An alternative approach to designing stabilizing compensators for saturating linear time-invariant plants

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SUMMARY

We present a new methodology for designing low-gain linear time-invariant (LTI) controllers for semi-global stabilization of an LTI plant with actuator saturation, which is based on the representation of a proper LTI feedback using a pre-compensator-plus-static-output-feedback architecture. We also mesh the new design methodology with time-scale notions to develop lower-order controllers for some plants. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Low-gain output-feedback stabilization of linear time-invariant (LTI) plants subject to actuator saturation has been achieved using the classical observer-followed-by-state-feedback controller architecture (see e.g. [1–3], see also the text [4] for a thorough overview). In this note, we discuss an alternative controller architecture for designing low-gain output-feedback control of LTI plants with saturating actuators. Specifically, we use a classical result of Ding and Pearson to show that a dynamic pre-filtering together with static-output-feedback architecture can naturally yield a stabilizing low-gain controller under actuator saturation (Section 2). Subsequently, by using time-scale notions, we illustrate through a single-input single-output (SISO) example that lower-order controllers can be designed in the case where the \( j\omega \)-axis eigenvalues of the plant are in fact at the origin (Section 3).

The reader may wonder what advantage the alternate architecture provides. Fundamentally, the applicability of our alternate design stems from the fact that we can choose the pre-compensator dynamics \textit{arbitrarily} without consideration of the plant in the design process, and it is only in the static controller design step that the plant structure need be considered. This is in contrast to the standard observer-based design, where building the (dynamic) estimator requires intimate knowledge of the plant. This freedom allows us to postulate the architecture of the closed loop even when the full plant model is not known or cannot be used, such as in decentralized and adaptive control applications [5–10].

Our particular motivation for developing the alternative low-gain architecture stems from our ongoing efforts on decentralized controller design, and in particular our effort to develop a low-gain methodology for
decentralized plants [5–10]. What these studies make clear is that freedoms in the structure of the controller facilitate design, because they can permit design that fits the structural limitations of the problem (in this case, decentralization). In pursuing decentralized control, we have in fact needed to use several novel controller structures, in particular ones that utilize pre-compensators and output-derivative approximations together, see [7–10]. These various architectures are compelling for decentralized controller design problems, because they do not require the design of dynamic state estimators; instead, an appropriate pre-filter form can be postulated, and then the design of controller parameters for stabilization and performance shaping (including under saturation) can be pursued directly. With this decentralized controller design application (as well as adaptive control and lifting applications in mind [9]) in mind, we here give a comprehensive introduction to one key alternative controller architecture.

We note for the sake of completeness that low-gain methodologies are only one aspect of controller design for plants with saturating actuators. In particular, performance objectives can be met in controlling saturating plants through the application of low- and high-gain designs [11], or alternately through a posteriori shaping of the dynamics using anti-windup control schemes [12]. We stress, however, that low-gain design is the primary tool for the core stabilization under saturation—which has not yet been addressed for the decentralized controls applications of interest to us—and hence we focus on low-gain design.

2. LOW-GAIN OUTPUT-FEEDBACK CONTROL THROUGH PRE-COMPARTMENTATION

In this section, we demonstrate the design of low-gain proper controllers for semi-global stabilization of LTI plants subject to actuator saturation, using a novel pre-compensator-based architecture. We also briefly discuss the connection of our design to the traditional observer-based design, and expose that the design is deeply related to a family of pre-compensator-based designs that also permit, e.g. zero cancellation and relocation.

