrate these strategies for the difference  $62 - 28$ :

*Sequential:* "Sixty take away twenty is forty. Then put back the two; that's forty-two. Now take away the eight from the forty-two. Take away the two, that's forty, and then six more makes thirty-four . "

*Combining units separately:* "Sixty take away twenty is forty. You can't take eight from two. If you take two from the two, you still have six more to take away. Now take the six from the forty; that's thirty-four."

*Compensating:* "Sixty-two take away thirty is thirty-two, but we're taking away twenty-eight so it's two more. It's thirty-four."

Both standard algorithms and invented strategies reduce a complex procedure to simpler components that can be operated on using available knowledge or procedures. For example, most procedures for adding or subtracting multidigit numbers essentially reduce the calculation to a series of sums or differences of single-digit numbers (or corresponding multiples of 10, 100, etc.). There are, however, some fundamental differences between standard algorithms and the strategies that children construct to solve multidigit problems. Standard algorithms have evolved over centuries for efficient, accurate calculation. For the most part, these algorithms are quite far removed from their conceptual underpinnings. Invented strategies, on the other hand, generally are derived directly from the underlying multidigit concepts. For example, with standard addition and subtraction algorithms, numerals are aligned so that the ones, tens, hundreds, and larger digits can be added in columns. But in the addition of columns, no reference is made to the fact that the addition involves the same unit (ones, tens, hundreds, etc.); one simply adds numbers in a column. Most invented strategies, on the other hand, specifically label the units being combined. For example, in the above addition examples, the invented strategies are based on adding 30 and 20 or 3 tens and 2 tens rather than adding two numbers that appear in the same column.<sup>1</sup>

#### *Sequence of Development of Major Concepts and Procedures*

Longitudinal studies afford the opportunity to trace the sequence in which major concepts and procedures develop. Although there are alternative perspectives, most current theoretical arguments support the development of conceptual knowledge before students master algorithmic procedures (Hiebert 1986; Hiebert & Carpenter, 1992). If invented strategies provide a basis for understanding multidigit procedures, it might be expected that there would be critical

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<sup>1</sup> Under certain instructional conditions, children also construct variations of written algorithms involving base-ten number concepts (Fuson & Burghardt, 1993).

differences between students who used invented strategies before they learned standard algorithms and students who initially learned standard algorithms without first constructing invented strategies. A major focus of our analysis deals with distinctions between students who used standard algorithms before using invented strategies and students who used invented strategies before or concurrently with their use of standard algorithms.

*Development of base-ten number concepts.* The sequence in which students develop an understanding of base-ten number concepts and use invented strategies is also of theoretical and practical interest. Because invented strategies are based on base-ten number concepts, it might be expected that students' use of invented strategies would be related to understanding of these concepts, but there is an ongoing debate whether children should master fundamental base-ten number concepts before they learn procedures for adding and subtracting multidigit numbers or whether learning of these concepts and procedures might be more integrated (Baroody, 1990; Fuson, 1990). The debate centers around the level of understanding of base-ten concepts that is a prerequisite. One perspective is that understanding multidigit procedures depends on a reasonably well-developed understanding of base-ten number concepts; the other is that learning procedures for adding and subtracting multidigit numbers provides a context for motivating and supporting the development of base-ten number concepts. Given that invented strategies are based on children's understanding of base-ten concepts, the argument can be made that children need to understand base-ten number concepts before they can invent their own procedures for adding and subtracting multidigit numbers. On the other hand, because grouping by tens is made so explicit in invented strategies, the use of invented strategies may provide a context for furthering the development of base-ten understanding.

Hiebert and Wearne (1996) found that understanding of base-ten number concepts and the use of appropriate multidigit addition and subtraction strategies were related. Children who demonstrated understanding of base-ten number concepts were more successful in inventing and modifying strategies for adding and subtracting multidigit numbers than children who demonstrated little or no understanding of base-ten number concepts. Their study did not, however, specifically examine whether students developed understanding of base-ten number concepts before they used invented strategies or whether children's multidigit procedures developed concurrently with their base-ten number concepts.

### *Consequences of Using Invented Strategies*

A primary goal of this study was to examine the role that invented strategies play in developing understanding of multidigit addition and subtraction. One important feature of knowledge that is acquired with understanding is that it can be extended to solve unfamiliar or nonroutine problems more readily than strictly procedural knowledge (Hiebert 1986; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1996). If invented strategies play a central role in the development of understanding of multidigit procedures, it would be expected that the development of invented strategies would be related to children's ability to apply their multidigit knowledge flexibly to problems that required more than the routine application of established procedures.

