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Constructing Consensus Controllers for Networks with Identical General Multi-Input Multi-Output Linear Agents

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Abstract

We use a high-gain methodology to construct linear decentralized controllers for consensus, in networks with identical but general multi-input multi-output (MIMO) linear time-invariant (LTI) agents and quite-general observation topology, including numerous time-invariant and time-varying ones.

1 Introduction

A multitude of networks in nature automatically synchronize, i.e. states of individual network components or agents dynamically evolve toward a common value or trajectory (that depends on the initial states of all the components). In complement, control-theorists have recently sought to develop decentralized algorithms/controllers that bring a network’s components into agreement, i.e. that deliberately drive the states of network components to a common value that depends on the initial component states in a prescribed manner.

Thus far, the literature on consensus control has been limited to the case where the agents’ open-loop internal dynamics are described by an integrator chain (e.g., single- or double-integrator models [1, 2, 3, 4, 5, 6, 7]). In this article, we address consensus control for networks whose agents have identical but arbitrary MIMO LTI open-loop dynamics. Specifically, we exploit a high-gain decentralized control scheme to obtain consensus for this general agent model, and for a broad class of network topologies.

Let us briefly overview the literature on consensus. We note that the consensus problem actually has a long history in the computer science community [8]. The control-theoretic approach to consensus - i.e., the

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use of a feedback methodology to synchronize agents’ local states in a network to a prescribed function of their initial states - is relatively new, but has been extensively studied in the control community during the last five years and has yielded some advances in e.g., sensor networking ([1, 2, 3, 4]) and autonomous vehicle control applications ([9, 5, 10, 7]). Although this literature is extensive, however, much of it fundamentally derives from a classical work of Chua ([11, 12]) that gives conditions on a network’s topology and agent dynamics for synchronization. Pogromsky ([13, 14]) has given a control-theoretic interpretation of the classical synchronization result, that captures the essence of the consensus problem. Beyond these core results, explicit design of controllers for consensus (i.e., design of controllers such that the closed-loop meets Chua’s condition) has been achieved for very simple agent models (integrator chains) and largely for simple network-topology models, e.g., those described by a Laplacian matrix [1, 2, 3, 5, 10, 7] (see our work [4, 9] for a more general network model). Also, consensus in networks with time-varying topologies has been studied extensively; we refer the reader to Blondel’s summary [15], which shows that general results in the time-varying case can be extracted from an early result of Tsitsiklis [16]. Yet another focus of the consensus literature has been on prescribing the dependence of the agreed-upon value on the initial conditions, or agreement-law design, see [4]. Finally, cursory studies of consensus under delay ([1, 2, 3, 17]) and actuator saturation [10] are available.

Noting that the ongoing research on consensus is progressing toward models of increasing generality (from first- [4, 1, 2, 3, 5] to second-order [9, 7] to integrator-chain internal dynamics [10]), we view the problem of constructing consensus controllers when the agents have general LTI internal dynamics as a key open problem. To this end, we develop decentralized controllers for consensus in a network of identical agents which have general MIMO LTI internal dynamics. Our design permits consensus for a very broad class of sensing/communication topologies (not only ones specified by Laplacian matrices). Also, we show how the agreement law can be assigned during consensus. To solve the consensus problem for general MIMO LTI agents, we apply a high-gain controller design methodology. This methodology provides a general approach to solving the consensus problem, and so in essence shows how the simultaneous-stabilization condition of Chua can be met when feedback control for the agents is permitted.

The remainder of the article is organized as follows. In section 2, we model in detail the agent internal dynamics, sensing/communication topology, controller architecture, and the consensus task considered in our development. In doing so, we also describe the sense in which our model generalizes and encompasses those in the literature. In section 3, we give network and agent theoretic conditions for completion of the consensus tasks using the described controller architecture for time invariant topology. In doing so, we draw extensively on the classical time-scale-based design of control systems, which permits us to study consensus in the broad class of network models introduced here. In section 4, we give network and agent theoretic conditions for
completion of the consensus task problem with varying topologies.

2 Problem Formulation

In this section, we introduce a general model for networked autonomous agents (Section 2.1), for which we seek consensus control. We then comprehensively introduce the consensus control problem, and present a controller architecture for achieving consensus (Section 2.2).

2.1 A Model for Networked Autonomous Agents

We study a network of identical agents with general linear time-invariant internal dynamics, that interact through an arbitrary linear observation topology. The autonomous-agent-network model that we introduce encompasses and generalizes many of the models considered in the consensus literature and more generally the autonomous-agent control literature (with respect to both the agents’ internal dynamics and their interactions). Of particular interest, it encompasses models for both distributed computational processes in networks (such as are used in sensor networking applications, see e.g. [1, 2, 3, 4]) and networks with mechanical or electromechanical hardware (such as autonomous-vehicle teams [9, 5, 10, 7]).

Here, let us describe the agents’ internal dynamics, their networked observations, and the framework for control in the model. Subsequently, we will also find it convenient to assemble the agents’ dynamics into a single state-space representation, and to introduce some terminology regarding the whole dynamics.

2.1.1 The Agent Model

We consider a network of \( N \) identical agents, which we label \( 1, \ldots, N \). We assume that each agent \( i \) has a local state \( \dot{x}_i \in \mathbb{R}^n \) which evolves in continuous time \( (t \in \mathbb{R}^+) \) according to the differential equation

\[
\dot{x}_i = \hat{A}x_i + \hat{B}u_i,
\]

where \( u_i \in \mathbb{R}^m \) is agent \( i \)’s local input. Without loss of generality, let us assume that the matrix \( \hat{B} \) has full column rank, and pair \((\hat{A}, \hat{B})\) is controllable.

2.1.2 Network Interactions

In many application areas, the fundamental challenge in achieving consensus among autonomous agents stems from the decentralization of the agents’ observations, i.e., from the fact that each agent only has partial and complex information about the local states in the network. To permit consensus control for a broad family of applications, we thus consider a quite-general model for the observations made by the agents.
In particular, we consider the rather general case that each agent observes a linear combination of multiple agents’ local states. That is, we assume that each agent $i$ makes the observation
\[ \hat{y}_i = \sum_{j=1}^{N} g_{ij} \hat{x}_j, \]
where we term $\hat{y}_i \in \mathbb{R}^n$ as the agent $i$’s observation and term the scalars $g_{ij}$ as observation weights. Noting that the observation weight $g_{ij}$ represents the influence (through sensing or networked communication) of each agent $j$’s state on agent $i$’s observation, we find it natural to assemble the weights into an $N \times N$ topology matrix $G = [g_{ij}]$. We note that the topology matrix $G$ entirely describes the observation model of the agents.

