The required text is

- Ideals, Varieties and Algorithms: an Introduction to Computational Algebraic Geometry and Commutative Algebra, by D. Cox, J. Little, D. O’Shea, 2nd Ed., Undergraduate Texts in Mathematics, Springer (it will be denoted by [CLO] in this course);

but other books that might be helpful are

- Elementary Algebraic Geometry, by K. Hulek, Student Mathematical Library, Vol. 20, AMS Publications
- Undergraduate Algebraic Geometry, by M. Reid, Cambridge University Press
- Geometry, by E. Rees, Notes on Geometry, Universitext, Springer-Verlag
- Algebraic Geometry, by R. Hartshorne, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag
- Algebraic Geometry: a First Course, by J. Harris, Graduate Texts in Mathematics, Vol. 133, Springer-Verlag
- The Geometry of Schemes, by D. Eisenbud and J. Harris, Graduate Texts in Mathematics, Vol. 197, Springer-Verlag
- Principles of Algebraic Geometry, by P. Griffiths and J. Harris, Wiley and Sons
- Introduction to Commutative Algebra and Algebraic Geometry, by E. Kunz, Birkhaüser
- Algebraic Geometry, (3 books) by K. Ueno, Translations of Mathematical Monographs, AMS Publications
- Algebraic Curves and One-Dimensional Fields, by F. Bogomolov and T. Petrov, AMS Publications
- Elementary Algebraic Geometry, K. Kendig, Graduate Texts in Mathematics, Vol. 44, Springer-Verlag

In the following questions, \( \mathbb{C} \) denotes the complex numbers, \( \mathbb{R} \) denotes the real numbers, \( \mathbb{Z} \) denotes the set of all integers, \( \mathbb{N} \) denotes the set of positive integers, \( \mathbb{Q} \) denotes the rational numbers and \( k \) denotes a field.

1. Read pages 1 & 2 of [CLO].

2. Check that \( \mathbb{R} \) is a field by using the definition on page 497 of [CLO].

3. Check that if \( k \) is a field, then \( k \) is a vector space over \( k \).

4. Check that \( \mathbb{R}[x] \) is a vector space over \( \mathbb{R} \), but not a field.

5. Suppose \( V \) is a vector space over \( k \) and that \( W \) is a subspace of \( V \). If \( \dim(V) = n < \infty \) and \( \dim(W) = m \), find \( \dim(V/W) \).
6. Suppose $V$ is a vector space over $k$ and that $W_1$ and $W_2$ are subspaces of $V$. Define a map 
$\phi : W_1 + W_2 \to W_2/(W_1 \cap W_2)$ by $w_1 + w_2 \mapsto [w_2]$, where $w_i \in W_i$.
(a) Prove that $\phi$ is well defined.
(b) Find the image of $\phi$.
(c) Find the kernel of $\phi$.
(d) Establish that $(W_1 + W_2)/W_1 \cong W_2/(W_1 \cap W_2)$.

7. Let $\phi : \mathbb{R}[x, y] \to \mathbb{R}[y]$ be defined by $\phi(x^i y^j) = \begin{cases} 0 & \text{if } i \neq 0 \\ y^j & \text{if } i = 0 \end{cases}$ and extend linearly to $\mathbb{R}[x, y]$. Prove that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathbb{R}[x, y]$.

8. [CLO, Page 225, #8] Let $R$ and $S$ denote commutative rings with identity.
(a) Let $\phi : R \to S$ be a ring epimorphism. Show that $\phi(1) = 1$.
(b) Let $\phi : R \to S$ be a ring epimorphism. Suppose $r \in R$ has a multiplicative inverse, $r^{-1}$. Show that $\phi(r^{-1})$ is a multiplicative inverse for $\phi(r)$ in $S$.
(c) Suppose that $R$ and $S$ are isomorphic as rings. Show that if $R$ is a field, then $S$ is also a field.

9. Read the first half of page 221 of [CLO] and understand it.

10. [CLO, Page 224-5, #4] In this problem, we will give an algebraic construction of a field containing $\mathbb{Q}$ in which 2 has a square root. Let $R = \mathbb{Q}[x]$.
(a) Show that every element of $R$ is congruent modulo the ideal $I = R(x^2 - 1)$ to a unique polynomial of the form $ax + b$, where $a, b \in \mathbb{Q}$.
(b) Show that the class, $[x]$, of $x$ in $R/I$ is a square root of 2 in the sense that $[x]^2 = [2]$ in $R/I$.
(c) Show that $F = R/I$ is a field. (Hint: Using [CLO, Page 221, Theorem 6], it remains to prove that every nonzero element of $F$ has a multiplicative inverse in $F$.)
(d) Using (a), find a subfield of $F$ that is isomorphic to $\mathbb{Q}$.

