

## Special Session 42: Delay Differential Equations

Tibor Krisztin, University of Szeged, Hungary

### Delay-induced destabilization of entrainment of nerve impulses on ephaptically coupled nerve fibers

**Mohit Adhikari**

University of New Mexico, USA  
mohit@unm.edu

**Evangelos A. Coutsias and John K. McIver**

We study the effect of delay on the synchronization of two nerve impulses traveling along two ephaptically coupled, unmyelinated nerve fibers. Two coupled nonlinear diffusion equations of the Fitzhugh-Nagumo type, originally proposed by Luzader and Scott, are used to model this system with the delay introduced in the coupling terms. A multiple-scale perturbation approach is used for the analysis of these equations in the limit of weak coupling. In the absence of delay, two pulses with identical speeds are shown to be entrained precisely. However, as the delay is increased beyond a critical value, we show that this precise entrainment becomes unstable. We make quantitative estimates for the actual values of delay at which this can occur in the case of squid giant axons and compare them with the relevant time-scales involved.

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### Periodic solutions and global attractor of a class of delay differential equations

**Yuming Chen**

Wilfrid Laurier University, Canada  
ychen@wlu.ca

**Jianhong Wu and Taishan Yi**

We consider a class of delay differential equations modeling a system governed by delayed positive feedback and instantaneous damping. First, by applying the discrete Lyapunov functional, we show the uniqueness and absence of periodic orbits in the level sets of the discrete Lyapunov functional. Then, following the uniqueness of periodic orbits, we study the existence of periodic orbits. Finally, with the help of the Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay, we describe the structure of the global attractor as the finite union of the unstable sets of stationary points and periodic orbits.

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### A regularization for discontinuous differential equations with application to state-dependent delay differential equations

**Nicola Guglielmi**

Dipartimento di Matematica, Università dell'Aquila, Italy

guglielm@univaq.it

**Giorgio Fusco**

In this paper we consider a regularization for a class of discontinuous differential equations arising in the study of neutral delay differential equations with state dependent delays.

For such equations in fact the possible discontinuity in the derivative of the solution at the initial point may propagate along the integration interval giving rise to subsequent points, called “breaking points”, where the solution derivative is still discontinuous.

As a consequence, in a right neighbourhood of each such point we have to face a Cauchy problem where the equation has a discontinuous right-hand side. In this case the existence and the uniqueness of the solution is no longer guaranteed to the right of such points and hence the solution of the neutral equation may either cease to exist or bifurcate.

The regularization is based on the replacement in the r.h.s. of the derivative of the solution by its time average over an interval of length  $\varepsilon$  and then considering the limit as  $\varepsilon \rightarrow 0^+$ .

We show that the regularization we consider allow us to define a weak global solution on bounded time intervals, which agrees with Filippov's definition. Several properties of the solutions corresponding to small values of  $\varepsilon > 0$ .

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### Floquet Theory for Functional Differential Equations of Mixed Type

**Hermen jan Hupkes**

University of Leiden, Netherlands  
hhupkes@math.leidenuniv.nl

**S.M. Verduyn Lunel**

Recently, differential equations involving both delayed and advanced arguments have appeared in an increasing number of models, originating from a wide variety of scientific disciplines. We present recent

theoretical and numerical results concerning bifurcations from periodic solutions to such equations. In particular, we show how a smooth center manifold can still be constructed around periodic orbits, despite the absence of a monodromy operator.

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### On the stability of periodic solutions of a differential delay equation

**Anatoli Ivanov**

Pennsylvania State University, USA  
afil@psu.edu

One of the most studied differential delay equations (DDEs) is the following equation with a piece-wise constant nonlinearity  $f$ :

$$\varepsilon x'(t) = -x(t) + f(x(t-1)).$$

The simple form of  $f$  allows in many cases for an exact reduction of the dynamics of solutions of equation (\*) to those of finite dimensional maps. Important questions arise about the rigorous correspondence between the dynamics of the two, and in particular about the stability of periodic solutions in DDE (\*).

For the continuous and smooth nonlinearities  $f$ , which are close to the piece-wise constant ones, we study relationships between the stability of periodic solutions of DDE (\*) and the stability of corresponding fixed points in resulting finite-dimensional maps. Variational equations along periodic solutions are considered and necessary continuities on the singular parameter  $\varepsilon$  and small continuous/smooth perturbations of  $f$  are established.