It also might be expected that the use of invented strategies would eliminate or at least reduce the occurrence of systematic errors based strictly on procedural applications of symbolic procedures. A substantial body of research has focused on students' systematic errors in executing addition and subtraction algorithms (Brown & VanLehn, 1982; VanLehn, 1986). One hypothesis is that these systematic errors or *buggy algorithms*, as they frequently are called, are a result of students relying exclusively on rote manipulations of symbols, and that developing understanding of multidigit procedures would eliminate most buggy algorithms (Fuson, 1992a, 1992b; Resnick, 1982). If invented strategies provide a basis for developing understanding of multidigit procedures, it follows that students who use invented strategies before they learn standard algorithms should develop fewer buggy algorithms than students who start out using standard algorithms.

## METHOD

Eighty-two students were individually interviewed in the winter of first grade and in the fall and spring of second and third grades on a variety of tasks involving base-ten number concepts, addition and subtraction word problems, addition and subtraction computation problems, and some problems that required students to extend their knowledge of base-ten number concepts and multidigit addition and subtraction procedures.

### *Sample*

The initial sample for the study consisted of 120 students in three schools. Six boys and six girls were randomly selected from each of the 10 first-grade classes in the three schools. One fourth of the students in one school were lost to the study in a boundary change, and there was some natural attrition as students moved during the 3 years of the study. The final sample consisted of the 82 students who participated in all five interviews. Over the 3 years of the study, the students were distributed among the classes of 27 teachers: 10 in the first grade, 8 in the second, and 9 in the third.

The three schools served somewhat different populations. One school served a predominately rural, White (99%) population. During the first year of the study, 412 were enrolled in the school; 4% of the students received free or reduced-price lunches. The second school had a population of 273 students; 70% of the students were White, and 26% received free or reduced-price lunches. The third school enrolled 541 students; 91% of the students were White, and 8% received free or reduced-price lunches.

### *Instruction*

All students in the study were in classes of teachers who were participating in a 3-year intervention study designed to help teachers understand and build on children's mathematical thinking (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996). The emphasis of the program was on how children's intuitive mathematical ideas emerge to form the basis for the development of more formal concepts and procedures. Teachers learned about how children solve problems using base-ten materials and the various invented abstract strategies children often construct.

No curricular materials or specific guidelines for instruction were provided, and teachers had latitude to plan their own instruction. As a consequence, there was variation in the instruction of the teachers. However, several features characterized instruction in most classrooms (Fennema et al., 1996). Word problems were used extensively. Students generally were given opportunities to solve problems using a variety of strategies, and alternative strategies were discussed with the whole class or in small groups.

The first-grade teachers provided students opportunities to solve addition and subtraction word problems involving regrouping of two- and, in some cases, three-digit numbers. Tens blocks and other base-ten materials were available, and most children initially used them to solve multidigit problems. By the end of the year, some children in each of the classes used and discussed invented strategies, but most children continued to use counters and base-ten materials in solving multidigit problems. First-grade teachers generally did not formally teach standard addition or subtraction algorithms. Second-grade teachers initially provided children opportunities to solve problems using base-ten materials and to discuss a variety of procedures. At the time of the Grade 2 fall interview, standard algorithms had not been a focus of instruction. By the spring interview, almost all second-grade students had been exposed to the standard addition and subtraction algorithms. Although most third-grade teachers provided opportunities for students to use and discuss a variety of strategies, standard algorithms played a prominent role in students' calculations.

### *Interview Tasks*

The interview tasks assessed (a) students' knowledge of base-ten number concepts, (b) their strategies for solving addition and subtraction word problems and computation exercises, (c) their abilities to use specific invented strategies, and (d) their abilities to extend and use addition and subtraction procedures flexibly. Not all tasks were administered in each interview. Easier tasks from the earlier interviews were replaced by more difficult tasks in later interviews. Numbers and contexts were varied slightly for tasks that were used in multiple interviews.

*Base-ten number concepts.* Base-ten number concepts were assessed with five different tasks. In one task, students were given three bundles of 10 sticks and 5 loose sticks. The students took apart one bundle to determine that there were 10 sticks in the bundle, and they were told that the same number of sticks were in each bundle. They were then asked how many sticks there were altogether in the three bundles and the 5 loose sticks. Next, the interviewer covered up the 35 sticks, put out two more bundles of sticks, and asked how many sticks there were altogether. To demonstrate understanding of base-ten concepts on this task, students had to use the 10 groupings and respond without counting individual sticks.

