In the following, unless otherwise specified, \( \mathbb{F} \) denotes any field and \( V \) denotes any vector space over \( \mathbb{F} \).

**HOMEWORK 1**

1. Prove the following:
   (a) \(-(-\alpha) = \alpha \) for all \( \alpha \in \mathbb{F} \).
   (b) \((\alpha^{-1})^{-1} = \alpha \) for all nonzero \( \alpha \in \mathbb{F} \).
   (c) \((-1)\alpha = -\alpha \) for all \( \alpha \in \mathbb{F} \).
   (d) \((-1)(-1) = 1 \) in \( \mathbb{F} \).
   (e) \(-(-u) = u \) for all \( u \in V \).
   (f) \((-1)u = -u \) for all \( u \in V \).

2. Do Exercise 4 on page 15.

3. Read pages 1-5 and first half of page 6.

4. Let \( \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \). Prove that \( \mathbb{Q}[\sqrt{2}] \) is a field (you may use the fact that \( \mathbb{Q}[\sqrt{2}] \subset \mathbb{R} \)).

5. (a) Prove that if \( p \) is any prime number, then \( \{0, 1, 2, \ldots, p - 1\} \), mod-\( p \), is a field.
   
   (b) Is the same true if \( p \) were not prime, say \( p = 4 \)? Explain.

6. Let \( \mathbb{F}_2 = \{0, 1\} \) denote the field of two elements. Consider the set \( \mathbb{F} \) of four matrices of \( M_2(\mathbb{F}_2) \):
   \[
   \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.
   \]
   
   Prove that \( \mathbb{F} \) is a field. (You may use the properties of matrices which you know to hold for \( M_2(\mathbb{R}) \) (associativity etc), but you should state whichever ones you use.)

**HOMEWORK 2**

1. Do 1, 3(a),(c)-(f), 4-8, 10 on pages 32-33.

2. Show that the set of polynomials in one variable, where the polynomials have the same fixed degree, say \( n \in \mathbb{N} \), is not a vector space; that is, \( \{a_nx^n + \cdots + a_1x + a_0 : a_i \in \mathbb{F} \ \forall \ i, a_n \neq 0 \text{ or } a_i = 0 \ \forall \ i\} \) is not a vector space.
HOMEWORK 3

1. Prove that the set of $n \times n$ matrices, $M_n(F)$, as a vector space over $F$, has dimension $n^2$. What is the dimension of the set of $m \times n$ matrices as a vector space over $F$?

2. Prove that if $\{v_1, v_2\} \subseteq V$ is linearly dependent, then $v_1 \in Fv_2$ or $v_2 \in Fv_1$.

3. Do 1, 3-5 on page 37.

HOMEWORK 4

1. Read Example C on page 47.

2. Do Exercises 1, 2(a),(b),(d),(e), 3(a),(b),(d),(e), 5 on page 48.

3. Show that doing one elementary row operation to a matrix $A$ yields a matrix $B$ such that $B = CA$, where $C$ is the matrix obtained by doing the same row operation to an appropriate identity matrix.

HOMEWORK 5

- Do Exercises 1, 3-6 on page 52.

- Read pages 53-61.

HOMEWORK 6

- Do Exercises 1(b),(d),(e),(f), 2, 4 (think lin dep of columns) on page 61.

HOMEWORK 7

- Read examples B, C, D on pages 66-8.

- Do Exercises 1, 2(b),(d),(e),(f), 3 on page 68.

HOMEWORK 8

- Do Exercises 2-4, 5(a),(b), 9, 10 on pages 73/4.

- Read pages 75 & 76.

HOMEWORK 9

- Read pages 75-86.

- Do Exercises 1(b),(c),(d), 3-5, 6(a),(d), 10 on page 87.
1. Let \( \{v_1, v_2, \ldots, v_n\} \) be a subset of a vector space \( V \). Prove that the map \( T : \mathbb{F}^n \to V \) given by \( T((\alpha_1, \ldots, \alpha_n)) = \alpha_1 v_1 + \cdots + \alpha_n v_n \) is a linear transformation. (This could have been assigned in Homework 9.)

2. Do Exercises 2 and 4 on page 98.

3. Do Exercise 11 on page 108. (Use only definition of linear transformation, and matrix representing it.)

4. Find all linear transformations \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which map the line \( y = x \) to the line \( y = 3x \). (Think in terms of matrices.)