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### Large periodic orbits for delayed monotone positive feedback

**Tibor Krisztin**

University of Szeged, Hungary  
krisztin@math.u-szeged.hu

**G. Vas**

We consider the equation  $\dot{x}(t) = -x(t) + f(x(t-1))$  with a real  $C^1$ -smooth nonlinearity  $f$  so that  $f' > 0$ . Assume that  $f(x) - x = 0$  has the solutions  $x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2$  with  $f'(x_{\pm 1}) > 1$  and  $f'(x_i) < 1$  for  $i \in \{-2, 0, 2\}$ . It is known that there exist periodic solutions with ranges in  $(x_{-2}, 0)$  and  $(0, x_2)$  provided  $f'(x_{-1})$  and  $f'(x_1)$  are large enough,

respectively. We show that for certain nonlinearities  $f$ , large periodic solution exists, that is, the minimum of the periodic solution is in  $(x_{-2}, x_{-1})$  and the maximum is in  $(x_1, x_2)$ .

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### Dynamics of a delay differential equation model of hepatitis B virus infection

**Yang Kuang**

Arizona State university, USA  
kuang@asu.edu

**S. A. Gourley, Y. Kuang and J. D. Nagy**

We formulate and systematically study the global dynamics of a simple model of HBV virus in terms of delay differential equations. This model has two important and novel features compared to the well known basic virus model in the literature. Specifically, it makes use of the more realistic standard incidence function and explicitly incorporates a time delay in virus production. As a result, the infection reproduction number is no longer dependent on the patient liver size (number of initial healthy liver cells). For this model, the existence and the component values of the endemic steady state are explicitly dependent on the time delay. In certain biologically interesting limiting scenarios, a globally attractive endemic equilibrium can exist regardless of the time delay length.

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### The change of invariant manifolds for $x'(t) = f(x(t-1))$ , when the shape of $f$ changes from monotone to non-monotone.

**Bernhard Lani-wayda**

Mathematisches Institut Universitaet Giessen, Germany

Bernhard.Lani-Wayda@math.uni-giessen.de

It is known that, for monotone  $f$  with negative feedback ( $\text{sign}(f(x)) = -\text{sign}(x)$ ), the infinite-dimensional dynamical system generated by the delay equation  $\dot{x}(t) = f(x(t-1))$  possesses a two-dimensional invariant manifold  $W$  with Poincaré-Bendixson-like dynamics, which attracts all slowly oscillating solutions.

On the other hand, it is known that if  $f$  has only one extremum, chaotic dynamics in the set of slowly oscillating solutions is possible. It is therefore of interest how the surface  $W$  develops under the change of  $f$  into a more complicated object. Analytical results in the direction are currently not existent – we

present numerical observations which provide a visualization of the deformation of  $W$ .

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### Recent advances about the uniqueness of the slowly oscillating periodic solutions of Wright's equation

**Jean-philippe Lessard**

Rutgers University & VU University Amsterdam, USA

lessard@math.rutgers.edu

An open question in delay equations is whether or not Wright's equation  $y'(t) = -\alpha y(t-1)[1+y(t)]$  has a unique slowly oscillating periodic solution (SOPS) for every parameter value  $\alpha > \pi/2$ . In this talk, we introduce a new theorem that partially answer the question. Via a rigorous continuation method, we will rule out the existence of folds (up to a parameter value  $\alpha^*$ ) on the branch of SOPS that bifurcate from the trivial solution at  $\alpha = \pi/2$ .

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### On the interplay between delay differential equations and one-dimensional maps

**Eduardo Liz**

Universidad de Vigo, Spain

eliz@dma.uvigo.es

**G. Rost**

The aim of this talk is showing how one-dimensional maps may be used to get bounds of the global attractor for delay differential equations of the form  $x'(t) = -ax(t) + f(x(t-\tau))$ , with  $a \geq 0$ ,  $\tau > 0$ . In special, the case when  $f$  is decreasing and has negative schwarzian derivative is considered. In the particular case of decreasing exponential nonlinearities, Wright's equation ( $a = 0$ ) and the Lasota-Ważewska model ( $a > 0$ ) are well-known examples satisfying our conditions. We revisit some remarkable results in the literature, provide some numerical examples, and formulate related open problems.