In the second task (from Kamii, 1989), the interviewer showed the students a card with the numeral 17 written on it and asked the students to count out 17 chips. The interviewer then pointed to the numeral 7 on the card and asked the students to show with the chips what that part meant. Then the interviewer pointed to the numeral 1 and asked what that meant.

The rest of the base-ten-number-concept tasks were set in the context of word problems involving groups of 10. In one problem, students were asked to find the total number of pieces of gum in four packs of gum with 10 sticks of gum in each pack. In another problem, students were asked how many groups of 10 could be made from an initial set of 36. The final problem involved making teams of 10 each from a group of 241 children. Each problem was coded as demonstrating base-ten knowledge if students responded immediately on the basis of their knowledge of tens or counted by 10 without using counters or base-ten materials.

The bundles task was used in the first two interviews, the multiplication word problem in the first three, and the division problem involving 241 children in the last two. The other two tasks were used in all five interviews. In the first two interviews, students were classified as demonstrating knowledge of base-ten number concepts if three out of four of their responses were coded as demonstrating base-ten knowledge; in the last three interviews the criterion was two out of three.

*Addition and subtraction problems.* Straightforward, result-unknown addition and subtraction word problems involving joining, separating, and part-whole relations with two-digit numbers were administered in each interview. The fourth and fifth interviews included three-digit part-whole and separating word problems. For most problems, children had a choice of materials including single counters, base-ten blocks, and paper and pencil. In order to increase the likelihood that children would use invented strategies if they were capable, two addition word problems and one subtraction word problem were administered with no materials or paper and pencil. The addition problems without materials were administered in every interview, and the subtraction problem without materials was administered in the last two interviews. The second through the fifth interviews also included two-digit addition and subtraction computation exercises.

For each task children's strategies were classified using a predetermined coding system. Coding categories included *incorrect, modeling or counting by ones, modeling with tens materials, sequential invented strategies, combining units invented strategies, compensating invented strategies, other invented strategies, algorithms, and buggy algorithms.*

*Use of specific invented strategies.* In the fourth interview, the interviewer demonstrated how a hypothetical child solved a three-digit addition problem using a sequential invented strategy. Children were asked to use the same strategy to solve another problem. To be counted correct, the strategy the child used had to be the same strategy the interviewer demonstrated. For example, a combining-units-separately strategy would not be counted as correct. In the fifth interview, the same procedure was used for a subtraction problem. In this case, the invented strategy involved a compensating strategy. The interviewer said that a girl named Sarah subtracted  $65 - 19$  by subtracting 20 from 65 and then adding 1 back. The student was then asked to use the same strategy to subtract  $53 - 28$ . To have a correct response, a student had to recognize that Sarah rounded the subtrahend up to a decade number to make subtraction easier. In the new problem the student had to round up to 30 and add 2 back on rather than 1.

*Extension problems.* The fourth and fifth interviews included two problems that required some flexibility in calculating. Both problems involved three-digit numbers, and children were given no materials or paper and pencil to use to figure out the answer. One problem involved finding how much of \$4.00 would be left after a purchase of \$1.86. The other was a missing-addend problem (A child has \$398.00. How much more would he have to save to have \$500.00?).

### *Procedures*

The Grade 1 interview was conducted in January and February, the Grades 2 and 3 fall interviews were conducted in October and November, and the Grades 2 and 3 spring interviews were conducted in March and April. All students were interviewed individually by trained interviewers in a quiet place removed from the classroom. The complete interview lasted approximately 1 hour and was administered over 2 days. Each task was read to the child by the interviewer and reread as often as the child wanted. The child was given as much time as necessary to solve each problem. Except as noted above, a variety of materials were available including individual counters, base-ten blocks,

and paper and pencil. When the child had given the answer, the interviewer questioned the child about the strategy used if it was not obvious from observing the solution of the problem. The interviewer continued probing until either he or she understood the solution or it was clear that the child was not going to provide additional information.

The interviewer recorded the answer in predetermined coding categories when possible, and the interviews were audiotaped. In cases in which a child's response did not fit into a well-defined coding category, the interviewer recorded detailed notes on the child's solutions. On the basis of those notes and the audiotaped protocols of the child's explanations, new categories were constructed or the existing categories were expanded to include the new cases.