5. Find a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) such that \( T(\mathbb{R}^2) = \mathbb{R}e_1 \subset \mathbb{R}^3 \) with the property that \( \{x \in \mathbb{R}^2 : T(x) = 0\} = \mathbb{R}(1, 2) \). (Think matrices.)

6. Let \( A \) be an \( m \times n \) matrix. Prove that rank(\( A \)) \( \leq 1 \) if and only if \( A = BC \), where \( B \) is a column vector of length \( m \) and \( C \) is a row vector of length \( n \).

HOMEWORK 11

1. Do Exercises 1, 3(last line, as already did first part in class), 7-9 on pages 107/8.

2. For the maps in Exercise 1 on page 107, find their image/range, kernel/nullspace, nullity and rank.

3. Let \( V, W \) denote vector spaces over \( \mathbb{F} \) and let \( T \in \mathcal{L}(V, W) \). From lecture, you know that Ker(\( T \)) is defined to be Ker(\( T \)) = \( \{v \in V : T(v) = 0\} \). Prove that Ker(\( T \)) is a subspace of \( V \).

4. Read pages 109-117.

HOMEWORK 12 (Take \( \mathbb{F} = \mathbb{R} \).)

1. Prove that \( (0, v) = 0 \) for all \( v \in V \).

2. Do Exercises 1, 5 on page 129.

3. Prove that \( A \in M_n(\mathbb{R}) \) is orthogonal implies that \( A^T \) is orthogonal.

4. Do Exercise 6 on page 129.

5. Prove that the set of orthogonal transformations on \( V \) is closed under composition, contains the identity and contains the inverse of every element in the set.

6. Do Exercises 8, 9, 10, 12 on page 130.
HOMEWORK 13

1. Verify that \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \) is a basis for \( \mathbb{R}^3 \), and find the dual basis for \( (\mathbb{R}^3)^* \) relative to this basis.

2. Assume that \( V \) is finite-dimensional. Prove that \( T \in \mathcal{L}(V,V) \) is one-to-one if and only if \( T^* \in \mathcal{L}(V^*,V^*) \) is onto, and that \( T \) is onto if and only if \( T^* \) is one-to-one.

3. Let \( W \subseteq V \) be a subspace of \( V \) and let \( W^\perp \) denote
\[
W^\perp = \{ f \in V^* : f(w) = 0 \text{ for all } w \in W \}.
\]
Prove that \( W^\perp \) is a subspace of \( V^* \). Assuming \( \dim(V) < \infty \), prove \( \dim(W) + \dim(W^\perp) = \dim(V) \). (Hint: use bases.)

4. Let \( v \in V \) and let \( F_v : V^* \to \mathbb{F} \) be defined by \( F_v(f) = f(v) \) for all \( f \in V^* \).
   (a) Prove that \( F_v \) is a function and \( F_v \in (V^*)^* \).
   (b) Prove that the map \( \phi : v \mapsto F_v \) is well defined and is a linear transformation from \( V \) to \( (V^*)^* \).
   (c) Assume that \( V \) is finite-dimensional. Prove that \( \phi \) is an isomorphism.
   (d) Prove that (c) is false if \( V \) is infinite-dimensional. (Hint: consider elements of \( V^* \) and elements of \( (V^*)^* \) that assign the value 1 to each basis vector and use them to prove that \( \phi \) fails to be onto.)

HOMEWORK 14

1. For each of the following quadratic forms \( Q \), find a symmetric matrix \( A \) such that \( Q(x) = x^T A x \) for all \( x \in V \).
   (a) \( Q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2 \).
   (b) \( Q(x_1, x_2) = x_1x_2 \).
   (c) \( Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3 \).

2. For each of the following symmetric matrices, give the corresponding quadratic form.
   (a) \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \)
   (b) \( A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)
   (c) \( A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \).
HOMEWORK 15

For this homework, use only what has been taught, not what you know from undergraduate courses about determinants.

1. Read pages 132-4. Read proof of Theorem, part (e), on page 136-7.

2. Do Exercises 1(a)-(c) and 2 on page 139.

3. Let $D$ denote any determinant function. If $E$ is an elementary matrix, prove that $D(EA) = D(E)D(A)$.

4. Let $D$ denote any determinant function. If the rows of a square matrix $A$ are linearly independent, prove that $D(A) \neq 0$.

