Answer the following questions in groups of two, but turn in one solution sheet per student. Write neatly and orderly as points will be deducted for messy work. No work shown ⇒ partial/full credit not possible, so show as much work as possible.

**PART A** (Due Wed Jan 30)

If a function \( f \) satisfies \( f(x) \geq 0 \) on \([a, b]\), then \( \int_a^b f(x) \, dx \) represents the area under the graph \( y = f(x) \). If we can find an antiderivative for \( f \), then we can use the first fundamental theorem of calculus to find the integral, and hence find the area. However, if we cannot find an antiderivative of \( f \), then we can approximate the area using rectangles. Alternatively, we could use other shapes besides rectangles, such as **trapezoids**, which are four-sided figures with two parallel sides. The area for the trapezoid in the adjacent figure is given by the formula \( \frac{1}{2}h(b_1 + b_2) \).

1. We will now use trapezoids to estimate the area under the graph of \( y = x^2 \) on \([1, 2]\).
   
   (a) Divide the interval \([1, 2]\) into four equal subintervals. On each subinterval draw a vertical strip up to the graph and connect the two endpoints on the graph using a straight line. Find the coordinates of the points where the strips meet the graph.

   (b) Find the area of each of the four trapezoids in the above graph and use them to obtain an approximation for the area under the graph on \([1, 2]\). This method is called the Trapezoidal Rule.

   (c) Find the exact area and compare with your approximation.

2. Suppose that \( f \) is a function such that \( f(x) > 0 \) for all \( x \) and \( f''(x) < 0 \) for all \( x \). Using the Trapezoidal Rule, if we increase the number of subintervals, does the approximation of \( \int_a^b f(x) \, dx \) get larger or smaller? Explain your reasoning (as usual). Does your answer change if instead we assume that \( f(x) < 0 \) for all \( x \)? Explain.

3. The family of quadratic functions of one variable (say \( x \)) can be described by \( cx^2 + dx + g \), where \( c, d, g \in \mathbb{R} \) are constants. Find a family of functions \( f \) for which the Trapezoidal Rule is equal to the integral of \( f \) on \([a, b]\) no matter how many subintervals are used. Explain your reasoning.
The method of using trapezoids to approximate a definite integral is called the Trapezoidal Rule. One can also use genuine curves, such as parabolas, at the top of the strips instead of lines, and this method is called Simpson’s Rule. Methods such as the Trapezoidal Rule and Simpson’s Rule are part of the subject called numerical integration.

To compute a formula for Simpson’s Rule, we partition \([a, b]\) into \(n\) equal subintervals with \(x\)-coordinates \(a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\), but now we use only even values for \(n\). Across every two adjacent subintervals \([x_{i-1}, x_i]\) and \([x_i, x_{i+1}]\), we approximate the graph \(y = f(x)\) by a parabola that passes through the three points \((x_{i-1}, y_{i-1})\), \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\) (see Figure 1). To find a formula for the shaded area below such a parabola on two such subintervals, we may place the strips at the origin as in Figure 2, and simplify notation as shown.

The shaded area is \(\int_{-t}^{t} (Ax^2 + Bx + C) \, dx = \frac{2t}{3} (At^2 + 3C)\).

Since the parabola passes through the three points \((-t, h)\), \((0, k)\) and \((t, r)\), we also have \(h = At^2 - Bt + C\), \(k = C\), \(r = At^2 + Bt + C\), so that \(At^2 - Bt = h - k\), and \(At^2 + Bt = r - k\), so that we have \(2At^2 = r + h - 2k\) (why?). Hence,

the shaded area = \(\frac{2t}{3} (At^2 + 3C) = \frac{t}{3} (2At^2 + 6C) = \frac{t}{3} (r + h - 2k + 6k) = \frac{t}{3} (h + 4k + r)\),

and this formula uses only the \(y\)-coordinates of the three points and the width of each subinterval.

Shifting the subintervals back to their original position, the formula for the area under a parabola that passes through \((x_{i-1}, y_{i-1})\), \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\) (where these points lie on the graph \(y = f(x)\)) is 

\[\frac{\Delta x}{3} (y_{i-1} + 4y_i + y_{i+1}).\]

Adding together the areas under parabolas for every two consecutive subintervals, using each subinterval only once, is called Simpson’s Rule.
4. (a) Give the formula for the area under a parabola over the first two subintervals: $[x_0, x_1]$ and $[x_1, x_2]$, where, as above, the parabola passes through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

(b) Give the formula for the area under a parabola over the third and fourth subintervals: $[x_2, x_3]$ and $[x_3, x_4]$, where, as above, the parabola passes through $(x_2, f(x_2))$, $(x_3, f(x_3))$ and $(x_4, f(x_4))$.

(c) Find the sum of the areas computed in (a) and (b) and simplify your formula.

(d) Using $n$ subintervals on $[a, b]$, where $n$ is even, and parabolas on every two subintervals, as above, find a formula for the sum of the areas below those parabolas; this is Simpson’s Rule. (One could derive a formula for Simpson’s Rule with subintervals of unequal width, but it would be much more complicated than your formula.)

5. Use Simpson’s Rule to estimate the area under the graph of $y = \frac{1}{x^2}$ on $[1, 2]$ using $n = 6$.

6. In using Simpson’s rule, a certain fact was assumed implicitly: there is only one parabola that passes through the three points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$. Use this fact to decide whether or not the following statement is true; if it is true, explain why, but if it is false, give an example to show that it is false.

If $f(x) = cx^2 + dx + g$, where $c$, $d$ and $g$ are constants, then Simpson’s Rule will yield the same number no matter how many subintervals are used, and that number is the integral of $f$ on $[a, b]$.

Recall that Test 1 is on Wed Feb 6. Keep an eye on my website for information for the test. Also, turn to the next page for some study techniques.
Some Study Techniques

1. After each lecture, read/skim through your lecture notes and only then work on the assigned homework. Keep up with homework and use Padmini, Math Clinic or instructor for any questions on which you are stuck.

2. Do the first two questions assigned in the homework; then redo them both, and do the third; then redo those three, and do the fourth; then redo those four and do the fifth, etc. This study technique allows different, but related, ideas that arise in different questions to connect, and so allows students to see the big picture of what is going on while, simultaneously, learning intricate details.

3. The correct way to use a solution manual is to read the first one or two lines of a solution, and then try to continue the problem from there without looking at the manual’s solution again. If that is not possible, then read the solution all the way through, but then close the manual and try to reproduce the solution (or at least the main ideas of the solution) without the aid of the manual.

4. After you complete your homework, read the section(s) in the textbook that will be covered in the next lecture. It is likely that you will not understand everything you read; however, reading the sections before lecture will help you understand what is presented in the lecture.

5. A couple/few days before a test, look over all the homework. The day/night before the test, look over the homework again, and ask yourself what the key concepts and methods are per question (WITHOUT reworking the questions). Jot this down on a sheet of paper and, when you are done, compare with your solutions to see if you were correct. Any question in which you were on the wrong track, look over your solution to it to make sure you see what the main idea is. This work the day/night before will help get your brain to think faster and bring the material to the front of your brain to help you think faster during the actual test.