Variations in network’s observation topology are ubiquitous in a range of autonomous-agent applications, because of the harsh environments in which the agents operate or because of limitations in the agents’s sensing/communication capabilities, among other causes. Numerous articles have studied autonomous-agent control and/or synchronization under topological variation, using both deterministic and stochastic models for the variation. Here, we also study consensus control under topological variation, using a classical deterministic model for variation. In particular, we consider the case where each agent $i$ makes the observation
\[ \hat{y}_i = \sum_{j=1}^{N} g_{ij}(t) \hat{x}_j \]
at time $t$, where the time-$t$ topology matrix $G(t) = [g_{ij}(t)]$ is selected from the a finite set of $N \times N$ matrices $G_1, \ldots, G_z$ (i.e., $G(t) \in G_1, \ldots, G_z$ at all times $t$). For convenience, we also impose the technical condition that $G(t)$ is right-continuous. Thus, notice that there exists a (either finite or infinite) sequence of times such that, between any two subsequent times (and including the earlier one), the time-$t$ topology matrix is a constant $G_i$, $i \in 1, \ldots, z$.

2.1.3 Framework for Control

A decentralized feedback control paradigm is required, i.e. each agent $i$ only has available the observation $\hat{y}_i$ and can only set the input $\hat{u}_i$. In the broadest sense, we assume that each agent $i$ determines its input $\hat{u}_i(t)$ at time $t$ from concurrent and previous observations: $\hat{y}_i(\tau), 0 \leq \tau \leq t$. That is, the agent $i$’s controller constitutes a functional mapping from the signal $\hat{y}_i(\tau), \tau \in [0,t]$, to the vector $\hat{u}_i(t)$.

In achieving consensus, we will consider the family of static (memoryless) linear controllers. We will describe this specific controller architecture once we have introduced the agreement problem.
2.1.4 Assembled Dynamics and Terminology

We find it convenient to assemble the agents’ individual dynamics and observations into a single state-space equation. To this end, we define the full state vector as
\[ \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{bmatrix}, \]
the full input vector as
\[ \hat{u} = \begin{bmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix}, \]
and the full observation vector as
\[ \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix}. \]
In terms of these quantities, we obtain the following representation of the dynamics when the sensing topology is fixed:
\[ \dot{\hat{x}} = (I_N \otimes \hat{A})\hat{x} + (I_N \otimes \hat{B})\hat{u} \]
\[ \hat{y} = (G \otimes I_n)\hat{x}, \]  
(1)

where the notation ‘\( \otimes \)’ represents the Kronecker product. We refer to whole model—the dynamics (1) together with the decentralized feedback control paradigm—as a sensing-agent network, or SAN.

In the case where the topology may vary, the dynamics of the networks are as follows:
\[ \dot{\hat{x}} = (I_N \otimes \hat{A})\hat{x} + (I_N \otimes \hat{B})\hat{u} \]
\[ \hat{y} = (G(t) \otimes I_n)\hat{x}, \]  
(2)

where the characteristics of the evolving topology matrix \( G(t) \) were described above. We refer to the model in this as a sensing-agent network with topological variation, or SAN-VT.

2.2 The Consensus Control Problem and Static LTI Feedback Architecture

2.2.1 The Consensus Control Problem

At is essence, consensus control has to do with feedback design to achieve synchronization among networked agents. That is, we seek controllers for the SAN that make the manifold in which all the agents’ states are identical asymptotically stable. Beyond this fundamental goal, consensus control applications sometimes require more refined shaping of dynamics (for instance, designing the trajectory on the asymptotically stable manifold). Here, let us introduce the core consensus task, and then discuss the design of the trajectory on the asymptotically stable manifold.

Since consensus has to do with asymptotic stability of the manifold where all the agents’ states are identical, it is convenient for us to define relative state vectors that are nil when the agents’ states are identical. Formally, let us define the relative state vectors as \( \hat{q}_i = \hat{x}_i - \hat{x}_N, \) for \( i = 1, \ldots, N - 1 \) (where
we have chosen to measure the states relative to $\mathbf{x}_N$ for notational simplicity). Now that we have defined the relative state vectors, we are ready to formally define the consensus task:

**Definition 1**  
A SAN is said to achieve consensus, if its feedback controller has been designed so that the manifold $\hat{q}_1 = \ldots = \hat{q}_{N-1} = 0$ is asymptotically stable.

Let us make several comments on the definition for consensus:

1) Consensus controls are needed in a variety of application areas, ranging from satellite antenna alignment to vehicle-group formation and sensor fusion, see [3, 9, 18] for just some of the relevant literature.

2) Conceptually, a SAN essentially achieves consensus if the local states of the agents reach the same value, or in other words agree. Formally, however, we note that consensus is a stronger condition in that we require not only attractivity to the manifold where the local states are identical, but stability in the sense of Lyapunov of this manifold; this stronger definition is natural in feedback controller design, and matches with the existing literature on consensus. We kindly ask the reader to see the broad literature on nonlinear control for a careful deconstruction of the difference between attractivity and stability. For the linear dynamics that we study here, the notions are identical.

3) Asymptotic stability of the state $\hat{\mathbf{x}}(t)$ (with the origin as the equilibrium point) is sufficient for consensus. However, consensus is possible even when stability is not: only equalization of the various agents’ states is needed. In fact, our definition does not enforce any condition on the dynamics on the manifold where the states are equal; the dynamics on the manifold may depend on the initial conditions in an arbitrary way, and may be time-varying. Thus, our definition encompasses both the concepts of consensus and tracking-consensus introduced in the literature [1, 2, 3, 4, 5, 6, 7, 10].

In contrast to the traditional studies of synchronization, we explicitly allow for controller design in seeking consensus in SANs. This design freedom can potentially allow not only for stabilization of the manifold of interest, but shaping of the trajectory on the manifold. Motivated by numerous applications (in particular, computational applications such as sensor fusion ones), we are especially interested in shaping the dependence of a SAN’s asymptotics on its initial conditions. This task of shaping the dependence of the asymptotic dynamics on the initial conditions has been termed agreement law design, see the initial work of Olfati-Saber and co-workers [1, 2, 3] as well as the systematic treatment in our earlier work [4]. Here, let us formalize the notion of an agreement law (and of agreement law design) for SANs.

**Definition 2**  
Consider a SAN that achieves consensus upon use of a particular feedback controller. Now consider a functional mapping from the initial states of the agents and time to an $n$-component vector, say
This function is said to be the agreement law of the SAN (upon use of the particular controller), if \( \lim_{t \to \infty} (\hat{x}_i(t) - f(\hat{x}_1(0), \ldots, \hat{x}_N(0), t)) = 0 \) for \( i = 1, \ldots, N \).

We note that, when a SAN achieves consensus using a particular controller, it has a unique agreement law. We will be interested in characterizing and designing the agreement laws of SANs that achieve consensus.

Finally, let us discuss the controller architecture that we propose for achieving consensus.

