Classifying Quadratic Quantum Planes using Graded Skew Clifford Algebras

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(supported in part by NSF grant DMS-0900239)

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Journal of Algebra 346 (2011), 152-164
with Manizheh Nafari & Jun Zhang
Motivation

- 2010: Cassidy & Vancliff → graded skew Clifford algebras (GSCAs)
  geometry determines when GSCA is regular etc.

- How useful are GSCAs in classifying (quadratic) regular algebras?

- Regular algebras of gldim 2 (resp, 1) are GSCAs. Gldim 3?

- The case of quadratic AS-regular algebras of gldim 3 (i.e., quadratic quantum planes) is the goal of this talk & is joint work with Manizheh Nafari and Jun Zhang.

- Henceforth, $k$ = algebraically closed field.
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Henceforth, $\mathbb{k} = \text{algebraically closed field}$. 
**Definition**

Let $\mu = (\mu_{ij}) \in M(n, \mathbb{k})$ be such that $\mu_{ij}\mu_{ji} = 1$ for all $i, j$ such that $i \neq j$. 

Clearly, $\mu_{ij} = 1$ for all $i, j$ $\Rightarrow$ $\mu$-symmetric = symmetric 
$\mu_{ij} = -1$ for all $i, j$ $\Rightarrow$ $\mu$-symmetric = skew-symmetric (if char($\mathbb{k}$) $\neq 2$).

**Example**

$n = 3$:

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\begin{pmatrix}
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**Assumption**

For the rest of the talk, assume \( \mu_{ii} = 1 \) for all \( i \).
Graded Skew Clifford Algebras

Definition ([ Van den Bergh, Le Bruyn ] char(\(k\) \(\neq\) 2 )

Let \( M_1, \ldots, M_n \in M(n, \mathbb{k}) \) denote symmetric matrices.

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Skew polynomial rings on generators \( x_1, \ldots, x_n \) with relations
\( x_i x_j = -\mu_{ij} x_j x_i \), for all \( i \neq j \), are GSCAs.
Graded Skew Clifford Algebras

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Definition ([ Van den Bergh, Le Bruyn ] \( \text{char}(k) \neq 2 \))

Let \( M_1, \ldots, M_n \in M(n, k) \) denote symmetric matrices. A \textit{graded Clifford algebra}, associated to \( M_1, \ldots, M_n \), is a graded \( k \)-algebra \( A \) on degree-1 generators \( x_1, \ldots, x_n \) and on degree-2 generators \( y_1, \ldots, y_n \) with defining relations given by:

(i) \[ x_i x_j + \mu_{ij} x_j x_i = n \sum_{k=1}^{n} (M_k)_{ij} y_k \] for all \( i, j = 1, \ldots, n \), and

(ii) the existence of a normalizing sequence \( \{ y'_{1}, \ldots, y'_{n} \} \subset A_2 \) that spans \( k y_{1} + \cdots + k y_{n} \).

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Skew polynomial rings on generators \( x_1, \ldots, x_n \) with relations \( x_i x_j = -\mu_{ij} x_j x_i \), for all \( i \neq j \), are GSCAs.
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Definition ([ Van den Bergh, Le Bruyn ] char(\(k\)) \(\neq 2\))

Let \(M_1, \ldots, M_n \in M(n, \kappa)\) denote symmetric matrices. A graded *Clifford algebra*, associated to \(M_1, \ldots, M_n\), is a graded \(\kappa\)-algebra \(A\) on degree-1 generators \(x_1, \ldots, x_n\) and on degree-2 generators \(y_1, \ldots, y_n\) with defining relations given by:

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Definition ([ Van den Bergh, Le Bruyn ] \( \text{char}(k) \neq 2 \))

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Graded Skew Clifford Algebras

Definition ([ Cassidy & Vancliff ] char(\(\mathbb{k}\)) \(\neq\) 2 )

With \(\mu\) as above, let \(M_1, \ldots, M_n \in M(n, \mathbb{k})\) denote \(\mu\)-symmetric matrices. A graded Clifford algebra, associated to \(M_1, \ldots, M_n\), is a graded \(\mathbb{k}\)-algebra \(A\) on degree-1 generators \(x_1, \ldots, x_n\) and on degree-2 generators \(y_1, \ldots, y_n\) with defining relations given by:

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Definition ([Cassidy & Vancliff] char(\( \mathbb{k} \)) \( \neq 2 \))

With \( \mu \) as above, let \( M_1, \ldots, M_n \in M(n, \mathbb{k}) \) denote \( \mu \)-symmetric matrices. A graded skew Clifford algebra, associated to \( M_1, \ldots, M_n \), is a graded \( \mathbb{k} \)-algebra \( A \) on degree-1 generators \( x_1, \ldots, x_n \) and on degree-2 generators \( y_1, \ldots, y_n \) with defining relations given by:

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Definition ([ Cassidy & Vancliff ] char($\mathbb{k}$) $\neq 2$ )

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Example

Skew polynomial rings on generators $x_1, \ldots, x_n$ with relations $x_i x_j = -\mu_{ij} x_j x_i$, for all $i \neq j$, are GSCAs.
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**Example**

Skew polynomial rings on generators \( x_1, \ldots, x_n \) with relations \( x_i x_j = -\mu_{ij} x_j x_i \), for all \( i \neq j \), are GSCAs.
Following [CV], to the data $\mu$ & $M_1, \ldots, M_n$ in the definition of GSCA, we associate

1. the skew polynomial ring $\mathcal{S}$ on generators $z_1, \ldots, z_n$ with defining relations:
   
   $z_j z_i = \mu_{ij} z_i z_j$,

   for all $i \neq j$, and

2. the elements $q_k = z^TM_k z \in \mathcal{S}$ where $z = [z_1 \ldots z_n]^T$.

Definition (Cassidy, Vancliff)

We call any (nonzero) element of $\mathcal{S}$ a quadratic form, and define the quadric $V(q)$, determined by any quadratic form $q$ to be the set of points in $P(\mathcal{S}^*) \times P(\mathcal{S}^*)$ on which $q$ and the defining relations of $\mathcal{S}$ vanish.

If $Q_1, \ldots, Q_m \in \mathcal{S}$, we call their span a quadric system. A quadric system $Q$ is said to be basepoint free (BPF) if $\bigcap q \in Q V(q)$ is empty; $Q$ is said to be normalizing if it is given by a normalizing sequence of $\mathcal{S}$.
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**Definition ([Cassidy, Vancliff])**

We call any (nonzero) element of $S_2$ a **quadratic form**, and define the **quadric**, $V(q)$, determined by any quadratic form $q$ to be the set of points in $\mathbb{P}(S_1^*) \times \mathbb{P}(S_1^*)$ on which $q$ and the defining relations of $S$ vanish.
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Theorem ([Cassidy, Vancliff])

A GSCA $A = A(\mu, M_1, \ldots, M_n)$ is a quadratic, Auslander-regular algebra of global dimension $n$ that satisfies the Cohen-Macaulay property with Hilbert series $1/(1 - t)^n$ iff

the quadric system associated to $M_1, \ldots, M_n$ is normalizing & BPF; in this case, $A$ is a noetherian AS-regular domain and is unique up to isomorphism.
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Quadratic Quantum Planes

Returning to the classification of quadratic regular algebras $D$ of global dimension 3....
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Returning to the classification of quadratic regular algebras $D$ of global dimension 3.... The classification depends on the point scheme $X$ of $D$: either $X \subseteq \mathbb{P}^2$ contains a line or it does not. The latter case, splits into 3 subcases, so in total we have 4 cases:

- $X$ contains a line
- $X$ is a nodal cubic curve in $\mathbb{P}^2$
- $X$ is a cuspidal cubic curve in $\mathbb{P}^2$
- $X$ is an elliptic curve in $\mathbb{P}^2$.

Note: our work attempts to classify all quadratic regular algebras $D$ of global dimension 3; not only the generic ones.
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J. Algebra 346 (2011), 152-164
uta.edu/math/vancliff
Theorem (char(\(k\)) \(\neq 2\))

If \(X\) contains a line, then either \(D\) is a twist, by an automorphism, of a GSCA,

\[\lambda x_1 x_2 = x_2 x_1, \lambda x_2 x_3 = x_3 x_2 - x_2^2, \lambda x_3 x_1 = x_1 x_3 - x_2^2,\]

where \(\lambda \in k\) and \(\lambda (\lambda^3 - 1) \neq 0\).