Formally, we demonstrate the design of a proper output-feedback compensator that achieves semi-global stabilization of the following plant $\mathcal{G}$:

$$\dot{x} = Ax + B\sigma(u)$$
$$y = Cx$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $\sigma(\cdot)$ is the standard saturation function, $A$ has eigenvalues in the closed left-half-plane (CLHP), and the triple $(C, A, B)$ is observable and controllable.\(^\dagger\) Our design is fundamentally based on (1) positing a control architecture comprising a pre-compensator with a zero-free and uniform-rank structure together with a feedback of the output and its derivatives (see Figure 1); Figure 2 designing the controller using this architecture; and Figure 3 arguing that the designed controller admits a strictly proper feedback implementation. This controller design directly builds on two early results: (1) Ding and Pearson’s result [13] for pole placement that is based on a dynamic pre-compensation + static-feedback representation of a proper controller (Figures 1(a) and 1(b)) and (2) Lin and Saberi’s effort [1] on stabilization under saturation using state feedback. For clarity, we cite the two results in the lemmas before we present our main result.

Lemma 1 concerns pre-compensator and feedback design for pole placement in a general LTI system, i.e. one of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$.

Lemma 1

Consider a plant of the form (2) that is controllable and observable, with observability index $v$. Pre-compensation through the addition of $v-1$ integrators to each plant input permits computation of the plant’s state $\mathbf{x}$ as a linear function of the plant’s output $\mathbf{y}$, its derivatives up to $y^{(v)}$ and the pre-compensator’s state.

\(^\dagger\)In fact, the methods developed here trivially generalize to the case where the dynamics are stabilizable and detectable. We consider the observable and controllable case for the sake of clarity.
Figure 1. Compensator architectures: (a) and (b) show the compensator architectures presented in Ding and Pearson [13]; in particular, a pre-compensator together with static-feedback viewpoint. (b) is used to design a proper compensator of (a). (c) and (d) show the compensator architectures that stabilize a plant under input saturation.

A consequence of this computation capability is that it permits design of a strictly proper feedback controller $C(s)$ that places the poles of the compensated plant at arbitrary locations (that are closed under conjugation).

When the matrix $C$ in the system (2) is not invertible, the classical method to obtain the state information from output is through observer design. This lemma of Ding and Pearson gives an alternative design for state estimation and feedback controller design, that is based on viewing certain proper compensators $C(s)$ as a dynamic pre-compensation together with static feedback (Figure 1(a) and (b)). Specifically,
the methodology of design is as follows: first, from the pre-compensator-based representation (Figure 1(b)), a computation of the plant state from the plant output and its derivatives together with the pre-compensator state can be directly obtained. Second, the classical state-feedback methodology thus permits us to compute the static feedback in the pre-compensator-based representation, so as to place the closed-loop eigenvalues at desired locations. Third, the equivalence between the pre-compensator-based representation and a proper feedback controller is used to obtain a realization of the feedback control (Figure 1(a)). We kindly ask the reader to see [13, 14], both for the details of the state computation and the equivalence between the pre-compensator-based architecture and the proper feedback controller. In our development, we broadly replicate the design methodology of Ding and Pearson, but use a stable rather than neutral pre-compensator in order to obtain a controller that works under input saturation. We stress that, while Ding and Pearson’s methodology (as well as our methodology, see below) is based on the concept of output-derivative feedback, the final compensator obtained through the design is a proper one—an output-derivative feedback viewpoint is helpful for conceptualizing the design, but derivatives do not explicitly need to be computed.

Lemma 2 is concerned with using linear state-feedback control to semi-globally stabilize the plant

\[ \dot{x} = Ax + B\sigma(u) \quad (3) \]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \( \sigma() \) is the standard saturation function. Please see Lin and Saberi’s work [1] for the proof of the lemma.

Lemma 2

Consider a plant of the form (3) that satisfies two conditions: (1) all the eigenvalues of \( A \) are located in the CLHP and (2) \( (A, B) \) is stabilizable. Then the plant can be semi-globally stabilized using linear static state feedback. That is, a parameterized family of compensators \( u = K(\varepsilon)x \) can be designed such that, for any specified ball of plant initial conditions \( \mathcal{W} \), there exists an \( \varepsilon^*(\mathcal{W}) \) such that, for all \( 0 < \varepsilon \leq \varepsilon^*(\mathcal{W}) \), the compensator \( K(\varepsilon) \) achieves local exponential stabilization of the origin and contains \( \mathcal{W} \) in its domain of attraction.

