A Stochastic Modeling and Analysis Approach to
Strategic Traffic Flow Management under Weather
Uncertainty

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In this article, we introduce a promising framework for representing an air traffic flow (stream) and flow-management action operating under weather uncertainty. We propose to use a meshed queuing and Markov-chain model—specifically, a queuing model whose service-rates are modulated by an underlying Markov chain describing weather-impact evolution—to capture traffic management in an uncertain environment. Two techniques for characterizing flow-management performance using the model are developed, namely 1) a master-Markov-chain representation technique that yields accurate results but at relatively high computational cost, and 2) a jump-linear system-based approximation that has promising scalability. The model formulation and two analysis techniques are illustrated with numerous examples. Based on this initial study, we believe that the interfaced weather-impact and traffic-flow model analyzed here holds promise to inform strategic flow contingency management in NextGen.

I. Introduction

Strategic air traffic management, which is concerned with decision-making for traffic management 2-15 hours in advance, is considered a critical component of the Next-Generation Air Transportation System (NextGen). This longer strategic time-scale (as compared to the shorter 2-hour time horizon for tactical decision-making) is promising for allowing the coordination of traffic management at a broader spatial scale, and even National Airspace System (NAS)-wide. Despite many significant advances in the modeling and optimization techniques for coordinated traffic management (see e.g., 2,10,15,21), the effort towards a careful understanding and systematic treatment of management at the strategic time-frame is sparse. The most pivotal difficulties arise from the uncertainty of weather events, especially convective weather, that is unavoidable and plays a significant role at this time frame. Specifically, the barrier to developing a safe, robust, yet efficient strategic management plan under weather uncertainty can be summarized into three interrelated aspects: 1) the lack of precise weather forecasts several hours in advance, 2) the difficulty in evaluating uncertain weather impact on traffic, and 3) the difficulty in finding the optimal management solution that is robust to the variability of weather. In this paper, we seek a stochastic modeling approach to weather, traffic flow, and their integration, as a step towards developing a systematic treatment of the analysis and design of strategic management under weather uncertainty.

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Let us briefly review the existing literature relevant to management at the strategic time-frame, with the aim of summarizing the current research status, stressing research needs, and also motivating our approach. Because deterministic traffic models lack the capability to capture uncertainty at the strategic time-frame, **stochastic flow network models** (and in particular **queuing network models**) are considered to be valuable for strategic decision-making. Some recent efforts, including our group’s, in using queuing models to capture uncertain traffic and obtain insights for planning can be found in e.g., \(^8,9,11,18,22,24,25,28\) Of most interest to us, the article\(^{22}\) used a center-level open Jackson network model to evaluate path efficiency. In the studies\(^{24,27}\) with the aim of designing network-level enroute flow rates, a M/D/1 queuing network model was constructed, and various abstractions of it were sought to facilitate design. Articles\(^{24,27}\) were extended to provide insights into optimal routing design through a sensitivity study on queuing models\(^{25}\). These modeling and analysis efforts using queuing-network models provide a natural framework for the evaluation and design of management actions at the strategic time-frame, because they consider traffic as flows and ignore the schedule details of individual aircraft. However, these studies do not address the performance of the NAS in response to *dynamic* and *uncertain* weather events.

In a parallel vein, there have been advances in the evaluation and design of management actions under dynamic and uncertain weather. One straightforward approach to address this problem is Monte-Carlo simulations, i.e., using ensembles of uncertain weather to evaluate management actions and select optimal ones. An intelligent way of selecting a minimum set of ensembles was given\(^{26}\); these numerical methods however are not appropriate for robustly solving large-scale design problems. In order to improve efficiency, systematic analysis and design has been sought using stochastic programming approaches (see e.g.\(^3,12,13\)). For instance, weather was modeled using Markov chain models, and routing design was formulated as a Markovian Decision Processes\(^{13}\). These studies are valuable in providing systematic designs that take into account inherent weather uncertainty. However, they address the management of individual aircraft instead of flows, while flow-level designs are necessary at the strategic horizon considering the large dimension of the decision space at this horizon. Moreover, the uncertainties present in not only the weather but also the demand estimates at this timeframe do not warrant the computational expense incurred by employing a flight specific model.

Because of the significant role that weather plays in strategic decision-planning, analytical tools that permit the evaluation and design of strategic management actions under uncertain weather are urgently needed. To meet this need, we take the perspective that uncertain weather-impact models must be seamlessly *interfaced* with aggregated traffic flow models, and methods for analyzing the integrated models must be developed. Pursuing this direction, in this paper, we model traffic as stochastic flows, and model management actions as flow restrictions or queues that shape downstream flows (to comply with capacity constraints) at the cost of incurring delay/backlog upstream. Moreover, we model dynamic and uncertain weather impact using Markov chains. To capture the impact of uncertain weather events on flow, we consider the parameters of the queuing models as being *modulated* by uncertain weather. In particular, the service rates of the queuing models are viewed as being randomly *reduced* by convective weather impact. Such an integrated weather and flow modeling perspective develops the modeling foundation for the performance analysis under weather uncertainty that we pursue here.

