Special Session 39: Adaptive and Iterative Decomposition Methods for Differential Equations: Stability, Error Analysis and Applications

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Dimensional Splitting Methods for Exotic Option Pricing

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In this talk, we will present the theory behind some realistic financial mathematics problems and also discuss dimensional and operator splitting algorithms applied to these problems.

A typical multidimensional PDE from financial mathematics is a *basket option*. There exist various types of basket options but a generic equally weighted basket, $\alpha_i = \alpha_j, 1 \leq i,j \leq d$, can be governed by the following $d-dimensional$ reaction-convection-diffusion equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \frac{1}{2}\rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 u}{\partial S_i \partial S_j} + \sum_{i=1}^{d} r S_i \frac{\partial u}{\partial S_i} - ru$$

$$u(\mathbf{S},0) = \max\left(\sum_{i=1}^{d} \alpha_i S_i - K, 0\right)$$

These types of equations are ubiquitous in financial mathematics and fast, efficient, stable numerical solvers for these types of equations are very much in demand.

In this talk, we will present the state-of-the-art techniques for exotic option pricing where the underlying computational models are each described by a multidimensional PDE, which is typically a reaction-convection-diffusion equation. We will discuss the mathematical theory concerning consistency and stability of various dimensional splitting methods applied to various models and introduce the audience to some complex *exotic* options. It is these types of complex exotic options that are relevant and realistically used in the industry.

Finally, we will illustrate all the material covered in the talk with a wide variety of realistic numerical experiments.

1. P.G. Zhang "Exotic Options: A Guide to Second Generation Options" - World Scientific Pub Co Inc; 2 edition (August 1, 1998)
2. L. Andersen and J. Andreasen "Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods for Option Pricing" - Review of Derivatives Research, Volume 4, Number 3 / October, 2000
3. K.W. Morton and R.D Richtmyer "Difference Methods for Initial-Value Problems" - Krieger Pub Co; 2 edition (June 1994)
4. P.G. Ciarlet and J.L. Lions "Handbook of Numerical Analysis" - Volume I - Finite Difference Methods (Part 1), Solution of Equations in Rn (Part 1), January 1990

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Modified Jacobian Newton iterative method with embedded Domain Decomposition Method

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In this article a new approach is proposed for constructing a domain decomposition method based on the iterative operator splitting method for nonlinear differential equations. The convergence properties of such a method are studied. The main feature of the proposed idea is the decoupling of space and time. We present a multi-iterative operator splitting method that combines iteratively the space and time splitting and include the Newton iterative method to solve the nonlinearity. We confirm with numerical applications the effectiveness of the proposed iterative operator splitting method in comparison with the classical Schwarz waveform relaxation method as a standard method for domain decomposition. We provide improved results and convergence rates.

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To Be Determined

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To be determined soon.

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Dimension splitting for quasilinear parabolic equations

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Alexander Ostermann

In this talk, we derive a rigorous convergence analysis for a broad range of splitting schemes applied to abstract nonlinear evolution equations, including the Lie and Peacemann–Rachford splitting. The analysis is in particular applicable to (possibly degenerate) quasilinear parabolic problems and their dimension splittings. The abstract framework is based on the theory of maximal dissipative operators. The derived convergence is illustrated with a set of numerical experiments.

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Positive solutions of singular initial value problems for integro-differential equations

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We study singular initial value problems for integro-differential equations of the form

$$u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))ds, \quad t \in (0, 1], \\ u(0) = 0,$$

where $f(t, x)$ is singular at $x = 0$. We prove the existence of a positive solution by means of the lower and upper solutions method and the Brouwer fixed point theorem in conjunction with perturbation methods to approximate regular problems.

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Integrate Matrix Differential Riccati Equations Whose Solutions May Have Singularities

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Matrix Differential Riccati Equations (MDRE) arise frequently throughout applied mathematics, science and engineering. In this talk we shall demonstrate the ability of simple reflexive numerical formulas that are able to pass over the solution poles and still produce meaningful numerical results so long as the integration does not accidentally step onto the poles.

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Adaptive grids and application