### 2.2.2 Static LTI Control Architecture

Our goal is to design a controller for a SAN, so as to achieve consensus and (additionally) set the agreement law of the SAN. Classical research on state feedback controller design, together with our recent efforts on stabilization through decentralized control ([19, 20, 21]), suggest that a linear static controller design should permit consensus under broad conditions on the network topology. Thus, we focus in this paper on a linear static (memoryless) feedback control architecture for consensus. We note that non-linear control architecture have also been considered, see [22].

In particular, we consider applying the controller \( \hat{u}_i = \hat{K}_i \hat{y}_i \), where \( \hat{K}_i \in \mathbb{R}^{m \times n} \), for each agent \( i \in 1, \ldots, N \). We will study how the gain matrices \( \hat{K}_i \) can be designed, to achieve consensus and shape the agreement law.

We find it convenient to assemble the control laws for each agent into a single relation. Doing so, we find that \( \hat{u} = \hat{K} \hat{y} \), where \( \hat{K} = \begin{bmatrix} \hat{K}_1 & & \\ & \ddots & \\ & & \hat{K}_N \end{bmatrix} \) is a block-diagonal matrix.

We notice that the control architecture that we consider is fundamentally a decentralized architecture, in that each agent can only use local observations and govern local actuators’s inputs.

### 3 Constructing Controllers for Consensus

In this section, we develop broad conditions under which the SAN achieves consensus, in the process explicitly specifying the static decentralized controllers that can achieve consensus. Our efforts here significantly enhance existing research on consensus control, in that 1) consensus is achieved for general agent internal dynamics, 2) a systematic high-gain methodology for designing consensus control is obtained, and 3) connections to ongoing research on synchronization and dynamical-network control/design are made explicitly.

Let us first present a key implicit condition on the network topology under which consensus can be achieved for a SAN, where proof allows construction of the high-gain consensus controller. After doing so, we will show that this condition encompasses a very broad range of network topologies, including not only the
Laplacian topology matrices commonly considered in the literature but a wide family of asymmetric topology matrices.

Here is the key condition:

**Theorem 1** Consider a SAN with topology matrix $G$. A static LTI decentralized controller can be designed for the SAN to achieve consensus, if there exists a diagonal matrix $D$ such that either 1) the eigenvalues of $DG$ are all in the OLHP; or 2) the eigenvalues of $DG$ are in the CLHP, only an eigenvalue is on the $j\omega$-axis and it is at the origin, and the corresponding right eigenvector is 1.

**Proof:** We prove this theorem by first transforming the (identical) agent’s open-loop system dynamics into a special form, which facilitates the design of a high-gain controller and the use of time-scale ideas to achieve and prove consensus. For case 1) of the theorem, we use a time-scale design technique ([23, 24]) to show that we can place the closed-loop eigenvalues in the OLHP, thus proving that consensus is achieved. For case 2) of the theorem, we consider the dynamics of the relative state, and then the time-scale analysis tells us that we can place the eigenvalues of the relative state’s closed-loop system matrix in the OLHP. Thus, we prove that consensus is also achieved.

From [25], we know that for any controllable pair $(\hat{A}, \hat{B})$, there exist non-singular input and state transformations:

$$x_i = T_s \hat{x}_i$$
$$u_i = T_i \hat{u}_i$$

such that $x_i = [x_{i,1}^T, \ldots, x_{i,m}^T]^T \in \mathbb{R}^n$, $x_{i,j} = [x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,l_j}]^T \in \mathbb{R}^{l_j}$, and $u_i = [u_{i,1}, u_{i,2}, \ldots, u_{i,m}]^T \in \mathbb{R}^m$ satisfy

$$\dot{x}_{i,j} = A_j x_{i,j} + B_j (u_{i,j} + \sum_{l=1, l\neq j}^{m} E_{j,l} x_{i,l}) \quad j = 1, 2, \ldots, m$$

where matrices $A_j \in \mathbb{R}^{l_j \times l_j}$, $B_j \in \mathbb{R}^{l_j \times 1}$ have the following special structures:

$$A_j = $$

$\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1 \\
E_{j,j,1} & E_{j,j,2} & \cdots & E_{j,j,l_j-1} & E_{j,j,l_j}
\end{bmatrix}$
\[ B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \]

and

\[ E_{j,l} = \begin{bmatrix} E_{j,l,1} & E_{j,l,2} & \cdots & E_{j,l,l} \end{bmatrix}. \]

We note that the state and input transformations \( T_s \) and \( T_i \) transform each agent’s model into \( m \) integrator chains, with the length of the \( j \)-th chain being \( l_j \). The triple subindex \( x_{i,j,l} \) denotes the state variable of the \( i \)-th agent, \( j \)-th chain and \( l_j \)-th level. The chains for each agent are coupled only at the bottom layer, and the input signal \( u_{i,j} \) is injected into the bottom layer of each integrator chain.

Now let us consider the design of feedback controller architecture. The controller in the new coordinates for agent \( i \) can be expressed as

\[ u_i = K_i \sum_{j=1}^{N} g_{i,j} x_j \tag{3} \]

where \( K_i = T_i \hat{K}_i T_s^{-1} \in \mathbb{R}^{m \times n} \). Here we design a high-gain controller \( K_i \) of the following form:

\[ K_i = \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \cdots & \frac{\beta_{1,l_1}}{\epsilon_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{m,1}}{\epsilon_m} & \cdots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} D_i \]

where \( \epsilon_j \) is sufficiently small and \( D_i \) is a scalar. We limit ourselves to the case where \( K_i \) is block diagonal so that the scalar input \( u_{i,j} \) of \( j \)-th chain of the agent \( i \), only feeds back the state information of its local chain. Also, the gain matrix \( K_i \) for different agents only differs by a scalar factor \( D_i \). We find it convenient to assemble the agents’ individual states and inputs into a single-space equation. By defining the full state vector as \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \), the full input vector as \( u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \), and introducing a diagonal matrix \( D \) as
\[ D = \begin{bmatrix} D_1 \\ \vdots \\ D_N \end{bmatrix}, \] we express the feedback law for the SAN as
\[ u = (DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \cdots & \frac{\beta_{1,l_1}}{\epsilon_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{m,1}}{\epsilon_m} & \cdots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix}) x \] (4)

Now, let us consider case 1), where all the eigenvalues of \( DG \) are in the OLHP. We will use the time-scale technique to show that the high-gain controller can stabilize the SAN and hence consensus is achieved. First, let us assemble the last state variable of each chain of all agents’ into a vector \( \eta \in \mathbb{R}^{Nm} \),
\[ \eta = [x_{1,1,l_1}, \cdots, x_{1,m,l_m}, \cdots, x_{N,1,l_1}, \cdots, x_{N,m,l_m}]^T \]
and the rest of state variables into another vector \( \zeta \in \mathbb{R}^{N(n-m)} \),
\[ \zeta = [x_{1,1,1}, \cdots, x_{1,l_1-1}, \cdots, x_{1,m,1}, \cdots, x_{1,m,l_m-1}, \cdots, x_{N,1,1}, \cdots, x_{N,1,l_1-1}, \cdots, x_{N,m,1}, \cdots, x_{N,m,l_m-1}]^T \]