The performance analysis of queuing systems with random service rate reduction, \(^1\), is mostly studied outside the air traffic management domain. In the field of road traffic planning, random service rate reduction is instead caused by uncertain events such as traffic accidents, vehicle failures, and other emergent road conditions. In paper\(^1\) and the references therein, random service rate reduction was modeled as a markovian modulated process, and a queuing analysis with *markovian modulated services* was sought. These studies are mostly focused on the analysis of *steady-state* performance, i.e., the statistics of queuing performance in a long run. However, in air traffic planning, *transient* performance is typically of significant value. For instance, a predicted temporary surge of traffic due to a severe storm, and the resultant congestion, are key factors in developing a strategic plan. To some extent, we can say strategic planning is in essence a redistribution of resources in advance of an event to alleviate temporary potential congestion caused by uncertain transient weather. In this paper, we develop *analytical* tools that can evaluate the impact of transient convective weather on uncertain flows, and thus give insight toward optimal management strategy design under weather
uncertainty at the strategic time-frame. Let us briefly discuss the specific analyses of the integrated stochastic weather and flow model that we conduct here. In particular, we consider two approaches that allow the transient performance analysis for queuing models driven by uncertain weather. In the first approach, we track both weather and flow dynamics using Markov chains, and investigate the analysis of steady-state and transient statistics of traffic delay under weather uncertainty. Though accurate, the computational complexities associated with intensive (transient) Markov chain analysis makes it hard to generalize this approach to queuing network models at a broader spatial scale. We then suggest a novel scalable jump-linear approach to analyze the integrated weather and flow models. We will show that the jump-linear approach is capable of effectively evaluating and comparing management actions under uncertainty. We envision that the jump-linear approach developed in this paper is promising for the evaluation and design of optimal flow contingency plans at a broad spatial scale, and is robust to likely weather scenarios.

It is worthwhile to note that this work is part of our ongoing effort to develop a Flow Contingency Management (FCM) framework for NextGen.23 We developed the model for a tool capable of predicting uncertain weather impact using stochastic automation;19,29 we also established a queuing network framework that allows the design of several management actions in practice or envisioned in NextGen.28 This paper discusses our efforts in integrating the above two directions, by developing systematic analytical and design tools for management actions under weather uncertainty.

The remainder of the paper is organized as follows. In Section II, we overview the use of stochastic models to represent weather impact, and the use of queuing models to capture management actions. We thus formulate the problem of analyzing the integrated two models. In particular, a stochastic automaton known as the influence model is used to capture the dynamics of stochastic weather at a broad spatio-temporal scale. Weather dynamics in a single region can be predicted from the model, and approximated using a low-order Markov chain model. Moreover, queuing models are used to represent management restrictions (e.g., Miles-in-trail (MIT) or Minute-in-trail (MINIT)) acting on flows. In Section III, we provide an extended Markov modeling approach to analyze the performance of management actions on flows in the presence of uncertain weather. Specifically, the statistics concerning the backlog of traffic can be obtained from the markov analysis of the integrated flow and weather model. In Section IV, we re-formulate the model as a jump-linear system, and show that this formulation permits an efficient performance analysis. Moreover, we use an example to show a key insight that the jump-linear modeling approach provides, concerning the role of detailed stochastic weather information in the performance of management actions. Finally, we discuss several features of the jump-linear approach. In Section V, a brief conclusion is provided.

II. Stochastic Weather and Flow Restriction Modeling: Overview and Problem Formulation

In this section, we provide an overview of the stochastic modeling of uncertain weather/weather-impact and of traffic flows/restrictions. We then motivate the problem of evaluating delay and backlog under uncertain weather, using Monte Carlo simulations.

A. Modeling Restriction’s Impact on Stochastic Flow

Queuing models are widely used to capture the impact of management actions on air traffic flows. In our concurrent study,26 we discussed the use of queuing models to capture various management actions, including MINIT/MIT, Rerouting, Time-based metering (TBM), ground-delay programs (GDP), and Airspace flow programs (AFP), as part of a comprehensive network model for air traffic. In this paper, we focus on a single traffic flow entering a weather zone (see Figure 1).

To begin, we recall that a single en route restriction’s impact (e.g., an MIT/MINIT restriction’s impact) under fixed weather conditions can be modeled using an M/D/1 queue (Memoryless inflow, deterministic service rate, and single server queue). Here let us briefly review the modeling of a single restriction under fixed weather, and then describe the M/D/1 model and the approximation of it using a saturation model (please see the article24 for more illustration).
We assume that a flow enters the boundary of a weather zone with an inflow rate $\lambda$ (the number of coming aircraft per unit time). In this paper, we focus on Poisson flows, i.e., the distribution of the number of aircraft coming to the restriction per unit time is $P(\lambda, k) = \frac{\lambda^k e^{-\lambda}}{k!}$ (the article for detailed discussion). However, the study developed in this paper can be generalized to other stochastic flows. In those cases, additional parameters may be required to describe the flow. As a flow is approaching a boundary, it is considered as entering an imaginary buffer. The number of aircraft in the buffer at time $t$ is denoted as buffer length $b(t)$. Because the en route rate restriction is set either by management actions or by weather-impacted capacity constraints, only a portion of the aircraft in the buffer is allowed to cross. The relationship between the crossing flow $e(t)$ and buffer length is denoted by $e(t) = f(b(t))$. In general, $f()$ can be either a deterministic or a stochastic function, depending on the nature of the restriction. The Backlog $B(t)$, which captures the number of aircraft being delayed at time $t$, is defined as the number of aircraft in the buffer excluding the ones crossing the boundary at the current time.