11. [CLO, Page 225, #5] Let $R = \mathbb{R}[x]$. In this problem, we will consider the addition and multiplication operations in the quotient ring $R/R(x^2 + 1)$.
(a) Show that every element of $R$ is congruent modulo the ideal $I = R(x^2 + 1)$ to a unique polynomial of the form $ax + b$, where $a, b \in \mathbb{R}$.
(b) Construct formulas for the addition and multiplication rules in $R/R(x^2 + 1)$ using the polynomials in (a) as the standard representatives for equivalence classes.
(c) Is there another way to describe the ring $R/R(x^2 + 1)$? (Hint: what is $[x]^2$?)

12. [CLO, Page 225, #10] An element $r$ in a ring $R$ is said to be nilpotent if there exists $n \in \mathbb{N}$ such that $r^n = 0$. Let $R = k[x]$ and let $I = Rx^2$.
(a) Show that $[x]$ is a nilpotent element in $R/I$ and find the smallest $n \in \mathbb{N}$ such that $[x]^n = 0$.
(b) Show that every class in $R/I$ has a unique representative of the form $at + b$, where $a, b \in k$ and $t$ is shorthand for $[x]$.
(c) Given $at + b \in R/I$, we may define a mapping $R \to R/I$ by taking each element $f \in R$ and substituting $x = at + b$ in $f$. E.g., if $f = 3x^2 + 2$, then the image in $R/I$ under this mapping
is $3(at + b)^2 + 2 = 3(a^2t^2 + 2abt + b^2) + 2$, which can be simplified further. . . . Show that

$$f(at + b) = af'(b)t + f(b),$$

where $f'$ is the formal derivative of the polynomial $f$. (Thus, derivatives of polynomials may be constructed in a purely algebraic way.)

13. Let $R = \mathbb{C}[x]$ and $I = Rx$. Find an ideal $J$ of $R$ that does not contain $I$ and is not contained in $I$.

14. [CLO, Page 226, #14]
   (a) Let $R = \mathbb{R}[x]$ and $I = R(x^3 - x)$. Determine the ideals that contain $I$ by using [CLO, Page 223, Proposition 10]. Draw a diagram indicating which of these ideals are contained in which others.
   (b) How does your answer to (a) change if $I = R(x^3 + x)$ instead?

   (a) Let $I = Rx^2 + Ry^2$. Determine the ideals that contain $I$ by using [CLO, Page 223, Proposition 10].
   (b) Let $J = Rx^3 + Ry$. Is $R/I$ isomorphic to $R/J$?

16. Let $R = k[x]$, and $f = (x - a)(x - b)$, where $a, b \in k$. Show that $R/Rf$ is not an integral domain.

17. Read pages 3 & 4 of [CLO].

18. If $f \in W = kx_1 \oplus \cdots \oplus kx_n$, prove that evaluating $f$ at points of $\mathbb{A}^n = k^n$ yields a linear transformation $f : k^n \to k$.

19. [CLO, Page 5, #2] Let $k = \mathbb{F}_2$ denote the field with two elements.
   (a) Consider the polynomial $g(x, y) = x^2y + y^2x \in k[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in k^2$ and explain why this does not contradict [CLO, Proposition 5].
   (b) Find a nonzero polynomial in $k[x, y, z]$ that vanishes at every point of $k^3$. Try to find one that uses all three variables.
   (c) Find a nonzero polynomial in $k[x_1, \ldots, x_n]$ that vanishes at every point of $k^n$. Can you find one that uses all $n$ variables?

20. [CLO, Page 5, #5] In the proof of [CLO, Proposition 5], we took $f \in k[x_1, \ldots, x_n]$ and wrote it as a polynomial in $x_n$ with coefficients in $k[x_1, \ldots, x_{n-1}]$. To see what this looks like in a specific case, consider the polynomial

$$f(x, y, z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3.$$

   (a) Write $f$ as a polynomial in $x$ with coefficients in $k[y, z]$.
   (b) Write $f$ as a polynomial in $y$ with coefficients in $k[x, z]$.
   (c) Write $f$ as a polynomial in $z$ with coefficients in $k[x, y]$.

21. [CLO, Page 5, #6(a)] Inside of $\mathbb{C}^n$, we have the subset $\mathbb{Z}^n$, which consists of all points with integer coordinates. Prove that if $f \in \mathbb{C}[x_1, \ldots, x_n]$ vanishes at every point of $\mathbb{Z}^n$, then $f$ is the zero polynomial. Hint: adapt the proof of [CLO, Proposition 5].
22. Read pages 5-12 of [CLO].