Now we are ready to present the main result. Specifically, the following theorem formalizes that a family of proper controllers can be designed for semi-global stabilization of \( \mathcal{W} \), based on the pre-compensator together with derivative-feedback architecture shown in Figure 1. The proof of the theorem makes clear the design methodology.

**Theorem 1**

The plant \( \mathcal{W} \) (Equation (1)) can be asymptotically semi-globally stabilized using proper feedback compensation of order \( mv \), where \( v \) is the observability index of the plant. Specifically, a parameterized family of compensators \( C(s, \varepsilon) \) can be designed (Figure 1(c)) to achieve the following: for any specified ball of plant and compensator initial conditions \( \mathcal{W} \), there exists an \( \varepsilon^*(\mathcal{W}) \) such that, for all \( 0 < \varepsilon \leq \varepsilon^*(\mathcal{W}) \), \( C(s, \varepsilon) \) makes the origin locally exponentially stable and contains \( \mathcal{W} \) in its domain of attraction. The design can be achieved by developing a controller of the architecture shown in Figure 1(d)—i.e. comprising an \( m \)-input uniform rank, square invertible, zero-free pre-compensator \( P \) with input \( u_p \) together with a feedback of the form \( u_p = K_0(\varepsilon)y + K_1(\varepsilon)y^{(1)} + \cdots + K_{v-1}(\varepsilon)y^{(v-1)} \) (where \( K_0(\varepsilon), \ldots, K_{v-1}(\varepsilon) \) are matrices of dimension \( m \times p \)—and then constructing a proper implementation.

**Proof**

We shall prove that, for the given ball of initial conditions, a family of proper compensators \( C(s, \varepsilon) \) can be designed so that the actuator does not saturate, and further the closed-loop system without saturation is exponentially stable. Together, these two aspects show that the origin is locally exponentially stable with \( \mathcal{W} \) in the domain of attraction. We first note that, as long as the compensator permits a proper state-space implementation and the system operates in the linear regime, the additive contribution of the compensator’s initial condition on the input can be made arbitrarily small through pre- and post-scaling of the compensator by a large gain \( \Gamma \) and its inverse (see Figure 1). Thus, without loss of generality (WLOG), we seek to verify that \( \|u\|_\infty < 1 \) for the ball of initial states and assuming null compensator initial conditions. To do so, we will design a compensator of the architecture shown in Figure 1 that achieves the design goals, and then note a proper implementation.
To do this, let \( \tilde{P} \) be any asymptotically stable LTI system of the following form:

\[
\begin{bmatrix}
    y_P^{(1)} \\
    y_P^{(2)} \\
    \vdots \\
    y_P^{(v)}
\end{bmatrix} =
\begin{bmatrix}
    I_m & & & \\
    & \ddots & & \\
    & & I_m & \\
    0 & \ldots & \tilde{Q}_0 & \tilde{Q}_{v-1}
\end{bmatrix}
\begin{bmatrix}
    y_P \\
    y_P^{(1)} \\
    \vdots \\
    y_P^{(v-1)}
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    \vdots \\
    0 \\
    I_m
\end{bmatrix} u_P
\]

where \( u_P \in \mathbb{R}^m \) and \( y_P \in \mathbb{R}^m \) are the input and output to \( \tilde{P} \). Notice that \( \tilde{P} \) is square invertible, zero free, and uniform rank. Let us denote the \( \infty \)-norm gain of this plant as \( q \).