In the case of an $M/D/1$ queue, each aircraft takes a fixed service time (denoted as $T_c$) to cross the boundary. The service rate $u_c$ is defined as $\frac{1}{T_c}$. Specifically, if the buffer is not completely empty within $T_c$ time units after an aircraft leaves the boundary, the first aircraft in the buffer cannot cross the boundary until the $T_c$ duration is completed. The deterministic service time in the $M/D/1$ model forces a minimum separation distance/time between successive aircraft, and as such the $M/D/1$ model is natural to capture MINIT/MIT en route restrictions. Using standard queuing analysis, some steady-state backlog and delay statistics for the $M/D/1$ queue can be calculated. For instance, the mean backlog in steady-state can be calculated as

$$E(B) = \frac{\lambda^2}{2(u_c^2 - \lambda u_c)}$$

Unfortunately, it is not straightforward to find higher-order steady-state statistics or characterize transient dynamics for the $M/D/1$ model using standard queuing analysis, hence a saturation model was developed to approximate the $M/D/1$ model to permit richer analysis (especially for networks of restrictions). The saturation approximation is a discrete-time model that assumes the following: during any time interval $\Delta t$, a maximum number of $N_c = u_c \Delta t$ aircraft (denoted as saturation restriction) is allowed to cross the boundary. The saturation model can be mathematically described as:

$$e[k] = \begin{cases} 
  b[k-1], & (b[k-1] \leq N_c) \\
  N_c, & (b[k-1] \geq N_c)
\end{cases}$$

$$b[k] = b[k-1] + x[k] - e[k]$$

$$B[k] = b[k-1] - e[k]$$

where $e[k]$, $b[k]$ and $B[k]$ represent the crossing flow, the buffer length, and the backlog at time interval $k$. Clearly, the saturation model is a discrete-time version of the $M/D/1$ model. In the limit as $\Delta t$ is made small, the saturation model approaches the $M/D/1$ model. Let us briefly discuss validation of the model using an example. In Table 1, we approximate the steady-state mean backlog of an $M/D/1$ queue through Monte-Carlo simulation of the saturation model, when the inflow rate $\lambda = 9.5$ and service rate $u_c = 10$. As seen from the simulation results, as the time interval of the approximation is made smaller, the saturation model approaches the $M/D/1$ queuing model in predicting steady-state mean backlog, which is 9.025 according to Equation 1.
Table 1: Steady state mean backlog obtained from simulating the saturation model with different time intervals ($\lambda=9.5, u_c = 10$)

<table>
<thead>
<tr>
<th>Time Interval $\Delta t$</th>
<th>3 hour</th>
<th>1 hour</th>
<th>0.1 hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Backlog</td>
<td>6.87</td>
<td>7.90</td>
<td>9.025</td>
</tr>
</tbody>
</table>

In this paper, we use the saturation model to capture the impact of restrictions on flows. We will show that this discrete-time recursive description of an M/D/1 model permits systematic analysis of the queue’s transient dynamics, even under weather uncertainty.

**B. Modeling Stochastic Weather-Impact**

Properly modeling and predicting weather impact is critical for decision-making at the strategic time-frame. Existing ensemble and probabilistic forecast products are not suitable to be directly employed in strategic planning because of 1) they focus on describing weather rather than weather impact, 2) they lack spatio-temporal descriptions of weather dynamic, and 3) they are computationally incredibly complex. These limitations motivated us to develop a spatio-temporal weather-impact model using a stochastic automaton called the influence model.\(^{19,29}\)

Analysis of the spatio-temporal stochastic weather model can provide various weather-impact statistics of interest, e.g., the statistics of weather dynamics in a single weather zone. Such information may be of particular interest, for instance when a critical weather zone plays a significant role in delay performance (see the article\(^{19}\) for more illustration). When the statistics/pdf of the dynamical weather impact at a single zone is obtained from the stochastic weather model, continuous-time Markov chain models can be constructed to approximate the statistics (see e.g.\(^{6}\) for a technique for parameterizing Markov models so that state transition durations match desired pdfs). We are particularly interested in these Markov models for local weather impact, since we would like to study impact of weather on particular traffic flows; let us thus discuss these models in further detail. Specifically, in the Markov chain model, states represent different stages of weather or weather-impact evolution, and the weather-state probabilities are governed by

$$\dot{p}_w(t) = p_w(t)Q_w$$  \hspace{1cm} (3)

where $p_w(t) = [p_{w1}(t), ..., p_{wi}(t), ..., p_{wn}(t)]$, $p_{wi}(t)$ represents the probability of weather being at state $i$ at time $t$, $n$ is the number of states in the Markov chain, and $Q_w \in \mathbb{R}^{n \times n}$ is the continuous-time transition matrix.

In this development, we examine two classes of weather events separately, namely an extended-duration severe weather event (e.g. repeated occurrence of storms during a busy-traffic period, or a long-duration winter-storm) and a transient weather event that lasts for a while and then disappears (e.g. morning fog in San Francisco). In the first case, convective weather can be modeled using a recurrent Markov chain. In the second case, the weather is represented using a non-recurrent four-state continuous-time Markov chain ($Q_w$).

For instance, consider the case that severe weather is temporarily present in an airspace region, causing a decrease in the region’s capacity, but then disappears. The pdf of the duration of the capacity reduction at the single region due to the convective weather is shown in Figure 2a, as generated by the full spatio-temporal weather-impact model. The pdf can be well approximated as being generated from a non-recurrent four-state continuous-time Markov chain with $Q_w = \begin{bmatrix} -0.95 & 0.85 & 0 & 0.1 \\ 0 & -0.85 & 0.85 & 0 \\ 0 & 0 & -0.85 & 0.85 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (see Figure 2b). Specifically in this example, state 1 to 3 represent sever weather (the zone is at reduced capacity), and state 4 represents

\(^{a}\) A Markov chain is recurrent, if starting from any state in the Markov chain, there is a non-zero probability that the Markov chain will return to the starting state; otherwise, the Markov chain is non-recurrent.
that the severe weather is gone (the zone is at normal capacity). Once the Markov chain jumps into state 4, it stays at state 4 forever. The duration of capacity reduction is the length of time that takes the Markov chain to reach state 4 for the first time.