23. [CLO, Page 12, #1] Sketch the following affine varieties in $\mathbb{R}^2$.
   (a) $\mathcal{V}(x^2 + 4y^2 + 2x - 16y + 1)$
   (b) $\mathcal{V}(x^2 - y^2)$
   (c) $\mathcal{V}(2x + y - 1, 3x - y + 2)$.

24. [CLO, Page 12, #2] In $\mathbb{R}^2$, sketch $\mathcal{V}(y^2 - x(x - 1)(x - 2))$. Hint: for which $x$'s is it possible to solve for $y$? How many $y$'s correspond to each $x$? What symmetry does the curve have?

25. [CLO, Page 12, #3] In the plane $\mathbb{R}^2$, draw a picture to illustrate
   $$\mathcal{V}(x^2 + y^2 - 4) \cap \mathcal{V}(xy - 1) = \mathcal{V}(x^2 + y^2 - 4, xy - 1),$$
   and determine the points of intersection. Note that this is a special case of [CLO, Lemma 2].

26. [CLO, Page 12, #4] Sketch the following affine varieties in $\mathbb{R}^3$.
   (a) $\mathcal{V}(x^2 + y^2 + z^2 - 1)$
   (b) $\mathcal{V}(x^2 + y^2 - 1)$
   (c) $\mathcal{V}(x + 2, y - 1.5, z)$
   (d) $\mathcal{V}(xz - xy)$ (Hint: factor.)
   (e) $\mathcal{V}(x^4 - zx, x^3 - yx)$
   (f) $\mathcal{V}(x^2 + y^2 + z^2 - 1, x^2 + y^2 + (z - 1)^2 - 1)$.
   In each case, does the variety have the dimension you would intuitively expect it to have?

27. [CLO, Page 12, #5] Use the proof of [CLO, Lemma 2] to sketch
   $$\mathcal{V}((x - 2)(x^2 - y), y(x^2 - y), (z + 1)(x^2 - y))$$
   in $\mathbb{R}^3$. Hint: this is the union of which two varieties?

28. Consider the set $Z = \{(1, 0), (2, 0)\} \subset \mathbb{R}^2$. Show that it is an affine variety by expressing $Z$ as the zero locus of certain polynomials in $\mathbb{R}[x, y]$.

29. [CLO, Page 12, #6] We will show that all finite subsets of $k^n$ are affine varieties.
   (a) Prove that a single point $(a_1, \ldots, a_n) \in k^n$ is an affine variety.
   (b) Prove that every finite subset of $k^n$ is an affine variety. Hint: [CLO, Lemma 2] will be useful.

30. [CLO, Page 13, #8] It can take some work to show that something is not an affine variety. For example, consider the set
   $$X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2,$$
   which is the straight line $y = x$ with the point $(1, 1)$ removed. To show that $X$ is not an affine variety, suppose that $X = \mathcal{V}(f_1, \ldots, f_s)$. Each $f_i$ vanishes on $X$, so if we can show that each $f_i$ also vanishes at $(1, 1)$, we will have a contradiction. Thus, here is what you have to prove: if $f \in \mathbb{R}[x, y]$ vanishes on $X$, then $f(1, 1) = 0$. Hint: let $g(t) = f(t, t)$, which is a polynomial in $\mathbb{R}[t]$; now apply the proof of [CLO, §1, Proposition 5].
31. [CLO, Page 13, #11] So far we have discussed varieties over \( \mathbb{R} \) or \( \mathbb{C} \). It is also possible to consider varieties over the field \( \mathbb{Q} \), although the questions here tend to be much harder. For example, let \( n \in \mathbb{N} \), and consider the variety \( F_n \subset \mathbb{Q}^2 \) defined by
\[
x^n + y^n = 1.
\]
Notice that there are some obvious solutions when \( x \) or \( y \) is zero. We call these solutions \textit{trivial solutions}. An interesting question is whether or not there are any nontrivial solutions.

(a) Show that \( F_n \) has two trivial solutions if \( n \) is odd and four trivial solutions if \( n \) is even.

(b) Show that \( F_n \) has a nontrivial solution for some \( n \geq 3 \) if and only if Fermat’s Last Theorem is false. (Fermat’s Last Theorem states that, for \( n \geq 3 \), the equation \( x^n + y^n = z^n \) has no solutions where \( x, y \) and \( z \) are nonzero integers. The general case of this conjecture was proved by Andrew Wiles in 1994 using some very sophisticated number theory and algebraic geometry. The proof is about 400 pages long.)

32. [CLO, Page 14, #15(c),(d)] In [CLO, Lemma 2], we showed that if \( V \) and \( W \) are affine varieties, then so are their union \( V \cup W \) and intersection \( V \cap W \).

(a) Give an example to show that the set-theoretic difference \( V \setminus W \) of two affine varieties need not be an affine variety.