Let us first consider pre-compensating the plant \( G \) using \( \tilde{P} \), and then using feedback of the first \( v \) derivatives of the output along with the states of the pre-compensator (see Figure 1). That is, upon pre-compensation with \( \tilde{P} \), we consider using a feedback controller of the form

\[
u_P = \sum_{i=0}^{v-1} K_i y_P^{(i)} + \sum_{i=0}^{v-1} \tilde{K}_i y_P^{(i)},
\]

where we have presciently used the notation \( K_i \) for the output-derivative feedbacks since these will turn out to be the gains in the compensator diagrammed in Figure 1(d), and where we suppress the dependence on \( \varepsilon \) in our notation for the sake of clarity. For convenience, let us define \( K = [K_0 \ldots K_{v-1}] \), \( \tilde{K} = [\tilde{K}_0 \ldots \tilde{K}_{v-1}] \)

\[
y(\text{ext}) = \begin{bmatrix}
    y \\
    y^{(1)} \\
    \vdots \\
    y^{(v-1)}
\end{bmatrix}
\quad \text{and} \quad
y_P(\text{ext}) = \begin{bmatrix}
    y_P \\
    y_P^{(1)} \\
    \vdots \\
    y_P^{(v-1)}
\end{bmatrix}
\]

In this notation, the controller becomes

\[
u_P = [K \; \tilde{K}] \begin{bmatrix}
y(\text{ext}) \\
y_P(\text{ext})
\end{bmatrix}
\]

We claim that such a controller can be designed, so that (1) the closed-loop system without saturation is exponentially stable; (2) \( \|u_P\|_\infty \leq \varepsilon \) for the given ball of plant initial conditions and any \( 0 < \varepsilon \leq 0.9 \); and (3) the controller gains \( K \) and \( \tilde{K} \) are \( \mathcal{C}(\varepsilon) \). To see why, first note that, based on the fact that the relative degree of the pre-compensator equals the observability index, the state of the pre-compensated system \( \tilde{x} = [x \; y_P(\text{ext})] \) is a linear function of \( [y_P(\text{ext})] \). In particular, it is automatic that

\[
\tilde{x} = \begin{bmatrix}
x \\
y_P(\text{ext})
\end{bmatrix} = Z \begin{bmatrix}
y(\text{ext}) \\
y_P(\text{ext})
\end{bmatrix}
\]

where \( Z \) has the form \( \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \), see Ding and Pearson’s development [13] for the method of construction. Next, from Lemma 2, we see that a low-gain full state-feedback controller \( \tilde{K}(\varepsilon) \) of order \( \varepsilon \) can be developed for the pre-compensated plant, which achieves local exponential stabilization of the origin and also makes the \( \infty \)-norm of the input less than \( \varepsilon \) for any \( \varepsilon > 0 \), for the given ball of plant initial conditions. Thus, by applying the feedback

\[
\tilde{K}(\varepsilon) Z \begin{bmatrix}
y(\text{ext}) \\
y_P(\text{ext})
\end{bmatrix}
\]

we can meet the three desired objectives.

It remains to be shown that the plant input \( u \) does not saturate on the application of this compensation. To do so, simply note that \( \|u\|_\infty \leq q \|u_P\|_\infty \leq 0.9 \).

We can absorb the feedback of \( y_P(\text{ext}) \) into the pre-compensator, so that we obtain a control scheme comprising a pre-compensator \( P \) with dynamics

\[
y_P^{(1)} = \begin{bmatrix}
    I_m \\
    \vdots \\
    \tilde{Q}_0 + \tilde{K}_0 \\
    \vdots \\
    \tilde{Q}_{v-1} + \tilde{K}_{v-1}
\end{bmatrix}
y_P
\]

\[
y_P^{(2)} = \begin{bmatrix}
    I_m \\
    \vdots \\
    0 \\
    \vdots \\
    0
\end{bmatrix} u_P
\]

\[
y_P^{(v-1)} = \begin{bmatrix}
    0 \\
    \vdots \\
    0 \\
    I_m
\end{bmatrix}
\]

In this notation, the controller becomes

\[
u_P = \sum_{i=0}^{v-1} K_i y_P^{(i)}.
\]
Finally, exactly analogously to the design method in [13], we see automatically that the transfer function from $y$ to $u$ is in fact strictly proper, and hence the design admits a proper state–space implementation. Through appropriate scaling of the compensator, we thus see that saturation is avoided for the ball of plant and compensator initial conditions, while the dynamics without saturation are exponentially stable. Thus, semi-global stabilization has been verified.