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Abstract: We use the deformation method for moving grids to construct invertible transformation. We review the existing method by harmonic maps in 2D and discuss the difficulty it encounters in 3d domains. The proposed method is based on a div-curl-ode system, which can be used as constraints for a class of optimization problems. Preliminary numerical results demonstrated the effectiveness of the method. A theorem by Rado (1926) states that in 2d, any harmonic map from a domain to a convex domain is injective if it is one-to-one and onto between the boundaries. It turns out that its extension to 3d is false. Melas in 1993 constructed a harmonic map from a unit ball into another unit ball that is one-to-one and onto between the boundaries; but its Jacobian determinant is zero at the center. This counterexample was modified such that it maps two distinct points to the same point. The new method works for both 2d and 3d. Let $f > 0$ be any given function normalized to satisfy $\int f = |D|$, we construct an invertible transformation f by a div-curl-ode system such that $J(f(x)) = f(f(x))$. This problem is solved both theoretically and computationally. The solution method is based on Jurgen Moserś work in volume elements of Riemannian manifolds [Moser 1965]. The main technical advancement we made in [Cai 2004] is that we use div-curl in place of the Poisson equation, which enables the treatment of moving domains. It turns out that the flexibility provided by the curl equation can be explored in reconstruction of any given invertible transformation. Solution method: For a given function $f(x) > 0$ with $\int f = |D|$, we take $f(x) - 1$ as the right hand side of the div equation (Remark: the resulting divergence equation is the linearization of the equation $J(f) = f$). Then we take any divergence free vector field g (i.e. $\nabla \cdot g = 0$) as the right hand side of the curl equation. We solve for a vector field u from the div-curl equations: $\nabla \cdot u(x) = f(x) - 1, \nabla \times u(x) = g(x)$ with $u \cdot n = 0$, n = the outward normal on ∂D. Then, we form a velocity vector $v(x, t)$ by $v(x, t) = u(x)/(t + (1 - t)f(x))$ for a parameter $t \in [0, 1]$, $x \in D$. Now we solve a family of transformations $T(x, t)$, $t \in [0, 1]$, from the ordinary differential equation (ODE) for each fixed x in $D : \partial T(x,t)/\partial t = v(T(x,t),t)$ with $T(x,0) = x$. Finally, we define f by setting $f(x) = T(x,1)$. Application of this method to image registration problem will be described. Preliminary computational results will be presented to demonstrate the effectiveness of the approach. Reference: [Cai 2004] X. Cai, D. Fleitas, B.-N. Jiang, G. Liao, Adaptive grid generation based on the least square finite element method, Computers and Mathematics 48 (2004) 1077-1085 [Moser 1965] J. Moser, Volume elements of a Riemann Manifold, Trans AMS, 120, 1965

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Orthogonal H-type and C-type grid generation for 2-d twin deck bridge

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Orthogonal H-type and C-type boundary conforming grids for 2-d twin deck bridge are generated in this paper. The generation mainly consists of three steps. The first step of our approach aims to generate a simple but not smooth algebraic grid based on transfinite interpolation. According to the grids on the boundary of the physical domain and grids in the computational domain, grid distribution in the interior of physical domain can be obtained based on a pair of differential equations.

The second step is to make grid in physical domain to be smooth and stretched. We will have two elliptic transformation from physical domain to parameter domain.

By solving the equations using proper finite difference schemes, we generate smooth and stretched grid in the physic domain.

Our last step is to make the grid in the physical domain to be orthogonal around the boundary by varing the boundary condition. Suppose we want to get orthogonal grid on a special boundary E, we may first solve two Laplace equations and with following boundary condition to get new pair of s and t values.

The re-computation of s and t based on boundary function via Hermite interpolation can be formulated. Adaptive or iterative procedures can be used. Computational demonstrations will be given.

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Greens functions for Laplace equation and some new infinite product representations of elementary functions

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Derivation procedures for Greens functions are revisited in the trivial case of standard boundary-value problems for two-dimensional Laplace equation. Regions of a regular shape are considered, with Dirichlet and/or Neumann boundary conditions imposed on fragments of their boundaries. Classical closed analytic forms of Greens functions are reviewed and the method of images is used for obtaining their alternative representations in terms of infinite products. The latter are obviously less attractive computationally compared to the closed forms of Greens functions. But the point is, however, that a surprising aspect is disclosed when the two forms are compared bringing some new *summation* formulae for infinite products. These lead, in turn, to interesting results in the approximation of elementary functions. Alternative to the classical Eulers infinite product representations are obtained, in particular, for trigonometric sine and cosine functions. Very new infinite product representations are also derived for some other trigonometric and hyperbolic functions.

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A Posteriori Error Estimation and HP-Adaptivity for Fourth-Order Equations

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Marina Rangelova

I will present an extension of a hp-adaptive finite element strategy for second-order reaction-diffusion equations to fourth-order equations using a hierarchical C1 basis constructed from Hermite-Lobatto polynomials. A priori and a posteriori error estimates will be described, for both the solution at the current order and one order higher. These estimates are used to drive the hp-adaptivity. I will show that the a posteriori error estimates are asymptotically exact on grids of uniform order. Computational results for a linear equation and the Cahn-Hilliard equation will demonstrate estimator accuracy and the reliability of the adaptive strategy.

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Deflated Krylov Methods for Large Systems of Linear Equations

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Systems of linear equations are at the heart of the solution of many differential equations. We look at why Krylov subspace methods are often effective for large systems of linear equations and why the small eigenvalues of the matrix are so important.