(b) Let \( V \subset k^n \) and \( W \subset k^m \) be two affine varieties, and let
\[
V \times W = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in k^{n+m} : (x_1, \ldots, x_n) \in V, (y_1, \ldots, y_m) \in W \}
\]
be their cartesian product. Prove that \( V \times W \) is an affine variety in \( k^{n+m} \). Hint: if \( V \) is defined by \( f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \), then we can regard \( f_1, \ldots, f_s \) as polynomials in \( k[x_1, \ldots, x_n, y_1, \ldots, y_m] \), and similarly for \( W \). Show that this gives defining equations for the cartesian product.

33. Read pages 14-19 of [CLO].

34. [CLO, Page 22, #4] Consider the parametric representation \( x(t) = t/(1 + t) \), \( y(t) = 1 - 1/t^2 \).

(a) Find the equation (in terms of \( x \) and \( y \) only) of the affine variety determined by these parametric equations.

(b) Show that these parametric equations parametrize all points of the variety found in (a) except for the point \((1, 1)\).

35. Read pages 29-34 of [CLO].

36. [CLO, Page 35, #2] Let \( I \subset k[x_1, \ldots, x_n] \) be an ideal and let \( f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \). Prove that \( f_1, \ldots, f_s \in I \) if and only if \( \langle f_1, \ldots, f_s \rangle \subseteq I \).

37. [CLO, Page 35, #3(a)(b)] Use the previous question to prove the following equalities of ideals in \( k[x, y] \). (You may assume that \( n \neq 0 \) in \( k \) for all \( n \in \mathbb{N} \).)

(a) \( \langle x + y, x - y \rangle = \langle x, y \rangle \)

(b) \( \langle x + xy, y + xy, x^2, y^2 \rangle = \langle x, y \rangle \).
38. [CLO, Page 35, #4] Prove that if \( f_1, \ldots, f_s \) and \( g_1, \ldots, g_t \) are generating sets of the same ideal in \( k[x_1, \ldots, x_n] \), so that \( \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle \), then \( \mathcal{V}(f_1, \ldots, f_s) = \mathcal{V}(g_1, \ldots, g_t) \).

39. [CLO, Page 35, #5] Show that \( \mathcal{V}(x + xy, y + xy, x^2, y^2) = \mathcal{V}(x, y) \). (Hint: see the previous two questions.)

40. [CLO, Page 35, #6(b)(d)]
   
   (a) In linear algebra, a basis must span and be linearly independent over \( k \), whereas, for an ideal, a basis is concerned only with spanning – there is no mention of any sort of independence. The reason is that once we allow polynomial coefficients, no independence is possible. To see this, consider the ideal \( \langle x, y \rangle \subset k[x, y] \). Show that zero can be written as a combination of \( y \) and \( x \) with nonzero polynomial coefficients.

   (b) A consequence of the lack of independence is that when we write an element \( f \in \langle f_1, \ldots, f_s \rangle \) as \( f = \sum_{i=1}^s h_if_i, h_i \in k[x_1, \ldots, x_n] \), the coefficients \( h_i \) are not unique. As an example, consider \( f = x^2 + xy + y^2 \in \langle x, y \rangle \). Express \( f \) as a polynomial combination of \( x \) and \( y \) in two different ways. (Even though the \( h_i \) are not unique, one can measure their lack of uniqueness. This leads to the interesting topic of syzygies.)

41. Prove that if \( I \) is an ideal in a commutative ring, then \( \sqrt[\mathbb{Q}]{I} = \sqrt{I} \).

42. Prove that if \( I \) is an ideal in \( k[x_1, \ldots, x_n] \), then \( \sqrt[\mathbb{Q}]{I} \subseteq I(\mathcal{V}(I)) \) (the converse is also true if \( k \) is algebraically closed, but is much harder to prove, so we will cover it in class).

43. Prove that if \( I \) is an ideal in \( k[x_1, \ldots, x_n] \), then \( \mathcal{V}(\sqrt{I}) = \mathcal{V}(I) \).

44. Prove that if \( I \) is an ideal in \( k[x_1, \ldots, x_n] \), then \( \mathcal{V}(I(\mathcal{V}(I))) = \mathcal{V}(I) \).

45. An ideal \( M \) is maximal in a ring \( R \) if the only ideal that strictly contains \( M \) is \( R \). A division ring is a ring with 1 in which every nonzero element has a multiplicative inverse (i.e., it is a “noncommutative field”).

   (a) Suppose that \( R \) is a (not necessarily commutative) ring with 1 and that \( M \subset R \) is an ideal (meaning, 2-sided ideal). Prove that if \( R/M \) is a division ring, then \( M \) is maximal.