## RESULTS

The pattern of development of children's use of invented strategies and standard algorithms is summarized in Table 1. These data represent the percentage of students who used an invented strategy or standard algorithm in the given interview or in a previous interview. In either the Grade 1 interview or the Grade 2 fall interview, before standard algorithms were discussed in most classes, 65% of the students used an invented addition strategy. An additional 23% started using invented strategies after algorithms were discussed, so that by the end of the study 88% of the students had used invented addition strategies. The use of invented subtraction strategies lagged behind the use of invented addition strategies. Only 15% of the students used invented subtraction strategies by the time of the Grade 2 fall interview, and by the end of the study 32% of the students had never used an invented subtraction strategy in any interview.

Table 1

*Cumulative Use of Invented Strategies and Standard Algorithms for Addition and Subtraction Problems (n=82)*

Interview Time	Addition		Subtraction	
	Invented <sup>a</sup>	Algorithm <sup>b</sup>	Invented <sup>a</sup>	Algorithm <sup>b</sup>
Grade 1	29	2	0	0
Grade 2 fall	65	20	15	4
Grade 2 spring	74	79	27	50
Grade 3 fall	82	90	55	66
Grade 3 spring	88	99	68	92

<sup>a</sup> Percentage of students who used an invented strategy in a given interview or did so in a preceding interview.

<sup>b</sup> Percentage of students who used an algorithm in a given interview or did so in a preceding interview.

Most students, however, did not use invented strategies in lieu of standard algorithms. By the end of the study, all but one student had used the standard addition algorithm for at least one problem, and all but seven students had used the standard subtraction algorithm. Of the seven students who never used the standard subtraction algorithm, four consistently used invented strategies successfully and three could not use either an invented strategy or the algorithm. Not surprisingly, the largest jump in the use of standard addition and subtraction algorithms came between the Grade 2 fall and spring interviews, when the standard algorithms were introduced in most classes.

### *Relating the Development of Invented Strategies and Standard Algorithms*

A primary consideration in examining the role of invented strategies on the development of understanding of multidigit procedures is the sequence in which students started using invented strategies and standard algorithms. Our working hypothesis was that students who used invented strategies before or at the same time that they used standard algorithms would demonstrate higher levels of understanding students who started out using algorithms. In order to examine this question, we clustered the students into five groups:

1. One group was composed of 27 students who first used invented addition strategies and invented subtraction strategies in the same interview or in an interview preceding the interview in which they first used the corresponding standard algorithms. This group was called the invented-strategy group.
2. A second group was composed of 18 students who used algorithms before they used invented strategies for both addition and subtraction calculations. This group was labeled the algorithm group.

3. A third group was composed of 33 students who first used an invented addition strategy in the same interview or in an interview preceding the interview in which they first used the standard addition algorithm but used the subtraction algorithm before they used an invented subtraction strategy. This group was called the invented-addition-strategy group.

4. The fourth group was composed of the students who first used an invented subtraction strategy in the same interview or in an interview preceding the interview in which they first used the standard subtraction algorithm but used the addition algorithm before they used an invented addition strategy. Because only one student fell in this group it is not included in further discussion.

5. The final group included three students who demonstrated limited ability to use either invented strategies or algorithms. None of these students ever used an invented strategy, and none ever successfully used the standard subtraction algorithm.

Although the groups were defined by differences in the order in which students first used invented strategies and algorithms, the differences in initial strategy use were reflected in the use of invented strategies and algorithms throughout the study. Students in the invented-strategy group consistently relied more on invented strategies and less on algorithms than students in the algorithm group. The invented-addition-strategy group tended to fall between the other two groups in the use of invented strategies and algorithms.

In comparing the three groups' strategy use, it is informative to look at several critical times: (a) the beginning of second grade, before most students had started to use standard algorithms; (b) at the end of second grade, after the majority of students had started to use standard algorithms for at least some problems-, and (c) during third grade, when almost all students used standard algorithms, particularly for problems with larger numbers. The percentages of students in each group using invented strategies and algorithms at these critical points are summarized in Table 2.