The effectiveness of restarted Krylov methods such as GMRES can be improved by augmenting the subspace with approximate eigenvectors. Eigenvalues are deflated from the spectrum, which causes the convergence rate to improve. We will also look at how solution of problems with multiple right-hand sides can sometimes be much faster if eigenvector information generated in solving one right-hand side is used while solving the others. For symmetric problems, we give a restarted Lanczos method for both solving linear equations and computing eigenvectors. For multiple right-hand sides, a deflated conjugate gradient method is given.

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Restarting the Lanczos Algorithm for Large Eigenvalue Problems and Linear Equations

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Krylov methods are well-known for solving large eigenvalue problems and large systems of linear equations. This talk focuses on new iterative methods for large symmetric problems. Specifically we give a restarted Lanczos algorithm that solves both the eigenvalue and linear equations problem simultaneously. Also examples are given and convergence theorems are discussed.

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Splitting methods for nonautonomous evolution equations

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Eskil Hansen

In this talk we are concerned with splitting methods for the time integration of abstract evolution equations. We introduce an analytic framework which allows us to prove optimal convergence orders for various splitting methods. Our setting is applicable for a wide variety of linear equations and their dimension splittings. In particular, we analyze parabolic problems with time dependent coefficients and Dirichlet boundary conditions. We further illustrate our theoretical results with a set of numerical experiments.

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A matrix analysis for the z-stretching finite difference methods for interface problems

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Leonel Gonzalez, Shekhar Guha and Qin Sheng

A z-stretching finite difference method will be presented for simulating the paraxial light beam propagation through a lens in a cylindrically symmetric domain. By introducing proper domain transformations, we solve corresponding difference approximations on a uniform grid in the computational space for great efficiency. A specialized matrix analysis method is constructed to study the numerical stability. Interesting computational results are presented.

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Modified split shooting procedures for solving optical wave quenching-collapsing problems

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Recent electro-optical studies indicate that the spatial profile of a quenching, or collapsing, optical wave evolves to a specific circularly symmetric shape, known as the Townes profile, for elliptically shaped or randomly distorted input beams. Computations of such a Townes profile have been playing an important role in understanding of the wave collapse phenomenon, but the numerical procedures are sensitive due to features of the generalized nonlinear Schrödinger equation boundary value problems involved. This talk studies an effective semi-implicit split finite difference method equipped with a dynamic shooting strategy for the numerical solution of the optical boundary value problems. The numerical method proposed is simple in structure, easy to use, and weakly asymptotically stable.

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The left-definite spectral analysis of the fourth-order Legendre type differential equations

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In this talk, we develop the left-definite spectral theory associated with the self adjoint operator A in $L^2_\mu[-1,1]$, generated from the fourth-order Legendre type differential equation

$$\begin{aligned} \ell_{LT}[y](x) &= \left(\left(1-x^2\right)y''(x)\right)'' \\ &- \left(8+4A\left(1-x^2\right)y'(x)\right)' + ky(x) \\ &= \lambda y(x); \end{aligned}$$

here, $x \in (-1,1)$, A is a fixed, positive constant and k is a fixed, non-negative parameter. For certain values of the spectral parameter λ, this equation has a sequence $\{P_{m,A}\}_{m=0}^{\infty}$ of orthogonal polynomial eigenfunctions, called the Legendre type polynomials. More specifically, for each $n \in \mathbb{N}$, we explicitly determine the unique left-definite Hilbert-Sobolev space W_n and its associated inner product $(\cdot,\cdot)_n$. The key to determining these spaces and inner products is in finding the explicit Lagrangian symmetric form of the integral composite power of $\ell_{LT}[\cdot]$. In turn, the key to determining these powers is a remarkable new identity involving two sequences $\{a_{n,j}\}$ and $\{b_{n,j}\}$ of real numbers which we call Legendre type-Stirling numbers of types 1 and 2.

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On the superiority of the WCS method for scalar conservation law as truncation errors, dissipation and dispersion are concerned

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In this presentation we will derive the truncation errors, dissipation and dispersion terms for the 5th order Weighted Essentially Non-Oscillatory (WENO) Scheme and the Weighted Compact Scheme (WCS) for the scalar conservation law. Then these errors will be further analyzed, initially by a Fourier analysis.

A Fourier analysis of the errors associated with the WCS and WENO will be performed, firstly by assuming a smooth continuous function, which should generate results similar to the linear weight schemes. That is assuming that the spatial variable x is periodic over the domain [0, L] with h = L / N. By a decomposion, the dissipation and the dispersion errors can be analyzed through the plots of modified wavenumber versus wavenumber.

We analyze three different cases. For all these cases, the truncation error for WCS is better than the truncation error for WENO. We can also observe that WCS is best when the central stencil is used. In this case, WCS is an order higher than WENO, and it is only dispersive.

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