To track weather-state probabilities described by the Markov chain, it is simpler and more apt for air traffic decision-making to use a discrete-time approximation of the Markov chain. It is not difficult to obtain an accurate discretization, simply through choice of a small discretization interval, as shown in Equation 5 (see Figure 3 for a discrete time version of the example, with discretization time-step $\Delta t = 20$ min).

\[
\begin{align*}
\mathbf{p}_w[k + 1] &= \mathbf{p}_w[k] \mathbf{P}_w \\
&= \mathbf{p}_w[k](\mathbf{Q}_w \Delta t + \mathbf{I}),
\end{align*}
\]

where $\mathbf{p}_w[k] = [p_{w_1}[k], \ldots, p_{w_4}[k], \ldots, p_{w_n}[k]]$, $p_{w_i}[k]$ represents the probability of weather state $i$ at time interval $k\Delta t$, $\mathbf{P}_w \in \mathbb{R}^{n \times n}$ is the transition matrix, and $P_{w_{I,J}}$ represents the conditional probability that the Markov chain is at state $J$ given that the state is $I$ at the previous time step. For the weather model example discussed in this section, the probability that the duration of capacity reduction takes $k$ time steps can be obtained by recording $p_{w_4}[k]$ (the probability of being at state 4), and subtracting $p_{w_4}[k]$ from the probability being at state 4 at the previous time step $p_{w_4}[k - 1]$.

![Pdf of the Duration of Capacity Reduction](image1)

![Pdf Constructed from Markov Chain](image2)

Figure 2: (a) Pdf of the duration of capacity reduction at a single region (generated from the simulation of the influence model, (b) Pdf reconstructed from a 4-state Markov chain model

![Markov Model Diagram](image3)

Figure 3: Discrete time 4-state Markov model to generate the pdf of the duration of bad weather ($\Delta t = 20$ min)
C. Problem Formulation

The variability in weather events, and especially convective weather, creates significant difficulties in defining strategic traffic management actions as these uncertainties must be accounted for in the models developed. We take the perspective that by integrating the stochastic weather model (using the influence model) and the flow restriction model (using the queuing network model) we can analyze and design strategic management under weather uncertainty. In this development, we begin the investigation by addressing the backlog analysis when a single stream of flow intersects with a weather zone. Here, the bad weather reduces the service rate of the traffic restriction acting on the flow, to reflect deliberate flow management or intrinsic rate reduction due to the weather. In Figure 4, we show the dynamics of backlog statistics using Monte Carlo simulations. The comparison between Figure 4c and 4d reveals that the mean value and the mean duration of abnormal backlog caused by bad weather are enlarged if there is a greater chance of a prolonged severe weather event. The comparison between Figure 4d and 4e shows that larger inflow rates (representing more demand) also increase the mean value and the mean duration of the abnormal backlog. The Monte Carlo simulations demonstrate that both weather and inflow uncertainties produce significant impact on the number of aircraft delayed. Due to the computational cost associated with Monte Carlo simulations, and its limitations as a design tool for management actions, it is important to develop efficient analytical tools that permit the prediction of backlog dynamics with uncertain weather and flow.

Figure 4: Illustration of the impact of stochastic weather and inflow on backlog. Capacity/service rate (number of aircraft allowed to pass in a hour) is 3 under bad weather conditions, and 10 under good weather conditions. Plot (a) shows the pdf of the duration for a short span of bad weather. (b) shows the pdf of the duration for long span of bad weather. (c),(d), and (e) are the performance analyses for the two-weather-impact-distribution models, under two possible inflow conditions. Blue lines represent the backlog dynamics obtained from Monte Carlo sample runs, red lines represent the mean backlog, and yellow lines represent mean backlog plus the standard deviation of backlog. (c) shows the backlog corresponding to the inflow rate 3.1 and short span of bad weather. (d) is corresponding to the inflow rate 3.1 and prolonged span of bad weather. (e) is corresponding to the inflow rate 3.5 and long span of bad weather.
In the next two sections, we investigate the prediction of dynamical backlog statistics for a stochastic flow under the impact of uncertain weather. Specifically, uncertain weather is modeled using a discrete-time Markov chain model, and the weather-impact-modulated restriction is modeled using a saturation model that approximates a M/D/1 queue with time-varying service rates. The number of aircraft coming during each time interval follows a Poisson distribution, with the probability at time step \( k \) represented as

\[
P_x(x[k] = c) = \frac{(\lambda \Delta t)^c e^{(-\lambda \Delta t)}}{c!}, \ c \geq 0.
\]

Note: In all examples presented in the paper, we assume that the discretization interval \( \Delta t = 20 \text{min} \) for both the saturation model and the weather Markov model, considering that 20min allows a reasonable approximation and also that it is a reasonable time interval for planning at the strategic time frame.


To predict the statistics of backlog/delay caused by flow restrictions in the presence of uncertain weather, without using time-consuming Monte Carlo studies, we develop a discrete-time Markov approach to model and analyze the integrated weather and flow restriction models. Specifically, we use a Markov chain to track the dynamics of buffer length as described by the saturation model. We then construct a master Markov chain, whose states are the combined pairs of weather states and buffer lengths, so as to track the dynamics of buffer length under uncertain weather. Markov chain analysis permits the prediction of steady-state and transient backlog. Finally, we pursue approximation studies of the transient analysis for extreme weather/flow scenarios.