With some algebra, we can express the closed-loop system dynamics separated in the slow and fast time scales as
\[
\dot{\eta} = (I_N \otimes R) \zeta + (I_N \otimes S) \eta \\
\dot{\zeta} = (DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \cdots & \frac{\beta_{1,l_1}}{\epsilon_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{m,1}}{\epsilon_m} & \cdots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} + I_N \otimes P) \zeta + (DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \cdots & \frac{\beta_{1,l_1}}{\epsilon_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{m,1}}{\epsilon_m} & \cdots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} + I_N \otimes Q) \eta
\]

where
\[
R = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \in \mathbb{R}^{(n-m) \times (n-m)},
\]
\[
R_j = \begin{bmatrix} 0_{(l_j-2) \times 1} & I_{l_j-2} \\ 0 & 0_{1 \times (l_j-2)} \end{bmatrix} \in \mathbb{R}^{(l_j-1) \times (l_j-1)},
\]
\[
S = \begin{bmatrix} S_1 \\ \vdots \\ S_m \end{bmatrix} \in \mathbb{R}^{(n-m) \times m},
\]
\[
S_j = \begin{bmatrix}
0_{(l_j-2)\times 1} \\
1
\end{bmatrix} \in \mathbb{R}^{(l_j-1)\times 1},
\]

\[
P = \begin{bmatrix}
E_{1,1,1} & \cdots & E_{1,1,l_1-1} & \cdots & E_{1,m,1} & \cdots & E_{1,m,l_m-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
E_{m,1,1} & \cdots & E_{m,1,l_1-1} & \cdots & E_{m,m,1} & \cdots & E_{m,m,l_m-1}
\end{bmatrix},
\]

and

\[
Q = \begin{bmatrix}
E_{1,1,1} & \cdots & E_{1,m,1} \\
\vdots & \vdots & \vdots \\
E_{m,1,1} & \cdots & E_{m,m,1}
\end{bmatrix}.
\]

Since \(\epsilon_j\) for \(j = 1, \ldots, m\) are sufficiently small and \(DG\) is nonsingular, the time-scale methodology [24] shows that the \(Nm\) fast eigenvalues can be divided into \(m\) groups, and for each group \(j = 1\) to \(m\), the eigenvalues are located at

\[
\lambda_{f_j} = \frac{\beta_{j,1}}{\epsilon_j} \lambda(DG) + \mathcal{O}(1)
\]

Since the eigenvalues of \(DG\) are in the OLHP, we can choose all the parameters \(\beta_{1,1}, \ldots, \beta_{m,l_m}\) to be positive to ensure that the fast eigenvalues are in the OLHP.

Now, let’s consider the \(N(n - m)\) slow eigenvalues. From [24], since \(\epsilon_j\) are sufficiently small, we know that the slow eigenvalues are the eigenvalues of matrix \(A_0\) shown below plus some small perturbation.

\[
A_0 = I_N \otimes R - (I_N \otimes S)(DG \otimes \begin{bmatrix}
\beta_{1,1} & \cdots & \beta_{1,l_1-1} \\
\beta_{m,1} & \cdots & \beta_{m,l_m-1}
\end{bmatrix})^{-1}(DG \otimes \begin{bmatrix}
\beta_{1,1} & \cdots & \beta_{1,l_1-1} \\
\beta_{m,1} & \cdots & \beta_{m,l_m-1}
\end{bmatrix})
\]

\[
= I_N \otimes R - (I_N \otimes S)[(DG)^{-1} \otimes \begin{bmatrix}
\beta_{1,1} & \cdots & \beta_{1,l_1-1} \\
\beta_{m,1} & \cdots & \beta_{m,l_m-1}
\end{bmatrix}] (DG \otimes \begin{bmatrix}
\beta_{1,1} & \cdots & \beta_{1,l_1-1} \\
\beta_{m,1} & \cdots & \beta_{m,l_m-1}
\end{bmatrix})
\]

\[
= I_N \otimes R - (I_N \otimes S)[I_N \otimes \begin{bmatrix}
\frac{\beta_{1,1}}{\beta_{m,1}} & \cdots & \frac{\beta_{1,1}}{\beta_{m,l_m}} \\
\frac{\beta_{m,1}}{\beta_{m,1}} & \cdots & \frac{\beta_{m,1}}{\beta_{m,l_m}}
\end{bmatrix}]
\]

\[
= I_N \otimes \begin{bmatrix}
A_{0,1} & \cdots & A_{0,m}
\end{bmatrix}
\]

\[\text{11}\]
where the matrix $A_{0,j} \in \mathbb{R}^{(l_j-1) \times (l_j-1)}$ has the following special structure:

$$A_{0,j} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 1 \\
-\beta_{j,1} & \beta_{j,2} & \beta_{j,3} & \ldots & -\beta_{j,l_j-1}
\end{bmatrix}.$$ 

Therefore, the slow eigenvalues can be placed arbitrarily close to $n - m$ locations, and there are $N$ eigenvalues at each location. For $j = 1$ to $m$, we can choose $\beta_{j,1}, \ldots, \beta_{j,l_j-1}$ such that the slow eigenvalues

$$\lambda_s = \lambda(A_{0,j}) + \mathcal{O}(\epsilon_j)$$

are in the OLHP. Since all the eigenvalues of the closed-loop system are in the OLHP. We have proved that the consensus is achieved for the SAN.