   (b) Prove that if \( R \) is commutative with 1, then an ideal \( M \subset R \) is maximal if and only if \( R/M \) is a field.

46. A commutative ring \( R \) is called reduced if, for every \( f \in R \) and each \( n \in \mathbb{N} \), \( f^n = 0 \iff f = 0 \) (i.e., \( R \) is reduced if it contains no nonzero nilpotent elements). Prove that \( R \) is reduced if and only if \( \{0\} \) is radical.

47. Let \( R \) denote a commutative ring and let \( I \) be an ideal of \( R \). Prove that \( R/I \) is reduced if and only if \( I \) is radical. Hint: see the previous question.

48. [CLO, Page 36, #10] Use the argument given in class for the twisted cubic to show that \( I(\mathcal{V}(x - y)) = \langle x - y \rangle \) in \( k[x, y] \). Your argument should be valid for any infinite field \( k \).
49. [CLO, Page 36, #11] Let $V \subset \mathbb{R}^3$ be the curve $\{(\alpha, \alpha^3, \alpha^4) : \alpha \in \mathbb{R}\}$.
   (a) Prove that $V$ is an affine variety.
   (b) Adapt the method used in class for the case of the twisted cubic to determine $I(V)$.

50. Read pages 37-40 of [CLO].

51. [CLO, Page 45, #5] If $f, g \in k[x]$, prove that $\langle f - qg, g \rangle = \langle f, g \rangle$ for any $q \in k[x]$.

52. [CLO, Page 79, #18] If an ideal has a basis where some of the elements can be factored, we can use the factorization to help understand the variety.
   (a) Show that if $g \in k[x_1, \ldots, x_n]$ factors as $g = g_1g_2$, then, for any $f \in k[x_1, \ldots, x_n]$, $\mathcal{V}(f, g) = \mathcal{V}(f, g_1) \cup \mathcal{V}(f, g_2)$.
   (b) Show that, in $\mathbb{R}^3$, $\mathcal{V}(y - x^2, xz - y^2) = \mathcal{V}(y - x^2, xz - x^4)$.
   (c) Use (a) to describe and/or sketch the variety from (b).

53. Read pages 177.5-178 of [CLO].

54. [CLO, Page 180, #14] Let $J = \langle xy, (x - y)x \rangle$. Describe $\mathcal{V}(J)$ and show that $\sqrt{J} = \langle x \rangle$.

55. Read pages 180-184.5 of [CLO].

56. [CLO, Page 189, #9] Let $k$ denote an arbitrary field and let $R$ denote $k[x_1, \ldots, x_n]$. Prove that $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$. Give an example to show that the product of radical ideals need not be radical.
   Give an example to show that $\sqrt{IJ}$ can differ from $\sqrt{I} \sqrt{J}$.

57. [CLO, Page 200, #2] Prove that every prime ideal is radical.

58. [CLO, Page 200, #5] Express $f = x^2z - 6y^4 + 2xyz$ as $f = f_1(x, y, z)(x + 3) + f_2(x, y, z)(y - 1) + f_3(x, y, z)(z - 2)$ for some $f_1, f_2, f_3 \in k[x, y, z]$.

59. [CLO, Page 200, #11] Prove that if $f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible, then so is $\mathcal{V}(f)$.

60. Read pages 203.5-204 of [CLO].

61. Let $R$ denote a commutative ring with 1. Prove that if $I$ is an ideal of $R$, then $I$ is prime if and only if $R/I$ is an integral domain.

62. Prove that $\langle xy, xz \rangle$ is radical but not prime.

63. From class, we know that a radical ideal is an intersection of prime ideals. Give an example of a nonradical ideal that is not an intersection of prime ideals.

64. Read §8 on pages 210-211 of [CLO].

65. [CLO, Page 217, #1] Let $V$ denote the twisted cubic curve in $\mathbb{R}^3$ and let $W = \mathcal{V}(v - u - u^2)$ in $\mathbb{R}^2$. Show that $\phi(x, y, z) = (xy, z + x^2y^2)$ defines a regular function from $V$ to $W$. (Hint: consider a parametrization of $V$.)
66. [CLO, Page 217, #2] Let \( V = \mathcal{V}(y - x) \) in \( \mathbb{R}^2 \) and let \( \phi : \mathbb{R}^2 \to \mathbb{R}^3 \) be the regular function represented by \( \phi(x, y) = (x^2 - y, y^2, x - 3y^2) \). Prove that the image of \( V \) under \( \phi \) is an affine variety in \( \mathbb{R}^3 \); find a system of equations defining this image.