Table 2  
*Percentage of Students in Each Group Using Invented Strategies and Standard Algorithms (n=82)*

Strategy/time	Group		
	Invented <i>n=27</i>	Invented addition <i>n=33</i>	Algorithm <i>n=18</i>
Used invented strategies before algorithms <sup>a</sup>			
Addition	100	100	0
Subtraction	100	0	0
Used an invented strategy in some interview			
Addition	100	100	61
Subtraction	100	65	28
Used invented addition strategies			
By fall of Grade 2	89	79	0
With three-digit addends <sup>b</sup>	89	89	39
Used algorithms by spring of Grade 2			
Addition	70	91	100
Subtraction	33	67	61
Used invented strategies in Grade 3 interviews			
Never	0	24	50
Only when materials not available	33	61	44
With a variety of problems	67	15	6

<sup>a</sup> The three groups are defined on the basis of these data.

<sup>b</sup> Used an invented strategy when shown a hypothetical student's sequential strategy for a three-digit problem and asked to use the same strategy for another problem.

*The invented-strategy group.* The 27 students in the invented-strategy group not only used invented strategies before they used standard algorithms, they tended to use invented strategies relatively early. Of the students in this group, 89% used an invented addition strategy by the beginning of the second grade. The students in this group did not, however, use invented strategies to the exclusion of standard algorithms. By the end of the second grade, 70% of them used the standard addition algorithm and they all used the standard addition algorithm for some problems by the end of the study. However, only 33% used the standard subtraction algorithm by the end of second grade, and 11% consistently used invented subtraction strategies and never used the standard subtraction algorithm in any interview. Almost all the students in this group used the standard algorithm to add 2 three-digit numbers, but 89% of them used the appropriate invented strategy for the problem in which they were shown a hypothetical student's sequential strategy for a three-digit problem and asked to use the same strategy for another problem. Furthermore, the students in this group did not abandon invented strategies as a strategy of choice once they learned algorithms. They all continued to use invented strategies throughout the 3 years of the study. In Grade 3, 33% of them used invented strategies only when paper and pencil were not available, but the other 67% chose to use invented strategies on other problems as well.

*The algorithm group.* The algorithm group, which included 18 students, was almost the mirror image of the invented-strategy group. In this group, 94% did not use an invented strategy in any of the first three interviews and used the standard addition algorithm in the third interview in the spring of Grade 2. Not only did students in this group learn standard addition and subtraction algorithms before they ever used the corresponding invented strategies, they continued to rely primarily on algorithms. In this group, 39% never used an invented addition strategy throughout the study, and 72% never used an invented subtraction strategy. Generally the students in the group who did use invented strategies used them only if paper-and-pencil calculations were not an option. Furthermore, 50% of the students did not use any invented strategies in either of the third-grade interviews, and 44% used them only when paper and pencil were not available. Only one student used invented strategies on other problems. Not only did students in this group not choose to use invented strategies, many of them were unable to use them when asked to do so. Only 39% of the students in this group used an invented strategy on the problem in which they were shown a hypothetical student's sequential strategy for a three-digit problem and asked to use the same strategy for another problem.

*The invented-addition-strategy group.* The 33 students in the invented-addition-strategy group generally tended to use invented strategies less and algorithms more than students in the invented-strategy group. Most of the students used an invented addition strategy about as early as the students in the invented-strategy group, but they more readily abandoned invented strategies for the standard algorithm. In the fall of Grade 2, 79% used an invented addition strategy, and the rest did so by the spring. Although their use of invented addition strategies preceded or coincided with their use of the standard addition algorithm 91% of the students in this group used the addition algorithm by the spring of Grade 2.

The students in this group tended to rely more on the standard subtraction algorithm than on invented subtraction strategies. In the Grade 2 spring interview, 67% of the students used the standard subtraction algorithm; 35% of the students never used an invented subtraction strategy in any interview. By the Grade 3 interviews, the students in this group tended to use invented strategies only when paper and pencil were not available for calculation, if at all. In this group, 24% of the students used no invented strategies in either Grade 3 interview, and 61% used them only when paper and pencil were not available. Only 15% used them on other problems. All the students used the standard algorithm to calculate the sum of 2 three-digit numbers, but they were as successful as the invented-strategy group in using an invented strategy for three-digit calculations if required to do so. When students were shown a hypothetical student's sequential strategy for a three-digit problem and asked to use the same strategy, for another problem, 89% used the appropriate invented strategy.

*Knowledge of base-ten number concepts and performance on the extension problems.* There were significant differences between the two invented-strategy groups and the algorithm group in knowledge of base-ten number concepts and performance on the extension problems (Table 3). The use of invented strategies by students in the two invented-strategy groups generally was supported by relatively early knowledge of base-ten number concepts: By the beginning of the second grade, significantly more students in both invented-strategy groups met the criteria for demonstrating knowledge of base-ten number concepts than was the case for students in the algorithm group. By the spring of the second grade, almost all the students in both invented-strategy groups had demonstrated knowledge of base-ten number concepts, whereas one third of the students in the algorithm group still had failed to do so. These

differences also were significant. Both invented-strategy groups also had significantly higher performance on the extension problems than the algorithm group for both Grade 3 interviews.