Now, let us consider case 2) of the theorem. To begin, we find it convenient to give the closed-loop dynamics:

$$\dot{x} = (I_N \otimes A + DG \otimes \begin{bmatrix} F_1 \\
\ddots \\
F_m \end{bmatrix})x$$

(5)

where matrix $A \in \mathbb{R}^{n \times n}$ is the system matrix of the (identical) local agent

$$A = \begin{bmatrix}
A_1 & B_1 E_{1,2} & \ldots & B_1 E_{1,m} \\
B_2 E_{2,1} & A_2 & \ldots & B_2 E_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
B_m E_{m,1} & B_m E_{m,2} & \ldots & A_m
\end{bmatrix},$$

and

$$F_j = \begin{bmatrix} 0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 \\
-\beta_{j,1} & \beta_{j,2} & \beta_{j,3} & \ldots & -\beta_{j,l_j-1}
\end{bmatrix} \in \mathbb{R}^{l_j \times l_j}.$$ 

Next, let us study the relative state vectors for the system. Specifically, let us define a transformed relative state vector $q_i = x_i - x_N$, $i = 1, \ldots, N - 1$ (where, incidentally, $q_i$ can be obtained from $\hat{q}_i$, see Definition 1 from Section II, through the same transformation $T_s$). To do so, we find it convenient to assemble all
the transformed relative state vectors into a single global relative vector \( q = [q_1^T \ldots q_{N-1}^T]^T \). Through a state transformation of (5), one finds that the global relative vector satisfies an autonomous differential equation (see e.g. [2, 11] for similar analysis). In particular, with some algebra similar to that of ([11, 26, 27]), we can express the dynamics of the global relative vectors as

\[
\dot{q} = (I_{N-1} \otimes A + \overline{DG} \otimes \begin{bmatrix} F_1 & \vdots & F_m \end{bmatrix})q
\]

(6)

where \( \overline{DG} \) is formed by removing the last row and column from \( DG - D_N g_N^T \), and \( g_N^T \) is the last row of \( G \). Equivalently, \( \overline{DG} \) can be viewed as being formed by subtracting the last row of \( DG \) from all other rows, and then removing the last row and column. We see that \( \overline{DG} \) has \( N-1 \) eigenvalues, which are the non-zero eigenvalues of \( DG \). Thus all the eigenvalues of \( \overline{DG} \) are in the OLHP, and we automatically see that the problem of stabilizing the relative state dynamics is identical to the stabilization of the state dynamics achieved for case 1). Specifically using a high-gain controller, we can place all \((N-1)n\) eigenvalues of the state matrix of dynamics (6) in the OLHP. Hence, the manifold where the agents’ state are identical is made asymptotically stable. We have proved the consensus is achieved for the SAN. □

Let us make a couple remarks about the implicit condition for consensus:

- The proof of the Theorem 1 provides a high-gain methodology for achieving consensus, under the condition that a diagonal \( D \) can be found to put the eigenvalues of \( DG \) either in the OLHP or in the CLHP with a single zero eigenvalue and corresponding eigenvector \( 1 \).

- We note that our design methodology for case 1) of Theorem 1, allows us to place \( N(n-m) \) eigenvalues arbitrarily close to \( n-m \) locations in the complex plane, in groups of \( N \). Meanwhile, the remaining \( Nm \) eigenvalues are within \( O(1) \) of the eigenvalues of \( \frac{\beta_{ij}}{\epsilon_j} DG \).

- A simple eigen-analysis for case 2) of Theorem 1, where \( DG \) has one zero eigenvalue, shows that the state matrix of the closed-loop system (5) has \( n \) eigenvalues that are exactly the same as those of local agents’ open-loop system matrix \( A \). This remark will be useful for agreement law design.

- We stress that the above result is, to the best of our knowledge, the first to give conditions for agreement for a network of agents with general LTI internal dynamics (including possibly multiple inputs).

The condition and controller construction given in Theorem 1 makes it clear that consensus can be achieved, whenever a diagonal scaling \( D \) can be found to put the eigenvalues of \( DG \) in a single half plane. The problem of finding a diagonal \( D \) to shape the spectrum of \( DG \) has been explored in both the classical
numerical-methods and control literature [9], as well as in recent works on dynamical network control [28, 29]. Drawing on this literature, we can obtain a broad explicit condition on the matrix $G$ for a SAN to achieve consensus. This condition encompasses those given in the literature. Here is the result:

**Theorem 2** Consider a SAN with topology matrix $G$. A static decentralized controller can be designed for the SAN to achieve consensus, if either 1) $G$ has a nested sequence of $N$ principal minors (of dimensions $1 \times 1, 2 \times 2, \ldots, N \times N$) all of full rank or 2) $G$ has a nested sequence of $N-1$ principal minors (of dimensions $1 \times 1, 2 \times 2, \ldots, N-1 \times N-1$) of full rank and further the vector $1$ is in the null space of $G$.

**Proof:** Let us consider case 1) of the theorem. In the case that $G$ has a nested sequence of $N$ principal minors all of full rank, the papers [30] and [9] give a systematic method for constructing a diagonal matrix $D$, such that all the eigenvalues of $DG$ are in the OLHP. Hence, the result follows from Theorem 1.

For case 2) of the theorem, we can design a diagonal matrix $D$ such that $N-1$ eigenvalues of $DG$ are in the OLHP. Also the vector $1$ is in the null space of $DG$, since the vector $1$ is in the null space of $G$. Thus another eigenvalue of matrix $DG$ is zero and the corresponding right eigenvector is $1$. Hence, the result follows from Theorem 1. □

Notice that the first condition of Theorem 2, i.e. the sequential-full-rank condition, is in fact satisfied for a broad class of matrices, including grounded Laplacian ones and more generally diagonally-dominant matrices. The second condition of Theorem 2 encompasses a broad class of matrices, including Laplacian matrices of connected graphs.

We have constructed a high-gain controller for general agent internal dynamics to achieve consensus. In many cases, the agreement law - i.e., the dependence of the consensus value or trajectory on the initial conditions - is of importance.

Let us characterize the agreement law, when a particular controller is used under the conditions of Theorem 2. We will do so by characterizing the dynamics on the consensus manifold through eigen-analysis.

In case 1) of Theorem 2, we see automatically that agreement law is $f(x_1(0), \ldots, x_N(0), t) = 0$, i.e., the final state is nil for all initial conditions.

Now, let us consider case 2) of Theorem 2. WLOG, assume the local agent’s system matrix $A$ has $k \leq n$ distinct eigenvalues, $\lambda_1, \ldots, \lambda_k$, each with algebraic multiplicity $p_i$, $i = 1, \ldots, k$. Through eigenvalue decomposition, we can write $A$ as

$$A = VJV^{-1}$$

where the Jordan matrix of $A$ is $J = \text{Blkdiag}(J_1, J_2, \ldots, J_k)$, the Jordan Block $J_i$ can be subpartitioned as

$$J_i = \text{Blkdiag}(J_{i,1}, J_{i,2}, \ldots, J_{i,p_i}),$$

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where each \( n_{ij} \times n_{ij} \) subblock \( J_{i,j} \) is of the form below:

\[
J_{i,j} = \begin{pmatrix}
\lambda_i & 1 & 0 & \ldots & 0 \\
0 & \lambda_i & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \lambda_i & 1 \\
0 & \ldots & \ldots & \ldots & \lambda_i
\end{pmatrix},
\]

\( V = [V_1, V_2, \ldots, V_k] \) is the matrix whose columns are eigenvectors and generalized right eigenvectors of \( A \), the rows of \( V^{-1} \) are left eigenvectors and generalized left eigenvectors of \( A \), the partitioning of \( V \) and \( V^{-1} \) matches that of \( J \), \( \text{Blkdiag}(\quad) \) represents a block diagonal matrix with entries specifying the blocks.