67. [CLO, Page 218, #5] Show that \( \phi_1(x, y, z) = (2x^2 + y^2, z^2 - y^3 + 3xz) \) and \( \phi_2(x, y, z) = (2y + xz, 3y^2) \) represent the same regular function from the twisted cubic curve in \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \).

68. [CLO, Page 218, #6] Consider the regular function \( \phi : \mathbb{R}^2 \to \mathbb{R}^5 \) defined by \( \phi(u, v) = (u, v, u^2, uv, v^2) \).

(a) The image of \( \phi \) is a variety \( V \) known as an affine Veronese surface; find polynomial equations defining \( V \).

(b) Show that the projection \( \pi : V \to \mathbb{R}^2 \) defined by \( \pi(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2) \) is the inverse mapping of \( \phi \) taking \( \mathbb{R}^2 \) to \( V \). What does this imply about \( V \) and \( \mathbb{R}^2 \)?

69. [CLO, Page 243, #4] Let \( V = \mathcal{V}(y - x^n, z - x^m) \), where \( m, n \in \mathbb{N} \). Show that \( V \) is isomorphic as a variety to \( k \) by constructing explicit inverse regular functions \( \phi : k \to V \) and \( \chi : V \to k \).

70. [CLO, Page 218, #8] Let \( V = \mathcal{V}(xy, xz) \subset \mathbb{R}^3 \).

(a) Show that neither of the regular functions \( f = y^2 + z^3 \) and \( g = x^2 - x \) is identically zero on \( V \), but that their product is identically zero on \( V \).

(b) Find \( V_1 = V \cap \mathcal{V}(f) \) and \( V_2 = V \cap \mathcal{V}(g) \) and show that \( V = V_1 \cup V_2 \).

71. (a) Consider \( \langle x \rangle \lhd \mathbb{C}[x] \) and write \( V = \mathcal{V}(x) \).

(i) Find \( |V| \).

(ii) Find \( \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x]}{\langle x \rangle} \right) \).

(b) Consider \( \langle x^2 \rangle \lhd \mathbb{C}[x] \) and write \( V = \mathcal{V}(x^2) \).

(i) Find \( |V| \).

(ii) Find \( \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x]}{\langle x^2 \rangle} \right) \).

72. Let \( I \lhd k[x_1, \ldots, x_n] \) and write \( V = \mathcal{V}(I) \).

(a) Prove that if \( |V| < \infty \), then \( |V| \leq \dim_k \left( \frac{k[x_1, \ldots, x_n]}{I} \right) \). (Hint: (i) Write \( V = \{ p_1, \ldots, p_m \} \subset k^n \). Prove that there exist \( f_i \in k[x_1, \ldots, x_n] \) such that, for all \( i, f_i(p_i) = 1 \) and \( f_i(p_j) = 0 \) if \( j \neq i \) (see page 232.5 in [CLO]).

(ii) Prove that \( \left\{ [f_i] \in \frac{k[x_1, \ldots, x_n]}{I} \right\}_{i=1}^m \) is a linearly independent set and explain why (a) now follows.)

(b) Suppose that \( k \) is an algebraically closed field and \( |V| < \infty \). Prove that if \( \sqrt{I} = I \) (i.e., \( I \) is radical), then \( |V| = \dim_k \left( \frac{k[x_1, \ldots, x_n]}{I} \right) \). (Hint: prove that the \( [f_i] \) in part (a)(ii) span \( \frac{k[x_1, \ldots, x_n]}{I} \) )

(c) Let \( k = \mathbb{R} \) and \( V = \mathcal{V}(x^2 + 1) \). Show that \( |V| < \dim_{\mathbb{R}} \left( \frac{\mathbb{R}[x]}{x^2 + 1} \right) \), so that equality in (b) can fail if \( k \) is not an algebraically closed field, even if \( \sqrt{I} = I \).
(d) Let $k$ denote any field and suppose that $\dim_k \left( \frac{k[x_1, \ldots, x_n]}{I} \right) < \infty$.

(i) Prove that $\dim_k \left( \frac{k[x_1, \ldots, x_n]}{\sqrt{I}} \right) \leq \dim_k \left( \frac{k[x_1, \ldots, x_n]}{I} \right)$.

(ii) Prove that $|V| \leq \dim_k \left( \frac{k[x_1, \ldots, x_n]}{\sqrt{I}} \right)$.

73. [CLO, Page 243, #1] Let $C$ denote the twisted cubic curve in $\mathbb{R}^3$.

(a) Show that $C$ is a subvariety of the surface $S = V(xz - y^2)$.

(b) Assuming that $I(S) = \langle xz - y^2 \rangle$, find an ideal $J \triangleleft k[S]$ such that $C = V_S(J)$.