We have given an alternative low-gain controller design for semi-global stabilization under saturation. The crux of the design in concept is the ability to construct the plant’s full state as a static mapping of output derivatives together with pre-compensator variables, upon adequate dynamic pre-compensation. This observation yields a design strategy where a dynamic pre-compensator’s impulse response is designed followed by a low-gain static state feedback, with the goal of ensuring that the output of their cascade is small (for the given ball of initial conditions). This is a different viewpoint from the traditional one in limited-actuation output-feedback design [1, 2, 6], where actuation capabilities are divided between the observation and the state-feedback tasks. We again stress that, although derivatives of the output are conceptualized as being used in feedback, the entire design admits a proper implementation and derivatives do not need to be computed in practice.

**Remark**

By choosing the $\tilde{Q}_i$ appropriately, we can set the gain $q$ of the pre-compensator $P$ to an arbitrary value. An appropriate selection of the pre-compensator can potentially facilitate the selection of more numerically stable feedback gains, by permitting a larger input prior to the pre-compensator. We leave a careful analysis to the future work.

### 3. A COMPENSATOR THAT EXPLOITS TIME-SCALE STRUCTURE

Our philosophy for low-gain control using a pre-compensation-plus-feedback architecture also permits the construction of stabilizers that exploit time-scale structure in the plant. Specifically, we here demonstrate the design of pre-compensators for semi-global stabilization of the plant $G$, which are generally lower order than those in Section 2 because they exploit time-scale separation in the plant. Conceptually, when stabilization under saturation is the goal, low-gain state feedback only need be provided for the plant dynamics associated with $j\omega$-axis eigenvalues (see e.g. [6] for the use of this idea in observer-based designs). In the case where these eigenvalues are at the origin, the corresponding dynamics are in fact the slow dynamics of the system. Thus, through time-scale separation, we can design pre-compensation together with feedback so as to stabilize the slow dynamics under actuator saturation, and then obtain a proper feedback implementation. The use of time-scale separation ideas in the pre-compensator-based design becomes rather intricate, and hence we illustrate the design only for SISO plants for the sake of clarity. We shall use standard singular-perturbation notions to prove the result. Here is a formal statement:

**Theorem 2**

Consider a plant $G$ (as specified in Equation (1)) that further is SISO, and has $q$ poles at the origin. This plant can be semi-globally stabilized under actuator saturation using a proper dynamic compensator of order $q$.

**Proof 2**

We shall prove that, for any given ball of plant and controller initial conditions, there exists a proper compensator of order $q$ such that (1) the closed-loop system without saturation is exponentially stable and (2) actuator saturation does not occur. Together, these observations yield that the origin is locally exponentially stable for the given ball of initial conditions, and hence that semi-global stabilization is achieved.

To this end, let us begin by denoting the transfer function of the plant’s linear dynamics by $G(s)$. We note that the transfer function can be written as

$$G(s) = \frac{b_ms^m + \cdots + b_0}{s^q(s^n - q + a_n - q - 1s^{n-q} - 1 + \cdots + a_0)}$$

where $m$ is the number of plant zeros. We find it easiest to conceptualize the compensator as comprising a
zero-free dynamic pre-compensator of order \( q \), together with a feedback of the output \( y \) and its first \( q-1 \) derivatives, as shown in Figure 1. We choose the pre-compensator to be any stable system of this form. We denote the pre-compensator’s transfer function by

\[
C_p(s) = \frac{1}{(s^q + c_{q-1}s^{q-1} + \cdots + c_0)}
\]

and the feedback controller by

\[
K(s) = k_{q-1}s^{q-1} + \cdots + k_0.
\]

We note the entire compensator \( K(s)C_p(s) \) has order \( q \) and is proper.