Let \( \omega^T \) be the normalized left eigenvector of \( G \) associated with the zero eigenvalue. From the proof of Theorem 2, we know that \( DG \) has all except a single zero eigenvalue in the OLHP, the right eigenvector of \( DG \) associated with zero eigenvalue is \( 1 \), and the normalized left eigenvector associated with the zero eigenvalue is \( \omega_0^T = \frac{1}{\pi D} \omega^T D^{-1} \), where we have assumed that an invertible \( D \) is being used. For \( i = 1, \ldots, k \) and \( j = 1, \ldots, p_i \), let us denote the \( j \)-th right eigenvector of \( A \) associated with \( \lambda_i \) as \( v_{i,j,1} \), and the associated generalized right eigenvectors as \( v_{i,j,2}, \ldots, v_{i,j,n_{ij}} \). Similarly, let us denote the \( j \)-th left eigenvector of \( A \) associated with \( \lambda_i \) as \( \omega_{i,j,1}^T \), and the associated generalized left eigenvectors as \( \omega_{i,j,2}, \ldots, \omega_{i,j,n_{ij}} \).

With just a little algebra, we find that the closed-loop system matrix (5) has an eigenvalue \( \lambda_i, i = 1, \ldots, k \) with algebraic multiplicity \( p_i \). Further, the \( j \)-th left eigenvector associated with \( \lambda_i \) is \( \omega_0^T \otimes \omega_{i,j,1}^T \), and the corresponding generalized left eigenvectors are \( \omega_0^T \otimes \omega_{i,j,2}^T, \ldots, \omega_0^T \otimes \omega_{i,j,n_{ij}}^T \). Similarly, the \( j \)-th right eigenvector associated with \( \lambda_i \) is \( 1 \otimes v_{i,j,1} \), and the corresponding generalized right eigenvectors are \( 1 \otimes v_{i,j,2}, \ldots, 1 \otimes v_{i,j,n_{ij}} \).

WLOG, let’s assume that the first \( k_1 \leq k \leq n \) eigenvalues of the local agent system matrix \( A \), \( \lambda_1, \ldots, \lambda_{k_1} \) are in the CRHP. From the analysis in Theorem 1’s proof, we immediately find that

\[
\lim_{t \to \infty} (x(t) - \sum_{i=1}^{k_1} (1 \otimes V_i) e^{J_{i,t}} (\omega_0^T \otimes W_i^T) x(0)) = 0
\]

where for \( i = 1, \ldots, k_1 \) and \( j = 1, \ldots, p_i \)

\[
e^{J_{i,t}} = \text{Blkdiag}(e^{J_{i,1,t}}, e^{J_{i,2,t}}, \ldots, e^{J_{i,p_i,t}}),
\]

\[
e^{J_{i,j,t}} = e^{\lambda_{i,t}} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \ldots & \frac{t^{n_{ij}-1}}{(n_{ij}-1)!} \\ 0 & 1 & t & \ldots & \frac{t^{n_{ij}-2}}{(n_{ij}-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 1 \end{bmatrix},
\]
\[ V_i = \begin{bmatrix} v_{i,1} & \cdots & v_{i,p_i} \end{bmatrix}, \]

\[ V_{i,j} = \begin{bmatrix} v_{i,j,1} & \cdots & v_{i,j,n_{ij}} \end{bmatrix}, \]

\[ W_i^T = \begin{bmatrix} W_{i,1}^T \\ \vdots \\ W_{i,p_i}^T \end{bmatrix}, \]

and

\[ W_{i,j}^T = \begin{bmatrix} \omega_{i,j,1}^T \\ \omega_{i,j,1}^T \\ \vdots \\ \omega_{i,j,1}^T \end{bmatrix}. \]

Hence, for \( L = 1, \ldots, N \):

\[
\lim_{t \to \infty} (x_L(t) - \sum_{i=1}^{k_1} V_i e^{J_L t} (\omega_0^T \otimes W_i) x(0)) = 0
\]

Hence, the agreement law of the SAN (upon use of a particular high-gain controller with matrix \( D \)) is

\[
f(x_1(0), \ldots, x_N(0), t) = \frac{1}{\omega^T D^{-1}} \sum_{i=1}^{k_1} V_i e^{J_i t} ((\omega^T D^{-1}) \otimes W_i) x(0)
\]

When \( DG \) has a zero eigenvalue, we thus see that the asymptotic trajectory does depend on the initial states of the agents. Notice that the agreement law - the dependence of the asymptotic dynamics on initial conditions - in general is a time varying function, which depends on the CRHP modes of the agent’s internal dynamics; thus, tracking in consensus is also possible.

By selecting \( D \), we see that the agreement law can be designed. In [4], Roy and co-workers study selection of \( D \) so that a desired agreement law is achieved while the eigenvalues of \( DG \) are left in the CLHP. We can apply these results to pursue agreement assignment in the general studied here.

4 Consensus Controller Design under Topological Variation

In this section, we will consider controller design for consensus in the SAN-VT, i.e. for a sensing-agent network model that is subject to variations in the observation topology. Such controller design for consensus under topological variation is relevant in several application domains, including for control of autonomous
vehicle teams and sensor networks (which both tend to operate in harsh environments with limited actuation/power, and so maybe routinely subject to sensing failures and other topological variations). Our work on controller design under topological variation is complementary to numerous studies on modeling and analyzing synchronization/consensus under topological variation, see e.g. Blondel and co-workers’ recent article for a succinct overview [15]. We also note the connection of our work to several recent works on design of network controllers under arbitrary and stochastic topological variation [31]; in comparison, the results presented here permit design for much more general agent models and a broad class of network topologies.

In our efforts to design controllers for the SAN-VT, we distinguish between two paradigms regarding information dissemination on the topology changes. The first case we consider is that the controller can detect when the network topology changes, and so (formally) the controller has available the index of the topology at the current time; in this case, (switching) gain parameters that depend on the index of the current topology can be used. The second paradigm that we consider is that current network topology is unknown to the controller, and so a single set of gain parameters are used.

Here, we develop conditions under which the SAN-VT achieves consensus, for both information paradigms. As in our earlier development, we separately consider the case where the stable manifold is only the origin (i.e., all agents’ states converge to the origin) and the case of a more general consensus manifold.

Let us consider the paradigm that the current topology is known to the controller, and present two conditions under which consensus can be achieved. Here is the first:

**Theorem 3** Consider a SAN-VT, and assume that the controller has available the index of the current topology at each instant. A linear decentralized controller that switches with the network topology can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:

1) Assumption 1: At least one of the possible topologies \( G_i \), \( i = 1, \ldots, z \) has a nested sequence of \( N \) principal minors that all have full rank.