A. Integrated Markov model for weather and Poisson flow

Let us first consider tracking the dynamics of a saturation model with a fixed service rate \( u_c \). (or equivalently, with saturation restriction \( N_c = u_c \Delta t \) during a time interval \( \Delta t \)). The dynamics can be tracked using a Markov chain. Specifically, we construct an infinite-state Markov chain with each state \( i \in \{0, 1, 2, \ldots, \infty\} \) representing the buffer length, i.e., the number of aircraft in the buffer. The transition probability \( P_Q_{i,j} = P(s[k + 1] = j | s[k] = i) \) representing the probability of transiting from buffer length \( i \) to buffer length \( j \) can be calculated as

\[
P_Q_{i,j}(N_c) = \begin{cases} 
P_x(j), & 0 \leq i \leq N_c; j \geq 0 \\ 
P_x(j - i + N_c), & i > N_c; j \geq i - N_c \\ 
0, & i > N_c; j < i - N_c. \end{cases}
\]

We note that in simulation or analytical studies, we usually use a truncated finite-state Markov chain to approximate the infinite-state Markov chain. As long as the dimension of the finite-state Markov chain is sufficiently large, the approximation is accurate.

Now let us consider the full model with overlaid stochastic weather. When the flow enters a weather zone, we model the service rate of the flow-restriction as varying because of changing weather impact, and hence the transition probabilities in a Markov model for the queue will also vary. Recall from Section II.B that weather impact in a single region can be represented by a discrete-time Markov chain with transition probability \( P_{w_{I,J}} \), where \( I \) and \( J \) represent the states of weather as described by the Markov chain. For each weather state \( I \), we model the flow-restriction as having an (in general) different service rate. Therefore, when the weather state is \( I \), the queue length transitions according to the probability \( P_Q_{i,j}(N_I) \); where \( N_I \) is the saturation restriction value (which reflects the service rate) at weather impact state \( I \).

Now we integrate the queuing and weather Markov models by modeling flow restrictions (specifically, queue service rates) as being driven by the stochastic weather model. Specifically, we construct a larger-size master Markov chain, whose states are each defined as a combination of both a weather state \( I \) and a buffer...
length $i$ (which we denote as the pair $Ii$). The states transition in the master Markov chain according to both weather propagation and incoming flow. Since, given a current weather state and buffer length, weather and buffer length evolve independently, the transition probability of the master Markov chain is equal to the multiplication of the transition probabilities of the two individual Markov chains. Specifically, the transition probability of beginning at weather state $I$ when the number of aircraft in the buffer is $i$ (i.e., state $Ii$) and going to weather state $J$ and buffer length $j$ (i.e., state $Jj$) is represented as

$$P_{M_{Ii,Jj}} = P_{w}(s[k+1] = Jj | s[k] = Ii) \tag{7}$$

$$= P_{w}(s[k+1] = J | s[k] = I_i) P_Q(s[k+1] = j | s_w[k] = I, s_Q[k] = i)$$

Let us denote that the transition matrix for the master Markov chain as $P_M$.

Let us illustrate the construction of the master Markov chain using the weather impact model discussed in Section II.B. In the weather-impact Markov model, weather impact takes two states: for the good weather condition, the maximum number of aircraft allowed to pass during a time interval $\Delta t$ is $N_{c1}$; for the bad weather condition, the maximum number of aircraft allowed to pass during a time interval $\Delta t$ is $N_{c2}$. The integrated Markov chain for the 4-state weather model and M/D/1 queue model we considered is shown in Figure 5. In this case, $I = 1, 2, 3, 4$, $j = 1, 2, \ldots, \infty$, and $N_I = N_{c2}$ for $I = 1, 2, 3$, and $N_4 = N_{c1}$. When $I = 1, 2, 3$, the transition probability $P_{M_{Ii,Jj}} = P_{w,I} P_{Q,J}(N_{c2})$: when $I = 4$, $P_{M_{I4,Jj}} = P_{w,I} P_{Q,J}(N_{c1})$ For the good weather circumstance, the queue length transitions according to the probability $P_{Q,J}(N_{c2})$; and for the bad weather circumstance, the queuing length transitions instead according to $P_{Q,J}(N_{c2})$.

![Figure 5: Integrated master Markov chain](image)

**B. Steady-state and transient analysis**

The construction of the integrated Markov chain allows us to obtain statistics of backlog in a systematic fashion, instead of using intensive Monte-Carlo simulation. In this section, we summarize the steady-state and transient analysis of backlog statistics using the Markov chain approach.

Steady-state statistics can allow performance analysis over a long time horizon, e.g., computation of the average backlog or delay over a long time-span. For extended-duration severe weather, steady-state statistics of traffic backlog under weather uncertainty gives a valuable performance measure. However, for transient weather events, transient dynamics of the flow statistics is often of more interest, because of the short span of severe weather. We now examine both steady-state and transient analysis in greater detail.