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### Global dynamics for non-monotone delayed feedback

**Gergely Röst**

University of Szeged, Hungary

grost@yorku.ca

**E. Liz and J. Wu**

The dynamics generated by the delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-\tau))$$

with unimodal feedback is studied. The existence of the global attractor is shown and (in some situations the sharpest) bounds of the attractor are given. We give sufficient conditions that guarantee that all solutions enter the domain where  $f'$  is negative with respect to a positive equilibrium, then the results for delayed monotone feedback can be applied to describe the asymptotic behaviour of solutions. In particular, the existence of heteroclinic orbits from the trivial equilibrium to a periodic orbit oscillating around the positive equilibrium is established. Several numerical examples are provided to illustrate the main results.

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### An epidemic model with post-contact prophylaxis of distributed length

**Horst Thieme**

Arizona State University, USA

thieme@math.asu.edu

**Abdessamad Tridane and Yang Kuang**

A possible control strategy against the spread of an infectious disease is the treatment with antimicrobials that are given prophylactically to those that had a contact with an infective person. The treatment continues until recovery or until it becomes obvious that there was no infection in the first place. The dynamic consequences of such a control strategy are studied. First the conditions are established for the extinction of the infectious disease. If they are not given, it is investigated whether the disease dynamics converge towards an endemic equilibrium or towards undamped oscillations. Since treatment length are considered that have an arbitrary distribution, our model system consists of ordinary differential and integral equations. Results from dynamical systems and frequency domain theory are used.

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### Parameter Identification in a Respiratory Control System Model

**Janos Turi**

University of Texas at Dallas, USA

turi@utdallas.edu

## Ferenc Hartung

In this presentation we discuss parameter identification issues by computational means for a set of nonlinear delay equations which have been proposed to model the dynamics of a simplified version of the respiratory control system. We design specific inputs for our system to produce “information rich” output data needed to determine values of unknown parameters. We also consider the effects of noisy measurements in the identification process. Several case studies are included.

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## Center-stable manifolds for differential equations with state-dependent delays

### Hans-otto Walther

Universitaet Giessen, Germany  
Hans-Otto.Walther@math.uni-giessen.de

### Redouane Qesmi

Consider the functional differential equation (FDE)  $\dot{x}(t) = f(x_t)$  with  $f$  defined on an open subset of the space  $C^1 = C^1([-h, 0], \mathbb{R}^n)$ . Under mild smoothness assumptions, which are designed for the application to differential equations with state-dependent delays, the FDE generates a semiflow on a submanifold of  $C^1$  with continuously differentiable time- $t$ -maps. We show that at a stationary point continuously differentiable local center-stable manifolds of the semiflow exist. The proof uses results of Chen, Hale and Tan and of Krisztin about invariant manifolds of time- $t$ -maps and their invariance properties with respect to the semiflow.

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## Heteroclinic connections for neutral functional differential equations

### Marion Weeder mann

Dominican University, USA  
mweeder m@dom.edu

We consider scalar neutral differential equations of

the form

$$\frac{d}{dt} [x(t) - ax(t-1)] = \gamma x(t-1) + f(x_t) \quad (1)$$

with  $|a| < 1$ , where  $f$  is such that the equation has two equilibria, a focus and a saddle with a one-dimensional unstable manifold. After defining a cone  $\Gamma$  in the phase space motivated by the local behavior near the equilibria, we use integral representations of the invariant manifolds to set up a contraction on  $\Gamma$ . The fixed point of the contraction is a heteroclinic solution of the given equation connecting the two equilibria.

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## Nonlinear waves in a delayed BAM neural network

### Yuan Yuan

Memorial University of Newfoundland, Canada  
yyuan@math.mun.ca

### Shangjiang Guo, Yuan Yuan and Lihong Huang

In this paper, we consider a class of delayed bidirectional associative memory (BAM) neural networks. Linear stability of the model is investigated by analyzing the associated transcendental characteristic equation. By means of the symmetric bifurcation theory of delay differential equations coupled with representation theory of standard dihedral groups, we not only investigate the effect of synaptic delays of signal transmission on the pattern formation, but also obtain some important results about the spontaneous bifurcation of multiple branches of periodic solutions and their spatio-temporal patterns. Based on the normal form approach and the center manifold theory, we derive the formula to determine the bifurcation direction and stability of Hopf bifurcated periodic solutions. These theoretical results are important to complement the experimental and numerical observations made in living neurons systems and artificial neural networks, in order to better understand the mechanisms underlying the system's dynamics.

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