2) Assumption 2: Time epochs during which no topology satisfies the premise of Assumption 1 are upper bounded in duration (say by \( T_1 \)), while time epochs during which any particular possible topology \( G_i \) that satisfies Assumption 1 is in force are lower bounded by \( T_2 \).

To prove Theorem 3, we first find it convenient to develop the following lemma regarding controller design during a time interval when a particular topology (that is amenable to control) is in force.

**Lemma 1** Consider a particular topology \( G_i \) of a SAN-VT such that Assumption 1 of Theorem 3 is in force, i.e. \( G_i \) has a sequence of \( n \) nested principal minors all of full rank. Say that there is an epoch \( T = [t_A, t_B) \)
of duration greater than $T_2$ such that $G(t) = G_i$ for all $t \in T$. Then, for any $\gamma > 0$, a controller can be
designed for the SAN-VT so that $||x(t)|| \leq \frac{1}{\gamma}||x(t_A)||$ for all $t \in [t_A + T_2, t_B]$. Furthermore, for each such
design, there exists $\gamma_1$ and $\lambda_1 > 0$ such that $||x(t)|| \leq \gamma_1 e^{-\lambda_1(t-t_A)}||x(t_A)||$ for all $t \in T$.

Let us first prove the lemma:

**Proof:** Consider application of a high-gain stabilizing controller for consensus, as developed in Theorem 1, during the epoch $[t_A, t_B]$. Notice that, since $G_i$ has a sequence of leading principal minors all of full rank, we will design the controller based on the first assumption in Theorem 1. For any such controller, classical results on high-gain state feedback control clarify that, for any sufficiently high gain, $||x(t)||$ can be made less than $\frac{1}{\gamma}||x(t_A)||$ for any $\gamma$ and after any fixed interval of time (while the model remains in force). Thus, we immediately recover that a controller can be designed to achieve $||x(t)|| \leq \frac{1}{\gamma}||x(t_A)||$ for all $t \in [t_A + T_2, t_B]$. Whatever asymptotically-stabilizing high gain controller is used, the state $x(t)$ is bounded in the interim and the state approaches the origin exponentially (from properties of linear systems), and so the theorem is proved. ∎

We notice that reduction of the state’s norm to an arbitrary level within an interval is possible for any stabilizing controller developed through Theorem 1, and is achieved for any sufficiently high gain.

Let us now apply the lemma to prove Theorem 3:

**Proof:** Let us label the sequence of switching times for the observation topology as $t_0, t_1, \ldots$. We consider applying a feedback control of the following form: during the intervals $[t_i, t_{i+1}]$ such that the corresponding topology matrix $G_j$ satisfies the sequential full rank condition (which we call the “good” intervals), we apply a stabilizing linear high-gain controller as per Theorem 1. During the remaining intervals (which we call “bad” intervals), we set the feedback control to nil. If the gains during the good intervals are chosen sufficiently large, we claim that asymptotic stability and hence consensus is achieved.

To formalize this, let us first consider $||x(t)||$ at the end of each good interval; for convenience, we label these times as $\hat{t}_1, \hat{t}_2, \ldots$, and also label the initial time as $t_0 = t_0$. We will bound $||x(\hat{t}_{i+1})||$ with respect to $||x(\hat{t}_i)||$, for $i = 0, 1, 2, \ldots$. To do so, we note that, during the epoch $[\hat{t}_i, \hat{t}_{i+1}]$ the concluding good interval of interest (which has duration of at least $T_2$) may be preceded by several bad intervals with total duration of at most $T_1$. Using exponential bounds on the transition matrix norm during each bad interval and noting the bound on the total duration, we immediately can bound $||x(t)||$ before the beginning of the good interval (say $t^*_i$) as follows: $||x(t)|| \leq \mu ||x(\hat{t}_i)||$ for $\hat{t}_i \leq t \leq t^*_i$, for some $\mu > 0$. Next, from Lemma 1, we see that the high-gain controller during the concluding good interval can be selected so that $||x(\hat{t}_{i+1})|| \leq \frac{1}{\gamma}||x(t^*_i)||$ for any $\gamma > 0$. Choosing the controller to achieve $\gamma = 2\mu$, we immediately recover that $||x(\hat{t}_{i+1})|| \leq \frac{1}{\gamma}||x(\hat{t}_i)||$. Thus, we see that $||x(\hat{t}_i)|| \leq \left(\frac{1}{\gamma}\right)^i||x(t_0)||$.
Now let us consider \( ||x(t)|| \) for \( t \) between \( \tilde{t}_i \) and \( \tilde{t}_{i+1} \). Noting the bound on the state during the bad intervals and noting the exponential bound during the good intervals (from Lemma 1), we recover that 
\[
||x(t)|| \leq \mu \gamma_1 ||x(\tilde{t}_i)||
\]
for some fixed \( \gamma_1 > 0 \) (which for convenience we can take to be the largest among those given by Lemma 1 for the topologies satisfying Assumption 1), for \( \tilde{t}_i \leq t < \tilde{t}_{i+1} \). Thus, we automatically find that 
\[
||x(t)|| \leq \left( \frac{1}{2} \right)^i \mu \gamma_1 ||x(t_0)||
\]
for \( \tilde{t}_i \leq t < \tilde{t}_{i+1} \).

Now consider two cases. The first case is that there is an infinite sequence of topologies, in which case we obtain asymptotic stability directly from the expression 
\[
||x(t)|| \leq \left( \frac{1}{2} \right)^i \mu \gamma_1 ||x(t_0)||
\]
Alternately, if a (good) interval persists beyond a particular time \( \tilde{t} \), we can directly invoke the exponentially-decaying bound on the response upon stabilizing control together with boundedness in the earlier time period to verify asymptotic stability.\(^1\)

We note that Theorem 3 holds whether or not the open-loop agent plant has CLHP eigenvalues; in the case where it has ORHP or unstable eigenvalues, stabilization is still possible because the state can be driven to the consensus manifold at a faster rate during the good intervals than it escapes during the bad ones. In practice, various constraints may limit that capability to rapidly drive the state to the consensus manifold in short periods, and so the result is most apt for the (typical) case of open-loop poles in the CLHP.

Theorem 3 is concerned with the case that the consensus manifold is only the origin. We also seek to verify consensus for the more general case, i.e. in the case of a general consensus manifold. Here is the result:

**Theorem 4** Consider a SAN-VT, and assume that the controller has available the index of the current topology at each instant. A linear decentralized controller that switches with the network topology can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:

1) *Assumption 1:* At least one of the possible topologies \( G_i, i = 1, \ldots, z \), has a nested sequence of \( N-1 \) principal minors of full rank and further the vector \( 1 \) is in the null space of that topology matrix.

2) *Assumption 2:* Time epochs during which no topology satisfies the premise of Assumption 1 are upper bounded in duration (say by \( T_1 \)), while time epochs during which any particular possible topology \( G_i \) that satisfies Assumption 1 is in force are lower bounded by \( T_2 \).