In order to obtain the steady-state statistics of performance in terms of backlog, we need to first identify the steady state distribution of the integrated master Markov chain $p_M(s[k \rightarrow \infty])_{Ii}$ for $I = 1, \ldots, n$ and
\( i = 1, \ldots, \infty \), i.e., the steady-state probability of state \( I_i \) (where \( i \) is the queue length and \( I \) represents weather state). This steady-state probability can be found from the probabilistic recursion of the master Markov chain (or from one sample run, according to the ergodicity theory\(^{14}\)). From the steady-state probability distribution, the \( l \)th-order moment of backlog can be found as

\[
E(B^l) = \sum_{\forall i} \sum_{\forall I} \max(i - N_1, 0)^l p_M(s[k \to \infty])_{II}.
\]

(8)

where function \( \max(a, b) \) takes the maximum value of \( a \) and \( b \). We note that statistics of other performance metrics, such as delays, can be found in a very similar way.

The transient backlog at any time instance can be found by invoking the probability recursion of the combined Markov chain. Upon doing so, the \( l \)th-order moment of the backlog at time \( k \) can be found as

\[
\sum_{\forall i} \sum_{\forall I} \max(i - N_1, 0)^l p_M[k - 1]_{II},
\]

(9)

where \( p_M[k - 1]_{II} \) is the probability of the master Markov chain being at state \( I_i \) at time step \( k - 1 \).

Figure 6 shows that the mean and variance calculated from the the integrated Markov chain match those calculated from the Monte Carlo simulation. In this example, \( N_{c1} = 4, N_{c2} = 1, \Delta t = 20\text{min}, \) and \( \lambda = 3.5 \).

Figure 6: Comparison between the mean and variance calculated from the integrated Markov chain (b) and the Monte Carlo approach (a). The red curve shows the mean of backlog, and the yellow line shows the mean of backlog plus standard deviation.

The Markov approach allows the prediction of dynamical backlog statistics for both extended duration severe weather and transient weather. However, from the analysis, we see that the computation is not effective because of the use of infinite-state Markov chain (or a large truncated finite-state Markov chain for approximation) to track the saturation model. In order to obtain a good approximation, even for a single region, the dimension of the transition matrix is high. This limitation makes this approach hard to generalize to the evaluation of backlog for a network of regions, since the dimension of the Markov chain grows exponentially with the number of regions. Next, we seek for some lower-computation approximations of key performance metrics.

C. Approximations for transient statistics

In many cases, it may not be important to capture the complete transient dynamics, but only a few characteristics of the transient performance, such as the time that the maximum delay occurs, the maximum backlog, and the duration for the excessive backlog to vanish after a severe weather event passes. Our approach has been to use abstractions of the Markov-chain analysis to obtain simple approximations to the above quantities.
For instance, if we plot the peak of backlog with respect to a small range of varying parameters such as inflow rate, service rates, and the mean of bad weather duration (as shown in Figure 7a-c), we see that the peak of backlog can be predicted from these parameters using linear relationships. Similarly, the extended duration for the excessive backlog to vanish after a severe weather event passes can also be predicted using a linear relationship as shown in Figure 7d. In fact, under extreme conditions (i.e., inflow rate is much greater than service rate at bad weather, and much smaller than the service rate at good weather) and stochastic weather duration with small variance, we can verify that the peak of backlog (denoted as $B_p$) and the extended duration (denoted as $T_s$) can be roughly calculated from the following simple equations:

$$B_p = \left(\lambda - \frac{N_c}{\Delta t}\right) T_d$$

$$T_s = \frac{B_p - B_s}{\frac{N_c}{\Delta t} - \lambda}$$

where $B_s$ is the steady state backlog after the bad weather is gone and $T_d$ is the mean duration of bad weather. Such measures of backlog and excessive delay predicted using the linear relationships can be used to assist in the design of management actions to reach performance goals, under uncertain severe weather conditions.

For non-extreme conditions, the above linear relationships do not yield good predictions. We instead introduce a jump-linear approach to analyze the transient congestion in Section IV. The jump-linear approach allows novel and effective evaluation and possibly design of strategic management plans that take into account of all probable weather scenarios.

### IV. Integration of Weather and Flow Models for Performance Evaluation: A Jump-Linear Approach

To evaluate the *dynamical* impact of convective weather on uncertain flows at a broad spatial scale, we need analytical tools that are computationally-efficient. In this section, we develop a jump-linear approach to the modeling and analysis of the effect of uncertain weather’s impact on flows.

*Markovian Jump-linear systems*—i.e., linear systems whose parameters are modulated by an underlying Markov chain with finite state-space—are a broad class of stochastic hybrid models which have nice tractabilities (e.g., 4, 17, 20). In this section, we first introduce the jump-linear modeling of the integrated stochastic weather and flow restriction models, then present the prediction of impact statistics using this model, and finally discuss some features and benefits of the jump-linear approach.

#### A. Formulation of the dynamics as a jump-linear system

The jump-linear modeling of the integrated flow and weather model is based upon a linear abstraction of the impact of a restriction’s impact on flows. Linear abstractions are appealing for large-scale traffic flow modeling because of their tractability and scalability. Let us first describe the principles for developing a linear abstraction, and then the formulation of the integrated weather and flow restriction model into a jump-linear system based upon the linear abstraction. 24

A linear restriction model approximates the relationship between the crossing flow and buffer length as a linear (actually, affine) function. 24 Specifically, in a unit time, the crossing flow $e[k]$ is modeled as a fraction of buffer length $b[k]$ plus a constant, as shown in Equation 12.