74. Read pages 249-252 of [CLO].

75. [CLO, Page 253, #9] Let $V$ denote an irreducible variety and let $f \in k(V)$. If we write $f = g/h$, where $g, h \in k[V]$, then we know that $f$ is defined on $V \setminus V_V(h)$. However, as discussed in class, $f$ might make sense on a larger set. We will illustrate this phenomenon in this question by considering $V = V(xz - yw) \subset \mathbb{C}^4$.

(a) You may assume that $xz - yw$ is irreducible in $\mathbb{C}[x, y, z, w]$ and that $\langle xz - yw \rangle$ is a prime ideal in $\mathbb{C}[x, y, z, w]$. Show that it follows that $V$ is irreducible and that $I(V) = \langle xz - yw \rangle$.

(b) Let $f = [x]/[y] \in \mathbb{C}(V)$, so that $f$ is defined on $V \setminus V_V([y])$. Show that $V_V([y])$ is the union of planes $V(x, y) \cup V(y, z)$.

(c) Show that $f = [w]/[z]$ in $\mathbb{C}(V)$, and conclude that $f$ is defined on $V \setminus V(y, z)$, so that $f$ has domain larger than originally thought in (b). (Note that what makes this possible is that we have two fundamentally different ways of representing the rational function $f$. This is one reason why rational functions are subtle to deal with.)

76. Read pages 349-356 of [CLO].

77. Let $k = \mathbb{R}$ and consider the hyperbola $V(xy - 1) \subset \mathbb{R}^2$.

(a) Homogenize the defining polynomial to get a new variety $V(f) \subset \mathbb{P}^2$.

(b) Recognise the original hyperbola as a subset $V$ of $V(f)$.

(c) Find $V(f) \setminus V$, and describe it.

(d) Explain what $V(f) \setminus V$ is describing intuitively in the $\mathbb{R}^2$ picture.

78. Let $k = \mathbb{R}$ and consider the vertical line $V(x - c) \subset \mathbb{R}^2$, where $c \in k^\times$.

(a) Homogenize the defining polynomial to get a new variety $V(f) \subset \mathbb{P}^2$.

(b) Recognise the original vertical line as a subset $V$ of $V(f)$.

(c) Find $V(f) \setminus V$, and describe it.

(d) Explain what $V(f) \setminus V$ is describing intuitively in the $\mathbb{R}^2$ picture.

79. Let $k$ be an algebraically closed field. Show that every homogeneous polynomial in two variables with coefficients in $k$ can be completely factored into a product of linear homogeneous polynomials.

80. Read pages 364-366 of [CLO].
81. [CLO, Page 378, \#11(a)(b)] A homogeneous ideal is said to be \textit{prime} if it is prime as an ideal in \( k[x_0, \ldots, x_n] \).
   (a) Prove that a homogeneous ideal \( I \subset k[x_0, \ldots, x_n] \) is prime if and only if whenever the product of two \textit{homogeneous} polynomials \( f, g \) satisfies \( fg \in I \), then \( f \in I \) or \( g \in I \).
   (b) Let \( I \) denote a homogeneous ideal. Show that the projective variety \( \mathcal{V}(I) \) is irreducible if \( I(\mathcal{V}(I)) \) is prime.

82. Prove that if an affine variety \( V \) is irreducible then its projectivization \( \mathbb{P}(V) \) is irreducible.
   (Hint: use \( \mathbb{P}(V) \) is the smallest projective variety containing \( V \).

83. Prove that \( I(\mathbb{P}(V)) = \mathcal{I}(V) \) for any affine variety \( V \subseteq \mathbb{A}^n \). (Hint: adapt the proof of (b) in Proposition on \( \mathbb{P}(V) \).)

84. [CLO, Page 383, \#9] Prove that an affine variety \( V \) is irreducible if and only if its projectivization \( \mathbb{P}(V) \) is irreducible. (Hint: use [CLO, Page 383, \#7] without proof.) (Remark: this question extends Question 82.)

85. [CLO, Page 384, \#14(a)(c)] In this exercise, we will see that \( \mathbb{P}^n \times \mathbb{P}^m \) can be identified with a certain projective variety in \( \mathbb{P}^{n+m+mn} \) known as a \textit{Segre variety}. The idea is as follows. Let \( p = (x_0, \ldots, x_n) \) be homogeneous coordinates of \( p \in \mathbb{P}^n \) and let \( q = (y_0, \ldots, y_m) \) be homogeneous coordinates of \( q \in \mathbb{P}^m \). The Segre mapping \( \sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+mn} \) is defined by taking the pair \( (p, q) \in \mathbb{P}^n \times \mathbb{P}^m \) to the point in \( \mathbb{P}^{n+m+mn} \) with homogeneous coordinates
   \[ (x_0y_0, x_0y_1, \ldots, x_0y_m; x_1y_0, \ldots, x_1y_m; \ldots; x_py_m), \]
   the coordinates being all the possible products \( x_iy_j \) where \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \). The image of \( \sigma \) is a projective variety called a \textit{Segre variety}.
   (a) Show that \( \sigma \) is well defined. (Hint: show that we obtain the same point in \( \mathbb{P}^{n+m+mn} \) no matter what homogeneous coordinates we use for \( p \) and \( q \).)
   (b) Taking \( n = m = 1 \), write out \( \sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) explicitly and find homogeneous equations for the image. (Hint: you should show that a single quadratic equation suffices – this Segre variety is a \textit{quadric surface} in \( \mathbb{P}^3 \).)