**Proof:** Let us label the sequence of switching times for the observation topology as \( t_0, t_1, \ldots \). We consider applying a feedback control of the following form: during the intervals \( [t_i, t_{i+1}] \) such that the corresponding

\(^1\)Our proof here is for asymptotic stability rather than uniform asymptotic stability, as per the definition of consensus. However, uniform asymptotic stability can also be proved here with a little more effort, by exploiting the exponential decay of \( ||x(t)|| \) in long-duration “good” intervals.
topology matrix $G_j$ satisfies the sequential full rank condition (which we call the “good” intervals), we apply a linear high-gain controller that achieves consensus as per Theorem 1.

To show that such a controller achieves consensus, we progress as follows. We consider each interval such that the network topology is fixed. In the proof of Theorem 1, we have shown that the global relative vector $q = [q_1^T, \ldots, q_{N-1}^T]^T$ has the following dynamics:

$$\dot{q} = (I_{N-1} \otimes A + DG \otimes \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix})q$$

(7)

where $DG$ can be viewed as being formed by subtracting the last row of $DG$ from all other rows and then removing the last row and column, and $G$ is the particular $G_i$ in force during that interval. We see that $DG$ has $N - 1$ eigenvalues, which are the non-zero eigenvalues of $DG$. We thus automatically see that the stabilization of the relative state dynamics for the time varying topology is identical to the stabilization of the state dynamics for time varying topology achieved in Theorem 3. That is, during the good intervals, we apply a linear high-gain controller as per Theorem 1. During the remaining intervals (which we call “bad” intervals), we set the feedback control to nil. Then, following the proof of Theorem 3, we see that the relative state and hence the consensus manifold is made asymptotically stable for the time varying topology. □

Now, let us consider the paradigm that current network topology is unknown to the controller, and so a single set of gain parameters are used. To achieve stabilization/consensus in this case, we seek for a single set of gain parameters that causes the state during each constant-topology interval to either be exponentially decrescent at a fast rate or to be only slowly growing. We argue that such gains can be found if all the topology matrices $G_i$ either fall in the broad class of $D$-stable matrices or are nil. This model for the observation topology is a broadly applicable one, for instance in the case that the network has one or more designed modes of operation and also may be subject to global network failures.

**Theorem 5** Consider a SAN-VT, and assume that the current network topology is unknown to the controller. A linear time-invariant decentralized controller can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:

1) Assumption 1: At least one of the possible topology matrices $G_i$, $i = 1, \ldots, z$, is $D$-stable, and all topology matrices are either $D$-stable or the zero matrix.

2) Assumption 2: Time epochs during which the topology remains the zero matrix are upper bounded in duration (say by $T_1$), while time epochs during which any particular possible topology $G_i$ that is $D$-stable is in force are lower bounded by $T_2$.  

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**Proof:** Let us label the sequence of switching times for the observation topology as $t_0, t_1, \ldots$. We consider applying a time-invariant high-gain feedback control, and then show that the SAN-VT can achieve consensus with this controller.

Let us consider the dynamics during the intervals $[t_i, t_{i+1}]$ such that the corresponding topology matrix $G_j$ is $D$-stable (which we call the “good” intervals). For any particular good interval, all the eigenvalues of $DG_i$ for an arbitrary positive definite matrix $D$, are in the OLHP. Therefore, we can choose a single diagonal matrix $D$ for all good intervals so as to place the eigenvalues of the $DG_i$ in the OLHP. From Lemma 1, we notice that reduction of the state’s norm to an arbitrary level within an interval is possible for any stabilizing controller developed through Theorem 1, and is achieved for any sufficiently high gain. It thus is clear that an LTI controller can be designed that reduces that state by any desired fraction during each good interval. Let us consider applying this controller.

During the remaining “bad” intervals (for which the topology matrices are zero matrices), the closed-loop dynamics are entirely independent of the control used. Thus, we immediately recover that the norm of the state at the ends of these intervals (which are also upper-bounded in duration) can be bounded as a fixed multiple of the norm at the beginning. The remainder of the proof thus follows as in Theorem 3.

Theorem 5 only develops the case that the consensus manifold is the origin. Next, we consider the case of a more general consensus manifold. We find that the result is related to the notion of $D$-semistability [4]. Since this notion is not very widely used, let us recall that $D$-semistability is defined as follows: a matrix $G$ is said to be $D$-semistable if the eigenvalues of the matrix $DG$ are in the closed left half plane and the eigenvalues of $DG$ on the $j\omega$-axis are simple, for all positive diagonal matrix $D$.

Now, we are ready to present the result:

**Theorem 6** Consider a SAN-VT, and assume that the current network topology is unknown to the controller. A linear time-invariant decentralized controller can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:

1) **Assumption 1:** At least one of the possible topology matrices $G_i$, $i = 1, \ldots, z$ is $D$-semistable, and all topology matrices are either $D$-semistable or the zero matrix. Furthermore, there exists a single positive diagonal matrix $D$ such that, for each $G_i$ that is $D$-semistable, $DG_i$ has no eigenvalue on the $j\omega$-axis other than the single eigenvalue at the origin, and the corresponding right eigenvector is 1.

2) **Assumption 2:** Time epochs during which the topology remains the zero matrix are upper bounded in duration (say by $T_1$), while time epochs during which any particular possible topology $G_i$ that is $D$-semistable is in force are lower bounded by $T_2$. 


Proof: The proof closely follows the proofs of Theorem 4 and Theorem 5, and so we omit the details. Notice here Assumption 1 means that there must exists one $D$ for all possible topologies $G_i$ that satisfies $D$-semistable condition.

Let us make a couple remarks about our results:

• In general, it is hard to test whether a given matrix is $D$-stable or $D$-semistable. However, there are several important classes of matrices that are known to be $D$-stable or $D$-semistable, and are representative of many common network interactions: these include Laplacian, grounded Laplacian, and symmetric positive-definite matrices, among others. We strongly refer the reader to [4] for details.

• Regarding Theorems 4 and 6, we note that the network dynamics on the consensus manifold may be quite complex, and may be persistently dependent on the particular sequence of the underlying network topologies. We leave it to future work to pursue design of the trajectory on the consensus manifold in this case.

• We note that Theorems 5 and 6 encompass the case without network failure, i.e. the case that none of the topologies are zero matrices. In this case, of course the upper bound on the duration of time epochs such that the network topology matrix is a zero matrix may be ignored.

• We notice that our results (Theorem 5 and Theorem 6) are connected with results in [1, 2, 5, 15], however, we consider a broad network sensing model and local agent model. And our results differ from those, in the sense that we design the controller to achieve consensus rather than checking the stability of the existing algorithm.

References


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