With a little algebra, we find that the characteristic polynomial of the closed-loop system is

\[
p(s) = s^q + a_{n-q-1}s^{n-q-1} + \cdots + a_0(s^q + c_{q-1}s^{q-1} + \cdots + c_0) + (b_ms^m + \cdots + b_0)(k_{q-1}s^{q-1} + \cdots + k_0).
\]

We note that the polynomial \( p_f(s) = (s^q + a_{n-q-1}s^{n-q-1} + \cdots + a_0)(s^q + c_{q-1}s^{q-1} + \cdots + c_0) \) has roots in the OLHP, by assumption, for the chosen stable pre-compensator \( C_p(s) \).

Let us now consider a family of multiple derivative output feedbacks, parameterized by a low-gain parameter \( \varepsilon > 0 \). In particular, let us consider feedback with

\[
k_i = (a_0c_0/b_0)\gamma_i \varepsilon^{d-i},
\]

where \( c_0 = 1 \). In this case, a stable polynomial with roots \( \lambda_1, \ldots, \lambda_q \), and \( \varepsilon \) is a low-gain parameter.

We will verify that the characteristic polynomial has \( n \) roots that are within \( C(\varepsilon) \) of the roots of the feedback controller \( p_f(s) \), whereas the remaining \( q \) roots are within \( C(\varepsilon^2) \) of \( \varepsilon \lambda_1, \ldots, \varepsilon \lambda_q \). To prove this, notice first that

\[
p(s) = s^q + a_{n-q-1}s^{n-q-1} + \cdots + a_0(s^q + c_{q-1}s^{q-1} + \cdots + c_0) + (b_ms^m + \cdots + b_0)(k_{q-1}s^{q-1} + \cdots + k_0).
\]

Noting that the entire second term in this expression is \( C(\varepsilon) \), we see that the roots of \( p(s) \) are \( C(\varepsilon) \) perturbations of the roots of \( s^q p_f(s) \). Thus, we see that \( n \) roots are within \( C(\varepsilon) \) of the roots of \( p_f(s) \), whereas the remaining are within \( C(\varepsilon^2) \) of the origin.

To continue, let us consider the change of variables \( \tilde{s} = \varepsilon/s \). Substituting into the closed-loop characteristic polynomial, we find that

\[
p(\tilde{s}) = \frac{\varepsilon^q}{\tilde{s}^q} p_f(\tilde{s}) + (b_ms^m + \cdots + b_0)(\gamma_{q-1} \varepsilon^{q-1} + \cdots + \gamma_0 \varepsilon^d) a_0c_0/b_0.
\]

Scaling the expression by \( \varepsilon^{n-q}/a_0c_0 \), we obtain that the expression \( p(\tilde{s}) = 0 \) is the following degree- \( (n+q) \) polynomial equation in \( \tilde{s} : \varepsilon\gamma_q s^{q+n} + \cdots + \gamma_{q-1}s^{q-1} + \gamma_0 + r(\tilde{s}) = 0 \), where \( r(\tilde{s}) \) is a polynomial in \( \tilde{s} \) of degree no more that \( q + 1 \) with each term scaled by a coefficient of order \( \varepsilon^{d+1} \) or smaller. Thus, dividing by \( \varepsilon^d \), we find that the solutions \( \tilde{s} \) to the equation are within \( C(\varepsilon) \) of the solutions to \( \gamma_0 s^{q+n} + \cdots + \gamma_{q-1}s^{q-1} + \gamma_0 + r(\tilde{s}) = 0 \). However, the roots of this equation are precisely \( 1/\lambda_1, \ldots, 1/\lambda_q \), as well as \( 0 \) repeated \( n \) times. Noting that \( \tilde{s} = \varepsilon/\tilde{s} \), we thus recover that \( q \) roots of the characteristic polynomial are within \( C(\varepsilon^2) \) of \( \varepsilon \lambda_1, \ldots, \varepsilon \lambda_q \). Thus, we have characterized all the poles of the closed-loop system. We notice that all the poles are guaranteed to be within the OLHP.