HOMEWORK 16

For this homework, use only what has been taught, not what you know from undergraduate courses about determinants.

- Do Exercises 1 and 2 on page 146.

HOMEWORK 17

For this homework, use only what has been taught, not what you know from undergraduate courses about determinants.

1. Do Exercises 1, 2, 5 on page 149.

2. Read pages 152-6.

3. Do Exercises 1, 2, 3, 11 on page 160.

HOMEWORK 18

- Do Exercises 1-4, 7, 10 on page 175.

- Read pages 176-180.

HOMEWORK 19

- Do Exercises 1, 3 on page 182.
HOMEWORK 20

- Read pages 187-8, 191-2.
- Do Exercises 3-12 on pages 192-3 (recall that, in Question 7, \( S(v) = \text{span}(v) = \mathbb{F}v \)).

HOMEWORK 21

1. Let \( V = \sum_{i=1}^{t} V_i \), where the \( V_i \) are subspaces of \( V \); i.e., each \( v \in V \) may be written \( v = \sum_{i=1}^{t} v_i \) where \( v_i \in V_i \) for all \( i \). Prove that if \( 0 = \sum_{i=1}^{t} u_i \) (where \( u_i \in V_i \) for all \( i \)) forces \( u_i = 0 \) for all \( i \), then \( V = \bigoplus_{i=1}^{t} V_i \). (Hint: see page 195.)

2. Read pages 193-200.

3. Do Exercises 1-6, 8 on page 201.

4. Suppose \( T, S \in \mathcal{L}(V, V) \) are diagonalizable. Prove that if \( S \circ T = T \circ S \), then there exists a basis for \( V \) such that the basis vectors are eigenvectors of both \( S \) and \( T \), so that \( S \) and \( T \) are simultaneously diagonalizable. (Hint: see page 200.)

HOMEWORK 22

1. Suppose \( \dim(V) < \infty \) and \( V = U \oplus W \), where \( U \) and \( W \) are subspaces of \( V \).
   (a) Prove that \( \mathcal{B} = \{ \text{basis for } U \} \cup \{ \text{basis for } W \} \) is a basis for \( V \). (This could have been assigned in Homework 21.)
   (b) Prove that \( \dim(V) = \dim(U) + \dim(W) \). (This could have been assigned in Homework 21.)
   (c) Let \( T \in \mathcal{L}(V, V) \) be such that \( T(U) \subseteq U \) and \( T(W) \subseteq W \). We define \( T|_U : U \to U \) (read \( T \) restricted to \( U \)) by \( T|_U(u) = T(u) \) for all \( u \in U \), and similarly for \( T|_W \). Prove that the matrix that represents \( T \), with respect to the basis \( \mathcal{B} \) from (a), is
   \[
   \begin{bmatrix}
   \text{matrix representing } T|_U & 0 \\
   0 & \text{matrix representing } T|_W
   \end{bmatrix}.
   \]

2. Do Exercise 1(a)-(e) on page 215 and find the Jordan normal (canonical) form of \( T \) in that exercise.

3. Repeat the last question for the \( T \) given in Exercise 2 on page 215.

4. Do Exercises 3-7, 10(a) on page 215-6.
5. Do Exercises 4, 6, 7 on page 226.

6. Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix}
0 & 0 & 0 & 4 & 0 & 0 & 0 \\
-3 & 3 & 2 & 8 & -2 & 0 & -1 \\
0 & 0 & 2 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 1 \\
\end{bmatrix},$$

and find an invertible matrix $P$ such that $P^{-1}AP$ is the Jordan normal form of $A$. (Hint: refer to the model example on the class handout.)

7. Assume $\mathbb{F} = \mathbb{C}$. Find the Jordan normal (canonical) form of the matrix

$$A = \begin{bmatrix}
37 & 10 & -5 & 5 \\
10 & 22 & 10 & -10 \\
-5 & 10 & 37 & 5 \\
5 & -10 & 5 & 37 \\
\end{bmatrix},$$

and find an orthogonal matrix $P$ such that $P^TAP$ is the Jordan normal form of $A$. (Hint: notice that $A$ is symmetric, and apply the Gram-Schmidt orthogonalization procedure to each eigenspace of $A$ in turn.)

END HOMEWORK