$$e[k] = ab[k] + c$$

We note that the linear restriction can be made to resemble the saturation restriction well, through proper choice of the parameters $a$ and $c$. In essence, the linear restriction is a stochastic linearization of the nonlinear saturation model. For stringent saturation restrictions (i.e., $\lambda \Delta t$ is close to $N_c$), a is typically small while $c$ is moderate. Meanwhile, for loose saturation restrictions (i.e., $\lambda \Delta t$ is much smaller than $N_c$), $a$ is close to 1.
Figure 7: Linear prediction of the characteristics of transient dynamics: (a) The linear relationship between mean value of peak backlog and the saturation restriction under bad weather ($N_{c1} = 12, \lambda = 5.3$, prolonged bad weather (see Figure 2b)); (b) The linear relationship between mean value of peak backlog and inflow rate ($N_{c1} = 12, N_{c2} = 5$, prolonged bad weather (see Figure 2b)); (c) The linear relationship between mean value of peak backlog and the mean of bad weather duration ($N_{c2} = 4, N_{c1} = 7$, and $\lambda = 5.3$)); (d) The linear relationship between the extended duration of backlog and the saturation restriction under bad weather ($N_{c1} = 7, \lambda = 5.3$, prolonged bad weather (see Figure 2b)).
and $c$ is close to 0. The parameters $a$ and $c$ can be found by matching the statistical impact of the linear and saturation restrictions on flows. For instance, a procedure to find $a$ and $c$ by matching the steady-state mean backlog and downstream flow variance for Poisson flows was presented in.\textsuperscript{24} In cases where transient dynamics need to be matched, we can apply curve fitting tools to find the corresponding $a$ and $c$ for a particular combination of saturation restriction and incoming flow.

Now consider the case that the flow restriction is subject to modulation by stochastic weather. In this case, we model the linear restriction’s parameters as being modulated by the stochastic weather. That is, at different weather severities, the restriction strength and hence linear restriction parameters will be different. Since weather is modeled using a Markov chain, the parameters of the restriction are changing according to the weather Markov chain. The dynamics of the integrated model can thus be represented as a jump-linear system:

\begin{align*}
e[k] &= a([q[k]])b[k-1] + c([q[k]]) \\
b[k] &= b[k-1] + x[k] - e[k] \\
B[k] &= b[k-1] - e[k]
\end{align*}

where $q[k] \in \mathbb{R}^{n\times1}$ has only one entry as 1 and all other entries as 0, representing the state of the weather Markov chain at time step $k$, and $a(q[k])$ and $c(q[k])$ represent the values of the parameters $a$ and $c$ associated with the state of the Markov chain $q[k]$. For instance, for the weather Markovian model shown in Figure 3, $a$ and $c$ take the same value when the Markov chain is in states 1, 2, 3 representing a bad weather, and takes a different value when the chain is in state 4 representing good weather. The parameters $a(q[k])$ and $c(q[k])$ change values as the Markov chain jumps among states $q[k]$.

The jump-linear formulation presented above resembles the original saturation model integrated with weather, but allows nice tractability as shown in the next subsection. Here in Figure 8, we show the simulation of the jump-linear representation and the saturation representation for a particular Poisson flow with $\lambda \Delta t = 4.9$, $\Delta t = 20\text{min}$, and a particular weather ensemble where bad weather lasts for 5 hours on average and then disappears. The plots in Figure 8 demonstrate that the jump-linear abstraction captures the saturation restriction well.

![Figure 8: Comparison between the jump-linear model and the saturation model for a specific inflow sequence and weather sample](image)

### B. Statistical Analysis of the jump-linear model

In this section, let us demonstrate the prediction of backlog statistics using the jump-linear model. To do so, we write recursions for the moments the jump-linear model (Equation 13) into a moment-linear
representation, which allows us to trace the statistics of a jump-linear model.

Specifically, as an illustration, let us trace statistics of the backlog. To begin from Equation 13, the dynamics of backlog \( B[k] \) can be represented as

\[
B[k + 1] = (1 - a(q[k]))(B[k] + x[k]) - c(q[k])
\]

(14)

Now let us introduce the vector \( \sigma[k] \), which is defined as

\[
\sigma[k] = q[k] \otimes \begin{bmatrix} B[k] \\ 1 \end{bmatrix}.
\]

We note that the conditional expectation \( E[\sigma[k + 1] | \sigma[k]] \) can be written as

\[
E[\sigma[k + 1] | \sigma[k]] = E \left[ q[k + 1] \otimes \begin{bmatrix} B[k + 1] \\ 1 \end{bmatrix} | B[k], q[k] \right]
\]

\[
= E[\Delta q[k]] \otimes E \left[ \begin{bmatrix} B[k + 1] \\ 1 \end{bmatrix} | B[k], q[k] \right]
\]

\[
= P_w' q[k] \otimes \begin{bmatrix} 1 - a[q[k]] \\ 0 \\ 1 \\ 0 \end{bmatrix} + P_w' q[k] \otimes \begin{bmatrix} 1 - a[q[k]] \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} B[k] \\ 1 \end{bmatrix} + \lambda \Delta t
\]

\[
= P_w' q[k] \otimes \begin{bmatrix} \lambda \Delta t \\ 0 \\ 1 \end{bmatrix} + P_w' q[k] \otimes \begin{bmatrix} 1 - a[q[k]] \\ 0 \\ 1 \end{bmatrix} \sigma[k]
\]

(16)

From Equation 16, we can find the mean of \( \sigma[k + 1] \) as

\[
E[\sigma[k + 1]] = E[E[\sigma[k + 1] | \sigma[k]]]
\]

\[
= P_w'E[\sigma[k]] \otimes \begin{bmatrix} \lambda \Delta t \\ 0 \end{bmatrix} + P_w' \otimes \begin{bmatrix} 1 - a[q[k]] \\ 0 \\ 1 \end{bmatrix} E[\sigma[k]]
\]