Now consider the response for a ball of initial conditions \( \gamma \). As in the proof of Theorem 1, we notice that the initial state of the pre-compensator is of no concern in terms of causing saturation, since the pre-compensator can be pre- and post-scaled by an arbitrary positive constant. Thus, WLOG, let us consider selecting among the family of compensators, to avoid saturation for a given ball of plant initial conditions and assuming zero pre-compensator initial conditions. Through the consideration of the closed-loop dynamics associated with the slow eigenvalues \((\varepsilon \lambda_1, \ldots, \varepsilon \lambda_q)\), we recover immediately (see the proof of Lemma 1 in [1]) that, for any specified ball of initial conditions, \( \|y(t)\|_\infty \), is at most of order \( 1/\varepsilon^{d-1-i} \) for \( i = 1, \ldots, q-1 \). Thus, from the expression for the feedback controller, we find that the maximum value of the pre-compensator input \( \mathcal{V} \) is \( C(\varepsilon) \), say \( v_1 \varepsilon^2 + C(\varepsilon^2) \) for the given ball of plant initial conditions. Furthermore, the stable pre-compensator imparts a finite gain, say \( v_2 \); hence the maximum value of \( u(t) \) is \( v_1 v_2 \varepsilon + C(\varepsilon^2) \). Thus, for any given ball of initial conditions, we can choose \( \varepsilon \) small enough so that actuator saturation does not occur. As actuator saturation is avoided and the closed-loop poles are in the OLHP, stability is proved.

Conceptually, the reduction in the controller order permitted by Theorem 2 is founded on focusing the control effort on only the slow dynamics of the system. That is, the controller is designed only to place the eigenvalues at the origin at desired locations (that are
linear with respect to the low-gain parameter \( \varepsilon \); simply using small gains enforces that the remaining eigenvalues remain far in the OLHP. Thus, one only needs to add pre-compensation to permit the estimation of the part of the state associated with the slow dynamics. In this way, stability can be guaranteed and saturation avoided, without requiring as much pre-compensation as would be needed to estimate the whole state.

We notice that the time-scale-based design is aligned with the broad philosophy of our alternative low-gain design, in the sense that it provides freedom in compensator design. In particular, as with the design in Section 2, we notice that any stable pre-compensator can be used for the time-scale-exploiting design, and further design of feedback component in the architecture only requires knowledge of the DC gain of the plant.

4. EXAMPLE

In this example, we demonstrate the design of a low-gain controller that semi-globally stabilizes the following plant:

\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma(u) \quad (6)
\]

\[
y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \quad (7)
\]

Specifically, we show that the compensator of the form \( u(t) = \sum_{i=0}^{v-1} A_i u(t-i) + \sum_{i=0}^{(v-1)} B_i y(t-i) \), where \( v \) is the observability index of the plant, can stabilize the plant under saturation. As developed in the paper, the design is achieved by first designing a pre-compensator together with output-feedback control law, and then implementing the controller in the proper feedback representation above. We shall use the notation from the above development in our illustration.

Let us begin with the pre-compensator-plus-output-feedback design. To begin, we notice that the observability index of this system is 2. As per the proof of Theorem 1, let us thus choose \( \hat{P} \) to be

\[
\begin{bmatrix} \dot{y}_\hat{P} \\ \ddot{y}_\hat{P} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -I & -I \end{bmatrix} \begin{bmatrix} y_P \\ \dot{y}_P \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_P \quad (8)
\]

The eigenvalues of this system are located at \( -\frac{1}{2} \pm \sqrt{3}/2i \); hence, it is clearly asymptotically stable.