Table 3  
*Performance of Each Group on Base-Ten and Extension Problems (n=82)*

Problems/interview	Group		
	Invented	Invented addition	Algorithm
Base-ten knowledge <sup>a</sup>			
Grade 2 fall	81*	85*	22
Grade 2 spring	96*	97*	67
Extension problems <sup>b</sup>			
Grade 3 fall	1.15*	.88*	.17
Grade 3 spring	1.37*	1.15*	.44

<sup>a</sup> Percentage of students in each group meeting criteria for demonstrating knowledge of base-ten concepts by the Grade 2 fall interview. Dunn-Bonferroni planned comparisons were used to test three pairwise contrasts of proportions for significance. An  $\alpha$  of .05 was split evenly among the three contrasts so that each contrast was tested at .017.

<sup>b</sup> Mean for each group on the extension problems. Dunn-Bonferroni planned comparisons were used to test three pairwise contrasts for means of significance. An  $\alpha$  of .05 was split evenly among the three contrasts so that each contrast was tested at .017.

\*Significantly different from the algorithm group ( $p < .05$ ).

*Buggy algorithms.* The percentage of students in each of the three groups using buggy algorithms is summarized in Table 4. Subtraction bugs were more frequent than addition bugs and appeared over a more sustained period of time. No student used a buggy addition algorithm in more than one interview, whereas over 30% of the students did so for subtraction. There was relatively little difference in the use of addition bugs for the three groups, but significantly more students in the algorithm group used buggy procedures in more than one interview than was the case for the invented-strategy group. Although buggy subtraction algorithms were less prevalent for the invented-strategy group, a number of students in all groups used buggy algorithms. In many cases bugs appeared in the same interview in which students used invented strategies on similar problems.

Table 4  
*Percentage of Students in Each Group Using Buggy Algorithms (n=82)*

Buggy algorithm	Group		
	Invented	Invented addition	Algorithm
Addition <sup>a</sup>	30	35	38
Subtraction <sup>b</sup>			
None	48	26	22
One	41	32	28
More than one	11*	38	50

<sup>a</sup> Percentage of students in each group using a buggy algorithm on one or more problems.

<sup>b</sup> Percentage of students in each group (a) never using a buggy algorithm, (b) using a buggy algorithm in one interview, or (c) using a buggy algorithm in more than one interview. Three pairwise contrasts compared the invented, invented-addition, and algorithm groups for no bugs, and three pairwise contrasts compared the three groups for bugs in more than one interview. Dunn-Bonferroni planned comparisons were used to test the six contrasts of proportions for significance. An  $\alpha$  of .05 was split evenly among the six contrasts so that each contrast was tested at .008.

\*Significantly different from the algorithm group ( $p < .05$ ).

### *Other Sequences of Development*

The above analyses are based on the sequence in which children first used invented strategies and algorithms. Other issues regarding sequences of development are of theoretical interest as well. One involves the question of whether children must develop a good understanding of base-ten number concepts before they can use invented strategies. Another involves the question of whether certain types of invented strategies emerge before others.

*Base-ten number concepts and invented strategies.* The data summarized in Table 3 suggest that the use of invented strategies is related to the development of base-ten number concepts. Children who use invented strategies develop knowledge of base-ten number concepts earlier than children who rely more on standard algorithms. These data do not, however, resolve the question of whether the use of invented strategies depends on prior knowledge of base-ten number concepts. In fact, many students used invented strategies before they demonstrated knowledge of base-ten number concepts on other tasks. Of the 82 children in the study, 23 used invented strategies at least one interview before they met the criteria for demonstrating knowledge of base-ten number concepts, 21 demonstrated knowledge of base-ten number concepts before they used an invented strategy, 33 students used invented strategies and demonstrated knowledge of base-ten number concepts concurrently, and 5 neither used invented strategies nor demonstrated knowledge of base-ten number concepts.