(17)

Since all quantities in Equation 17 except the variables \( E[\sigma[k]] \) are known, this equation allows us to calculate the dynamical mean of \( \sigma[k] \) through an effective recursive fashion. In fact, if we define \( A = P_w'E[\sigma[k]] \otimes \begin{bmatrix} \lambda \Delta t \\ 0 \end{bmatrix} \) and \( B = P_w' \otimes \begin{bmatrix} 1 - a[q[k]] \\ 0 \\ 1 \end{bmatrix} \), \( E[\sigma[k + 1]] \) can be found using

\[
E[\sigma[k + 1]] = B^k (E[\sigma[0]] + (B - I)^{-1}A) - (B - I)^{-1}A.
\]

(18)

Moreover, from Equation 15, we can easily derive that \( E[B[k]] \) can be calculated from \( E[\sigma[k]] \) using

\[
E[B[k]] = 1_{1 \times n} E[\sigma[k]] - 1.
\]

(19)

In Figure 9, we show the prediction of mean backlog using the integrated Markov chain approach and the jump-linear approach. In this example, stochastic weather model is shown in Figure 2 and 3, \( \lambda \Delta t = 4.9 \), \( \Delta t = 20 \text{min} \), \( N_{c1} = 10 \) and \( N_{c2} = 5 \). The comparison between the two plots shows that the jump-linear approach allows a good prediction of mean backlog.

We note that higher-order statistics of the backlog, as well as statistics of other performance metrics, can be computed in similar fashion.
C. Example and Discussions

The major contribution of this section is the introduction of the jump-linear approach to the evaluation of air traffic system performance in the presence of uncertain weather. In this section, let us first use an example to show the insights the jump-linear approach provides, and then discuss some features/benefits of this approach.

It is very efficient to obtain various insights using the jump-linear approach. As an example, let us examine whether the mean weather duration is sufficient for performance prediction. In Figure 10, we compare backlog predicted from the full weather pdf and from the mean weather (i.e., assuming that the weather duration is equal to its mean value), using the simple recursions of the jump-linear approach. We see that there is a large offset between the two dynamics, as reflected by the measures such as the maximum mean backlog and the duration of excessive delay. This insight is indeed informative since it is typical in practice to use mean weather duration for traffic system performance evaluation, due to the difficulty in performance evaluation under uncertain dynamical weather. The use of mean weather condition as a deterministic condition for performance evaluation avoids dealing with the stochastic weather. However, this example shows that mean weather is not sufficient for a good prediction, and the availability of richer weather information (such as a pdf) allows a more precise prediction.

Here let us summarize the features of the jump-linear approach in air traffic system performance evaluation under weather uncertainty:

- **Efficiency.** As shown in the previous section, the jump-linear approach provides an effective way to evaluate traffic backlog (and other performance metrics) under uncertainty. As such, it is not necessary to perform intensive Monte Carlo simulations to evaluate the performance under uncertainty. The jump-linear approach is also much more effective than the integrated Markov chain approach in terms of computational time. The dimension of the recursion to find means in the jump-linear approach is $2n$, where $n$ is number of states in the Markov chain model for weather. Meanwhile, the integrated Markov chain requires a recursion with the order $mn$, where $m$ is the number of states in the truncated Markov chain that tracks the queue length. As the truncated Markov chain is used to approximate the infinite-state Markov chain, $m$ is large for a good approximation (i.e., $m >> 2$).

- **Precision.** The jump-linear approach allows the prediction of backlog statistics with weather modeled as a stochastic automaton. As shown by Figure 9, the use of a precise stochastic model for uncertain
weather provides an evaluation of backlog with nice precision. In fact, the only offset comes from the use of linear restriction to capture the saturation constraint. Moreover, the prediction is much more precise compared to the use of mean weather information as shown in the example in this section.

- **Scalability** The approach is promising to be generalized to evaluate performance at a broad spatial scale, with the whole system modeled as a big jump-linear system. The significant characteristic of the approach is that the dimension of computation grows linearly with the increase of the number of regions in consideration. As such the jump-linear approach has significant potential for the evaluation of NAS performance for the NextGen.

- **Designability** The design of optimal management actions using the jump-linear approach is concerned with choosing parameters of the linear restriction for best statistical performance under stochastic weather and flow. This task is in essence related to the control of jump-linear systems. The tractability of the jump-linear model makes the design problem tractable. We leave the design problem to future work.

V. Conclusions

Weather uncertainty plays a critical role in the performance of air traffic systems, especially in the strategic time-frame. Systematic and effective evaluation of system performance under dynamical weather uncertainty is a crucial step toward the design of strategic management actions. In this article, we have made two contributions to the study of strategic traffic management under weather uncertainty:

1) We have formulated integrated models of stochastic weather and traffic flow. Specifically, we have introduced models in which traffic flows and flow constrictions have parameters that are modulated by an underlying transient or long-duration weather process. We have argued that such models permit useful analysis of traffic performance metrics under weather uncertainty.

2) We have developed two methods, namely the integrated Markov approach and the jump-linear approach, that allow prediction of performance statistics like backlog under uncertain weather. Of particular note, the jump-linear approach models flow restriction using linear relationships, with parameters modulated by a Markov chain describing weather uncertainty. The tractability and scalability of the approach makes it promising for the evaluation and design of strategic management actions under uncertain weather at a broad spatial scale.
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