Now let us pre-compensate the plant (Equation (6)) using \( \hat{P} \) and design feedback of the form

\[
u_\hat{P} = \tilde{K}_0(\varepsilon) y_\hat{P} + \tilde{K}_1(\varepsilon) \dot{y}_\hat{P} + K_0(\varepsilon) y + K_1(\varepsilon) \dot{y} \quad (9)\]

to design the feedback, we first recover the state of the entire system including the pre-compensator from the outputs \( y_\hat{P}, \dot{y}_\hat{P}, y, \dot{y} \) through linear transformation, and then apply the low-gain state-feedback design (see [1]) to shift eigenvalues of the whole system left by \( -\varepsilon \). Doing so, we have

\[
\tilde{K}_0(\varepsilon) = \begin{bmatrix} -0.42\varepsilon^5 + 0.30\varepsilon^4 + 1.8\varepsilon^3 - 4.2\varepsilon^2 - 0.90\varepsilon \\ -0.11\varepsilon^5 + 0.28\varepsilon^4 - 0.55\varepsilon^3 - 6.0\varepsilon^2 - 2.2\varepsilon \end{bmatrix}
\]

\[
\tilde{K}_1(\varepsilon) = \begin{bmatrix} -0.0084\varepsilon^5 + 0.13\varepsilon^4 - 0.66\varepsilon^3 + 0.74\varepsilon^2 - 5.3\varepsilon \\ -0.0024\varepsilon^5 + 0.039\varepsilon^4 - 0.19\varepsilon^3 + 0.21\varepsilon^2 - 0.95\varepsilon \end{bmatrix}
\]

\[
K_0(\varepsilon) = \begin{bmatrix} -0.30\varepsilon^5 - 1.2\varepsilon^4 - 1.2\varepsilon^3 - 0.91\varepsilon^2 \\ -0.088\varepsilon^5 - 0.35\varepsilon^4 - 0.35\varepsilon^2 - 0.26\varepsilon \end{bmatrix}
\]

\[
K_1(\varepsilon) = \begin{bmatrix} 0.39\varepsilon^5 - 0.06\varepsilon^4 - 3.3\varepsilon^2 - 2.5\varepsilon^2 - 1.8\varepsilon \\ 0.091\varepsilon^5 + 0.020\varepsilon^4 - 0.95\varepsilon^3 - 0.72\varepsilon^2 - 0.53\varepsilon \end{bmatrix}
\]
Hence, the control scheme can be viewed as comprising a pre-compensator \( P \)
\[
\begin{bmatrix}
\dot{y}_P \\
\ddot{y}_P
\end{bmatrix} = \begin{bmatrix} 0 & I \\ -I + \tilde{K}_0(\varepsilon) & -I + \tilde{K}_1(\varepsilon) \end{bmatrix} \begin{bmatrix} y_P \\ \dot{y}_P \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_P
\]
(10)
and the feedback flow \( u_P = K_0(\varepsilon) y + K_1(\varepsilon) \dot{y} \).

Finally, the above procedure leads to the proper feedback compensator design
\[
u^{(2)}(t) = (-I + \tilde{K}_0(\varepsilon)) u(t) + (-I + \tilde{K}_1(\varepsilon)) u^{(1)}(t) + K_0(\varepsilon) y(t) + K_1(\varepsilon) y^{(1)}(t)
\]
(11)

Now let us show how \( \varepsilon \) can be chosen. WLOG, let us assume that the pre-compensator initial conditions to nil, with the understanding that scaling of the pre-compensator (with appropriately revised proper implementation) permits design with non-zero compensator initial conditions. In particular, consider the case where the initial conditions of the plant are in a ball \( \mathcal{W} \) with infinity-norm radius 1, i.e. where each initial condition has a magnitude less than or equal to 1. We find that \( \varepsilon(\mathcal{W}) \approx 0.5 \) through an exhaustive search. Thus \( \varepsilon \) can be chosen between 0 and 0.5. Trajectories of the two inputs are shown for an initial condition at the edge of the ball, for two different values of \( \varepsilon \).

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