*Distinctions among invented strategies.* A distinction is made in the literature between types of invented strategies and the underlying number concepts (Fuson, 1992a). The distinction rests on whether multidigit numbers are conceived primarily in terms of collections of units of 10 or as chunks within the number-word sequence (10, 20, 30; 16, 26, 36). A fully integrated conception of number includes both conceptions, but it is an open question which develops first and under what circumstances. A related question is whether one type of invented strategy emerges earlier than another: the combining-units-separately strategy, which is related to the collected-multidigit construct, or the sequential strategy, which is related to the sequence-multiunit construct.

Our data suggest that there is no explicit sequence in which invented strategies develop for addition problems, and most children tended to use them somewhat interchangeably. Of the 72 students who used invented addition strategies, 10 students used only sequential strategies, 15 used only combining-units-separately strategies, and 47 students used both types of strategies at one time or another. Fourteen of these 47 students used a sequential strategy before they combined units separately, 22 combined units separately before they used a sequential strategy, and 11 used both strategies for the first time during the same interview.

For subtraction, the picture is somewhat different. Relatively few students combined units separately for subtraction; no more than 4 students used this strategy on any problem. However, whereas virtually all addition invented strategies could be classified as either a combining-units-separately or sequential strategy, 10 students used compensating strategies for some subtraction problems. Although compensating strategies were used relatively infrequently, a number of students could use them if asked specifically. In the Grade 3 spring interview students were shown an example of a hypothetical student using a compensating strategy for a given subtraction problem and were asked to use the same strategy on a related problem. Under these conditions, 40 students used the compensating strategy.

## DISCUSSION

We started this paper with the proposition that invented strategies could be taken as representing understanding of multidigit operations. We now return to that proposition and consider the evidence from this study that supports it. First, the evidence suggests that invented strategies are closely connected to other concepts that might be taken as representing basic understanding, in particular the development of fundamental base-ten number concepts. Students who initially used invented procedures demonstrated knowledge of base-ten number concepts before students who relied primarily on algorithms. Second, invented strategies demonstrate a hallmark characteristic of understanding. Children who use them are able to use them flexibly to transfer their use to new situations as demonstrated by the fact that students in the invented-strategy groups were significantly more successful in solving the extension problems than students in the algorithm group. Finally, the students in the invented-strategy groups demonstrated significantly fewer

systematic errors than students in the algorithm group. These results are consistent with the theoretical perspective that supports the development of understanding before mastery of procedures (Hiebert, 1986; Hiebert & Carpenter, 1992).

It could be argued that the success of the invented-strategies groups in demonstrating knowledge of number concepts and solving extension problems simply reflected differences in ability between them and students in the algorithm group. Although that possibility cannot be discounted, that perspective is based on assumptions about how mathematical ability might be characterized. In practice, computational skill often is taken as the measure of mathematical ability. Students in the algorithm group actually learned algorithmic procedures earlier than students in the invented-strategy group. The limitation was that most of them could not use their procedures flexibly to solve the extension problems. Even if ability is an issue, it could be argued that the strategies used by the students who had the most success in applying their knowledge flexibly should serve as a model for the learning of all students.

#### *Development of Underlying Base-Ten Number Concepts*

Although the development of base-ten number concepts and the ability to use invented strategies were related, a number of students used invented strategies before they demonstrated knowledge of base-ten number concepts on the problems used to measure base-ten knowledge. A caveat is in order in interpreting these results. First, although the problems used to assess knowledge of base-ten number concepts are similar to problems frequently used for this purpose, they are not a perfect measure of base-ten number knowledge. A case can be made that the use of invented strategies itself is a measure of students' understanding of base-ten number concepts. It certainly appears to be a measure of students' ability to apply this knowledge. Most of the base-ten number-concept tasks seemed to involve concepts that students would need in order to use invented strategies successfully. Some students may have been more successful in using their base-ten knowledge in constructing invented strategies than in solving these tasks assessing base-ten knowledge. Some students may have been capable of using the strategies required to demonstrate knowledge of base-ten number concepts but chose to use a more concrete strategy on those particular tasks.

Whatever the case, it seems that at least some students are sufficiently tenuous in their response to problems assessing basic knowledge of base-ten number concepts that they do not appear to have mastered those concepts, but at the same time they are capable of using them in invented strategies for addition and subtraction problems. These results tend to support Fuson's (1990) contention that learning to add and subtract multidigit numbers can contribute to the development of base-ten number concepts.