86. Assume that \( k = \mathbb{C} \). Find the singular points and the tangent lines at the singular points for the following affine varieties in \( \mathbb{A}^2 \):
   (a) \( \mathcal{V}(y^2 - x^3 + x) \)
   (b) \( \mathcal{V}((x^2 + y^2)^2 + 3x^2y - y^3) \)
   (c) \( \mathcal{V}((x^2 + y^2)^3 - 4x^2y^2) \).

87. Given an affine variety \( V \), the tangent space to \( V \) at the origin \( p \in V \) is defined to be
   \[ T_p(V) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(p)\alpha_i = 0 \text{ for all } g \in I(V)\}. \]
   If \( 0 \neq f \in k[x_1, \ldots, x_n] \) is irreducible and if \( \mathcal{V}(f) \) contains the origin \( p \), prove that
   \[ T_p(\mathcal{V}(f)) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\alpha_i = 0 \}. \]
88. Let \( f = y - x^3, g = y \in \mathbb{C}[x, y] \). Homogenize \( f \) and \( g \). You should find that \( \mathcal{V}(h_f) \cap \mathcal{V}(h_y) = \{(0, 0, 1)\} \subset \mathbb{P}^2 \). Hence, \( U_0 \cap \mathcal{V}(h_f) \cap \mathcal{V}(h_y) = \emptyset \), so \( i(\mathcal{V}(h_f) \cap \mathcal{V}(h_y), p) = 0 \) for all \( p \in U_0 \). Verify this is true by applying the definition of \( i(\mathcal{V}(h_f) \cap \mathcal{V}(h_y), p) \) using \( U_j = U_0 \).

89. Let \( f = x^2 - y^2 - 1, g = y - x \in \mathbb{C}[x, y] \). Show that \( i(\mathcal{V}(h_f) \cap \mathcal{V}(h_y), p) = 2 \) for all \( p \in \mathcal{V}(h_f) \cap \mathcal{V}(h_y) \).

90. Suppose that \( k \) is algebraically closed and let \( R = k[x, y]/(x^2y) \). Find \( X = \text{Spec} R \), its closed sets and its open sets. Find \( R[x^{-1}] \) and \( R[y^{-1}] \).

91. Let \( V = \mathcal{V}(xyz) \subset \mathbb{P}^2 \); so \( V = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z) \), which is a “triangle” in \( \mathbb{P}^2 \). Suppose \( q \in k^\times \) and define \( \sigma : V \to V \) by

\[
\sigma((\alpha_0, \alpha_1, \alpha_2)) = \begin{cases}
(0, \alpha_1, q\alpha_2) & \text{if } \alpha_0 = 0 \\
(q\alpha_0, 0, \alpha_2) & \text{if } \alpha_1 = 0 \\
(\alpha_0, q\alpha_1, 0) & \text{if } \alpha_2 = 0.
\end{cases}
\]

(a) Show that \( \sigma \) is well defined (i.e., show that \( \sigma \) respects the fact \( V \) lies in \( \mathbb{P}^2 \) and that \( \sigma \) is well defined on each vertex of the triangle).

(b) Show that \( \sigma \) is one-to-one and onto. (The fact that each entry of \( \sigma(\alpha_0, \alpha_1, \alpha_2) \) is a polynomial in the \( \alpha_i \) means that \( \sigma \) is regular, so, since \( \sigma \) is one-to-one and onto, (b) implies that \( \sigma \) is an automorphism of the scheme \( V \).)

(c) Suppose that \( q^3 \neq 1 \). Show that it is not possible to write \( \sigma \) in a non-piecewise way. (It follows that \( \sigma \) is not an automorphism of the homogeneous coordinate ring of \( V \), which is \( k[x, y, z]/(xyz) \).)

92. Let \( R = \frac{k(x, y, z)}{(xy - qyx, yz - qzy, zx - qxz)} \), where \( q \in k^\times, q^3 \neq 1 \). Show that

\[
\dim_k \left( \frac{R}{R(y - x) + R(z - x)} \right) = 2.
\]