Moreover, for any such \(\lambda\), any quadratic algebra with these defining relations is regular & its point scheme \(X\) is a nodal cubic curve in \(\mathbb{P}^2\).
Theorem \((\text{char}(k) \neq 2)\)

*If* \(X\) *contains a line, then either* \(D\) *is a twist, by an automorphism, of a GSCA, or* \(D\) *is a twist, by a twisting system, of an Ore extension of a regular GSCA of gldim 2.*
Theorem (\(\text{char}(k) \neq 2\))

If \(X\) contains a line, then either \(D\) is a twist, by an automorphism, of a GSCA, or \(D\) is a twist, by a twisting system, of an Ore extension of a regular GSCA of gldim 2.

Theorem

If \(X\) is a nodal cubic curve, then \(D = k[x_1, x_2, x_3]\) with defining relations:

\[
\lambda x_1 x_2 = x_2 x_1, \quad \lambda x_2 x_3 = x_3 x_2 - x_1^2, \quad \lambda x_3 x_1 = x_1 x_3 - x_2^2,
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where \(\lambda \in k\) and \(\lambda(\lambda^3 - 1) \neq 0\).
Theorem ( $\text{char}(\mathbb{k}) \neq 2$ )

If $X$ contains a line, then either $D$ is a twist, by an automorphism, of a GSCA, or $D$ is a twist, by a twisting system, of an Ore extension of a regular GSCA of gldim 2.

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Theorem (char(\(k\)) ≠ 2)

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- If \( \lambda^3 \notin \{0, 1\} \), then \( D \) is an Ore extn of a regular GSCA of gldim 2;
- if \( \lambda^3 = -1 \), then \( D \) is a GSCA.
Theorem

\( X = \text{cuspidal cubic curve in } \mathbb{P}^2 \text{ iff } \text{char}(k) \neq 3 \) & \( D = k[x_1, x_2, x_3] \) with def rels:

\[
\begin{align*}
    x_1x_2 &= x_2x_1 + x_1^2, \\
    x_3x_1 &= x_1x_3 + x_1^2 + 3x_2^2, \\
    x_3x_2 &= x_2x_3 - 3x_2^2 - 2x_1x_3 - 2x_1x_2.
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(Moreover, any such algebra is regular, even if \( \text{char}(k) = 3 \).)
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It remains to consider \( X = \text{elliptic curve in } \mathbb{P}^2 \).
**Theorem**

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If \( \text{char}(k) \neq 2 \& X = \text{cuspidal cubic curve} \), then \( D \) is an Ore extn of a regular GSCA of gl.dim 2.

It remains to consider \( X = \text{elliptic curve in } \mathbb{P}^2. \)

In [AS, ATV1], such algebras are classified into types A, B, E, H, where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.
Theorem ( char(\( k \)) \neq 2 )

Suppose \( X \) is an elliptic curve.
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(iii) As in [AS, ATV1], regular algebras $D$ of type $A$ are given by

$D = \mathbb{k}[x, y, z]$ with def rels:

\begin{align*}
axy + byx + cz^2 &= 0, \\
ayz + bzy + cx^2 &= 0, \\
atz + bxz + cy^2 &= 0,
\end{align*}

where $a, b, c \in \mathbb{k}$, $abc \neq 0$, $3abc \neq (a^3 + b^3 + c^3)^3$, $\text{char}(\mathbb{k}) \neq 3$, and either $a^3 \neq b^3$, or $a^3 \neq c^3$, or $b^3 \neq c^3$.

- If $a^3 = b^3 \neq c^3$, then $D$ is a GSCA.
- If $a^3 \neq b^3 = c^3$ or if $a^3 = c^3 \neq b^3$, then $D$ is a twist, by an automorphism, of a GSCA.

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Remarks & Questions

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- If $D$ is a regular algebra of type A or E, then its Koszul dual is the quotient of a regular GSCA; so, in this sense, such algebras are weakly related to GSCAs.
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- Can cubic regular algebras of gldim 3 be classified using GSCAs?
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- Can quadratic regular algebras of gldim 4 be classified using GSCAs?