#### *Student Errors*

Students in the invented-strategy group demonstrated a fundamental understanding of base-ten number concepts and multidigit operations, and it is not surprising that they used fewer buggy algorithms than students in the other groups. It is striking, however, that a substantial number of students in all groups, including the invented-strategy group, used buggy algorithms for some problems in at least one interview. Fuson and Briars (1990) and Resnick and Omanson (1987) also found that students who had developed understanding of multidigit operations occasionally still used buggy algorithms. Resnick and Omanson hypothesized that buggy procedures are constructed within the realm of procedural knowledge and can coexist with a reasonable level of understanding of relevant concepts. But Fuson and Briars found the bugs used by students who had a reasonable understanding of multidigit operations were not robust and often could be eliminated by simple counter suggestion. The fact that only three students in the invented-strategy group used buggy algorithms in more than one interview is consistent with this finding.

#### *Effects of Instruction*

The characterization of patterns of development observed in this study cannot be generalized to all students. Clearly, instruction has a significant impact on what students learn and what strategies they use. Students in classes in which algorithms are the focus and in which no opportunity is provided for discussion of alternative strategies would undoubtedly show markedly different patterns, as would students in classes in which students had no exposure to standard algorithms. We are not proposing that this study maps out an invariant pattern of development. It does, however, provide an existence proof that children can invent strategies for adding and subtracting, what that invention affords, and the role that different concepts may play in that development.

### *Implications for Instruction*

Instruction was not a focus of this study and the study says very little about how students actually learned to use invented strategies. Students in this study were given opportunities to use and discuss alternative strategies for solving all problems, and open discussion of alternative strategies characterizes other classrooms in which there is widespread use of invented strategies (Carpenter et al., 1994). We have conducted year-long case studies of instruction and learning in several first-grade classrooms (Carpenter, Levi, Fennema, Ansell, & Franke, 1995). The findings of those studies suggest that invented strategies develop as abstractions of children's solutions using tens materials and that they are constructed over time as children discuss and reflect upon their solutions with the tens materials.

Although we have used the term *invented strategy* throughout this paper, we are not proposing that all children individually constructed their own invented strategies in a vacuum. From the case studies and from the classroom observations conducted in relation to a larger study of which this study was a part (Fennema et al., 1996), we know that is not the case. The strategies were constructed in a social context in which students shared strategies with one another. However, none of the teachers made an explicit effort to teach a particular invented strategy or gave any one invented strategy a place of prominence. Nevertheless, these strategies could be taught in the same way that standard algorithms are taught, and a case might be made that some version of these strategies would be an improvement over current standard algorithms in that the connections to fundamental base-ten concepts are more apparent.

Although we have no data regarding explicit instruction on specific invented strategies, we hypothesize that direct instruction could change the quality of children's understanding and use of invented strategies. If these strategies were the object of direct instruction, there would be a danger that children would learn them as rote procedures in much the way that they learn standard algorithms today. In this study, we saw relatively few conceptual errors in using invented strategies, whereas children exhibited a number of buggy procedures in using standard algorithms. The children in this study had the latitude to adapt their strategies to their level of understanding of base-ten number concepts and to use invented strategies that made sense to them. This probably would not be the case if all children were expected to use the same strategy at the same time.

Most of the students in this study were exposed to standard algorithms during the second and third grades. In working with their teachers, we remained neutral on whether they should teach standard algorithms. We proposed that invented strategies could be viewed as a means for developing understanding of multidigit concepts and procedures or as an end in their own right. Because of various pressures, most of the second- and third-grade teachers opted for standard algorithms at some point. We are not proposing, however, that as the only choice. On the one hand, a strong case could be made for not teaching the algorithm at all, given the widespread availability of calculators for more complex calculations or calculations requiring speed and efficiency. On the other hand, the results of this study illustrate that invented strategies can provide a basis for developing understanding of multidigit operations, even when algorithms are taught.

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### Authors

**Thomas P. Carpenter**, Professor of Curriculum and Instruction, University of Wisconsin-Madison, 1025 W. Johnson St., Madison, WI 53706

**Megan L. Franke**, Assistant Professor of Education, University of California-Los Angeles, 405 Hilgard Ave., Los Angeles, CA 90024-1521

**Victoria R. Jacobs**, Assistant Professor of Education, California State University-San Marcos, San Marcos, CA 92096-0001

**Elizabeth Fennema**, Professor Emerita, Department of Curriculum and Instruction, and Senior Scientist, Wisconsin Center for Education Research, University of Wisconsin-Madison, 1025 W. Johnson St., Madison, WI 53706

**Susan B. Empson**, Assistant Professor of Curriculum and Instruction, University of Texas at Austin, Austin, TX 78712-1294